Solution
Assignment #2

Question 1

(a)

$$E_x = \int_{-\infty}^{\infty} A^2 \sin^2 2\pi f_0 t \, dt$$

$$= \int_{-\infty}^{\infty} \left(\frac{A^2}{2} - \frac{A^2}{2} \cos 4\pi f_0 t \right) dt$$

$$= \left(\frac{A^2}{2} t - \frac{A^2}{8\pi f_0} \sin 4\pi f_0 t \right)_{-\infty}^{\infty}$$

$$= \infty$$

$$P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left(A^2 \sin^2 2\pi f_0 t \right) dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \left[\int_{-T}^{T} \frac{A^2}{2} \, dt - \int_{-T}^{T} \frac{A^2}{2} \cos 4\pi f_0 t \, dt \right]$$

$$= \lim_{T \to \infty} \frac{1}{2T} \left(\frac{A^2}{2} t \Big|_{-T}^{T} - \frac{A^2}{8\pi f_0} \sin 4\pi f_0 t \Big|_{-T}^{T} \right)$$

$$= \lim_{T \to \infty} \frac{1}{2T} \left(A^2 T - \frac{A^2}{4\pi f_0} \sin 4\pi f_0 T \right)$$

$$= \frac{A^2}{2}$$

$$R_x(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A^2 \sin 2\pi f_0 t \sin 2\pi f_0 (t-\tau)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\left[\int_{-T}^{T} \frac{A^2}{2} \cos 2\pi f_0 \tau \, dt - \int_{-T}^{T} \frac{A^2}{2} \cos 2\pi f_0 (2t-\tau)\, dt \right]$$

$$= \lim_{T\to\infty} \frac{1}{2T}\left(A^2 T \cos 2\pi f_0 \tau - \frac{A^2}{8\pi f_0} \sin 2\pi f_0 (2t-\tau) \Big|_{-T}^{T} \right)$$

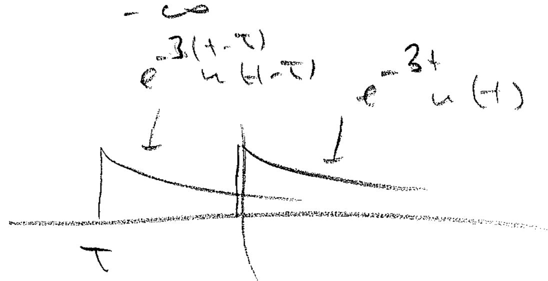
$$= \frac{A^2}{2} \cos 2\pi f_0 \tau$$

$$S_x(f) = \mathcal{F}\{R_x(\tau)\} = \frac{A^2}{4}\delta(f-f_0) + \frac{A^2}{4}\delta(f+f_0)$$

(b)

$$E_y = \int_0^\infty e^{-6t}\, dt = -\frac{1}{6} e^{-6t}\Big|_0^\infty = \frac{1}{6}$$

$$\varphi_y(\tau) = \int_{-\infty}^{\infty} e^{-3t} u(t)\, e^{-3(t-\tau)} u(t-\tau)\, dt$$

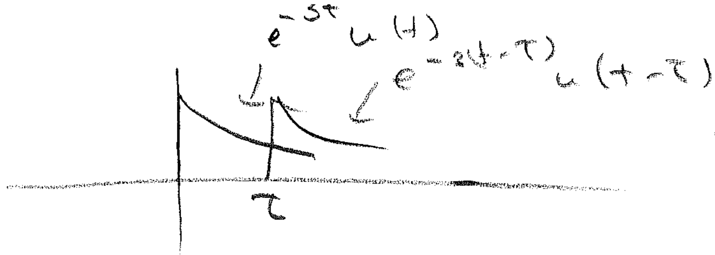


for $\tau < 0$

$$\varphi_y(\tau) = \int_0^\infty e^{-3t} e^{-3(t-\tau)}\, dt$$

$$= \int_0^\infty e^{-6t} e^{3\tau}\, dt = -\frac{1}{6} e^{-6t} e^{3\tau}\Big|_0^\infty$$

$$= \frac{1}{6} e^{3\tau} \qquad \tau < 0$$



for $\tau > 0$

$$\varphi_y(\tau) = \int_{\tau}^{\infty} e^{-3t} e^{-3(t-\tau)}\, dt$$

$$= \int_{\tau}^{\infty} e^{-6t} e^{3\tau}\, dt$$

$$= -\frac{1}{6} e^{-6t} e^{3\tau} \Big|_{\tau}^{\infty}$$

$$= \frac{1}{6} e^{-6\tau} e^{3\tau} = \frac{1}{6} e^{-3\tau}, \quad \tau > 0$$

$$\therefore \boxed{\varphi_y(\tau) = \frac{1}{6} e^{-3|\tau|}}$$

$$G_y(f) = \mathcal{F}\left\{ \frac{1}{6} e^{-3|\tau|} \right\}$$

$$= \frac{1}{6}\left(\frac{6}{9+(2\pi f)^2} \right) = \frac{1}{9+(2\pi f)^2}$$

Note: Since $y(t)$ is an energy signal,

$$G_y(f) = |Y(f)|^2$$

$$Y(f) = \mathcal{F}\{ e^{-3t} u(t) \} = \frac{1}{3+j2\pi f}$$

$$G_y(f) = \frac{1}{3+j2\pi f} \cdot \frac{1}{3-j2\pi f} = \frac{1}{9+(2\pi f)^2}$$

And $\varphi_y(\tau) = \mathcal{F}^{-1}\{G_y(f)\} = \frac{1}{6} e^{-3|\tau|}$

(easier solution this way)

(c) $z(t) = \Delta(t)$

$$E_z = \int_{-1}^{0} (1+t)^2 \, dt + \int_{0}^{1} (1-t)^2 \, dt$$

$$= \int_{-1}^{0} (1+2t+t^2) \, dt + \int_{0}^{1} (1-2t+t^2) \, dt$$

$$= t + t^2 + \tfrac{1}{3}t^3 \Big|_{-1}^{0} + \left(t - t^2 + \tfrac{1}{3}t^3\right)\Big|_{0}^{1}$$

$$= 0 - \left(-1+1-\tfrac{1}{3}\right) + \left(1-1+\tfrac{1}{3}\right)$$

$$= \tfrac{1}{3} + \tfrac{1}{3} = \underline{\underline{\tfrac{2}{3}}}$$

$$Z(f) = \mathrm{sinc}^2(f)$$

$$G_z(f) = \mathrm{sinc}^4(f)$$

$$\psi_z(\tau) = \Delta(\tau) * \Delta(\tau)$$

(good enough!)

(d) $w(t) = A\cos 2\pi 100t + B\sin 2\pi 200t$

$$P_w = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(A^2\cos^2 2\pi 100t + 2AB\cos 2\pi 100t \sin 2\pi 200t + B^2 \sin^2 2\pi 200t\right) dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(\frac{A^2}{2} + \frac{A^2}{2}\cos 4\pi 100t + \frac{B^2}{2} - \frac{B^2}{2}\sin 4\pi 200t + AB\sin 2\pi 300t - AB\sin 2\pi 100t\right) dt$$

all sinusoids have average = 0

$$\therefore P_w = \frac{A^2}{2} + \frac{B^2}{2}$$

$$R_w(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} w(t)\,w(t-\tau)\,dt$$

using trigonometric identities of

$$\cos a \cos b = \tfrac{1}{2}\cos(a+b) + \tfrac{1}{2}\cos(a-b)$$

$$\sin a \sin b = \tfrac{1}{2}\cos(a-b) - \tfrac{1}{2}\cos(a+b)$$

and $\cos a \sin b = \tfrac{1}{2}\sin(a+b) - \tfrac{1}{2}\sin(a-b)$

we can find that

$$R_w(\tau) = \frac{A^2}{2}\cos 2\pi 100\tau + \frac{B^2}{2}\cos 2\pi 200\tau$$

$$S_w(f) = \frac{A^2}{4}\delta(f-100) + \frac{A^2}{4}\delta(f+100)$$
$$+ \frac{B^2}{4}\delta(f+200) + \frac{B^2}{4}\delta(f-200)$$

see end of question 3

Question 2

$$\mathcal{F}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)\,e^{-j2\pi f t}\,dt$$

let $u = t - t_0$
$du = dt$

$$\mathcal{F}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(u)\,e^{-j2\pi f(u+t_0)}\,du$$

$$= \int_{-\infty}^{\infty} x(u)\,e^{-j2\pi f u}\,e^{-j2\pi f t_0}\,du$$

$$= \left(\int_{-\infty}^{\infty} x(u)\,e^{-j2\pi f u}\,du\right) e^{-j2\pi f t_0}$$

$$X(f)\,e^{-j2\pi f t_0}$$

Question 3

$$\int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

$$\therefore \int_{-\infty}^{\infty} x(t)\, y^*(t)\, dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df\, y^*(t)\, dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f)\, y^*(t)\, e^{j2\pi f t}\, dt\, df$$

$$= \int_{-\infty}^{\infty} X(f) \underbrace{\int_{-\infty}^{\infty} y^*(t)\, e^{j2\pi f t}\, dt}_{Y^*(f)}\, df$$

$$= \int_{-\infty}^{\infty} X(f)\, Y^*(f)\, df$$

from question 1(d), we have

$$\int_{-\infty}^{\infty} A\cos 2\pi 100 t \cdot B\sin 2\pi 200 t\, dt$$

$$= \int_{-\infty}^{\infty} \left[\frac{A}{2}\delta(f-100) + \frac{A}{2}\delta(f+100)\right]\left[\frac{B}{2}\delta(f-200) + \frac{B}{2}\delta(f+200)\right] df$$

$$= \underline{\underline{0}}$$