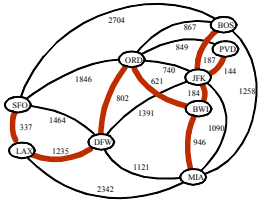
Minimum Spanning Tree



Outline and Reading

- Minimum Spanning Trees (§12.7)
  - Definitions
  - A crucial fact
- Prim-Jarnik's Algorithm (§12.7.2)
- Kruskal's Algorithm (§12.7.1)

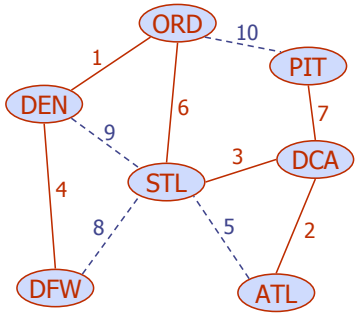
Minimum Spanning Tree

Spanning subgraph
- Subgraph of a graph G containing all the vertices of G

Spanning tree
- Spanning subgraph that is itself a (free) tree

Minimum spanning tree (MST)
- Spanning tree of a weighted graph with minimum total edge weight

Applications
- Communications networks
- Transportation networks

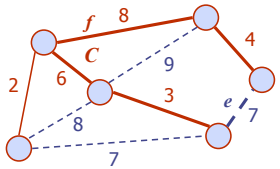


Cycle Property

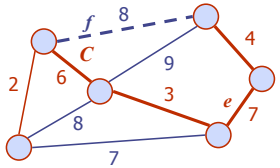
Cycle Property:
- Let T be a minimum spanning tree of a weighted graph G
- Let e be an edge of G that is not in T and let C be the cycle formed by adding e to T
- For every edge f of C, $weight(f) \leq weight(e)$

Proof:

By contradiction
- If $weight(f) > weight(e)$ we can get a spanning tree of smaller weight by replacing e with f

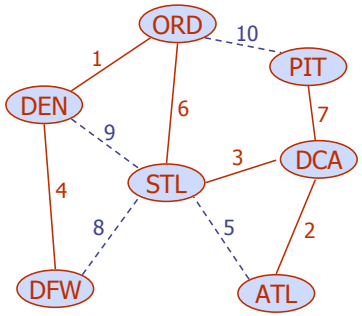


Replacing f with e yields a better spanning tree



Cycle Property

In other words:
in any cycle of the graph, the non-spanning tree edge (dotted line) has max weight.



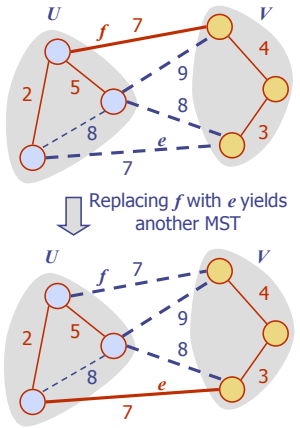
Partition Property

Partition Property:

Consider a partition of the vertices of G into subsets U and V. Let e be an edge of minimum weight across the partition. There is a minimum spanning tree of G containing edge e

Proof:

- Let T be an MST of G
- If T does not contain e, consider the cycle C formed by e with T and let f be an edge of C across the partition
- By the cycle property, $weight(f) \leq weight(e)$
- Thus, $weight(f) = weight(e)$
- We obtain another MST by replacing f with e

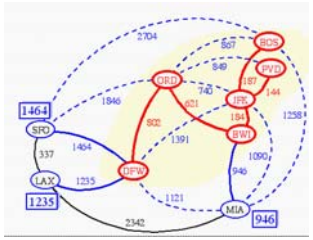


Prim-Jarnik's Algorithm

- Prim-Jarnik's algorithm for computing an MST is similar to Dijkstra's algorithm
- We assume that the graph is connected
- We pick an arbitrary vertex s and we grow the MST as a cloud of vertices, starting from s
- We store with each vertex v a label $d(v)$ representing the smallest weight of an edge connecting v to any vertex in the cloud (as opposed to the total sum of edge weights on a path from the start vertex to u).

Prim-Jarnik's Algorithm

- At each step
  - We add to the cloud the vertex u with (minimum-weight edge) outside the cloud with the smallest distance label
  - We update the labels of the vertices adjacent to u



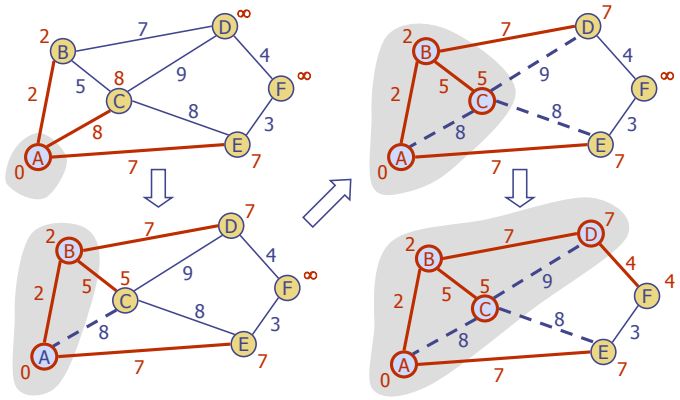
- Use a priority queue Q whose keys are D labels, and whose **elements are vertex-edge pairs**.
  - Key: distance
  - Element: vertex-edge pair

For example, an entry of Q is ((z, (u,z)), D[z]) for a vertex z, where (z, (u,z)) is the **element** and D[z] is the **key** of the vertex z.

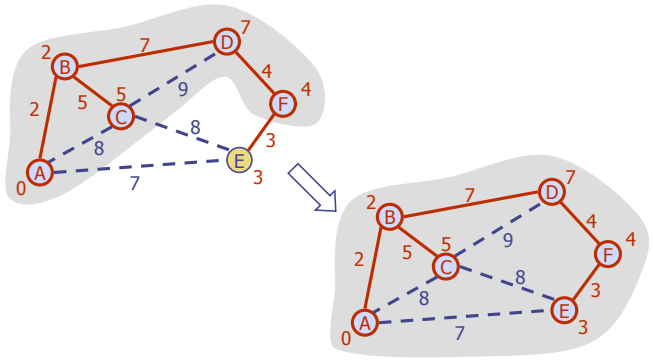
- Any vertex v can be the starting vertex.
- We still initialize all the D[u] values to INFINITE, but we also **initialize the edge associated with u to null.**
- **Return** the minimum-spanning tree T.
- *We can reuse code from Dijkstra's, and we only have to change a few things. Let's look at the pseudocode....*


```
Algorithm PrimJarnik(G):
        Input: A weighted graph G.
        Output: A minimum spanning tree T for G.
  pick any vertex v of G
  D[v] ← 0
  for each vertex u ≠ v do
    D[u] ← ∞
  Initialize T ← ∅
  Initialize priority queue Q with an entry ((u, null), D[u]) for each vertex u,
  where (u, null) is the element and D[u]) is the key.
  while Q ≠ ∅ do  {pull u into the cloud C}
     (u, e) ← Q.removeMin()
     add vertex u and edge e to T
    for each vertex z adjacent to u such that z is in Q do
     {perform the relaxation operation on edge (u, z) }
          if weight(u, z) < D[z] then
            D[z] ←weight(u, z)
            change to (z, (u, z)) the element of z in Q
            change to D[z] the key of vertex z in Q
          return tree T
```


Example



Example (contd.)



Prim-Jarnik… Why It Works

- This is an application of the Cycle Property!
- Let the minimum edge at some iteration be (u,v). If there is an MST not containing (u,v), then (u,v) completes a cycle. Since (u,v) was considered before some other edge connecting v to the cluster, it must have weight equal to or lower than that other edge. A new MST can be formed by swapping.

Analysis

- Graph operations
  - Method incidentEdges is called once for each vertex
- Label operations
  - We set/get the labels of vertex z $O(\deg(z))$ times
  - Setting/getting a label takes $O(1)$ time
- Priority queue operations
  - Each vertex is inserted once into and removed once from the priority queue, where each insertion or removal takes $O(\log n)$ time
  - The key of a vertex w in the priority queue is modified at most $\deg(w)$ times, where each key change takes $O(\log n)$ time
- Prim-Jarnik's algorithm runs in $O((n + m) \log n)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\sum_v \deg(v) = 2m$
- The running time is $O(m \log n)$ since the graph is connected

Dijkstra vs. Prim-Jarnik

```
Algorithm DijkstraShortestPaths(G, s)
  Q ← new heap-based priority queue
  for all v ∈ G.vertices()
    if v = s
      setDistance(v, 0)
    else
      setDistance(v, ∞)
    setParent(v, ∅)
    l ← Q.insert(getDistance(v), v)
    setLocator(v,l)
  while ¬Q.isEmpty()
    u ← Q.removeMin()
    for all e ∈ G.incidentEdges(u)
      z ← G.opposite(u,e)
      r ← getDistance(u) + weight(e)
      if r < getDistance(z)
        setDistance(z,r)
        setParent(z,e)
        Q.replaceKey(getLocator(z),r)
```

```
Algorithm PrimJarnikMST(G)
  Q ← new heap-based priority queue
  s ← a vertex of G
  for all v ∈ G.vertices()
    if v = s
      setDistance(v, 0)
    else
      setDistance(v, ∞)
    setParent(v, ∅)
    l ← Q.insert(getDistance(v), v)
    setLocator(v,l)
  while ¬Q.isEmpty()
    u ← Q.removeMin()
    for all e ∈ G.incidentEdges(u)
      z ← G.opposite(u,e)
      r ← weight(e)
      if r < getDistance(z)
        setDistance(z,r)
        setParent(z,e)
        Q.replaceKey(getLocator(z),r)
```


Kruskal's Algorithm

- Each vertex is initially stored as its own cluster.
- At each iteration, the minimum weight edge is added to the spanning tree if it joins 2 distinct clusters.
- The algorithm ends when all the vertices are in the same cluster.

Kruskal's Algorithm… Why It Works

- This is an application of the Partition Property!
- If the minimum edge at some iteration is (u,v), then if we consider a partition of G with u in one cluster and v in the other, then the partition property says that there must be an MST containing (u,v).

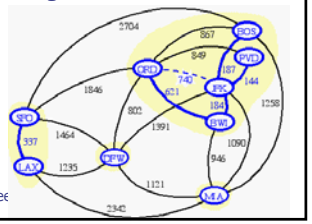
Kruskal's Algorithm

- A priority queue stores the edges outside the cloud
  - Key: weight
  - Element: edge
- At the end of the algorithm
  - We are left with one cloud that encompasses the MST
  - A tree T which is our MST

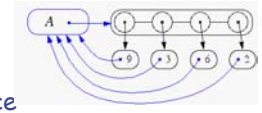
```
Algorithm KruskalMST(G)
  for each vertex V in G do
    define a Cloud(v) of ← {v}
  let Q be a priority queue.
  Insert all edges into Q using their
  weights as the key
  T ← ∅
  while T has fewer than n-1 edges do
    edge e = T.removeMin()
    Let u, v be the endpoints of e
    if Cloud(v) ≠ Cloud(u) then
      Add edge e to T
      Merge Cloud(v) and Cloud(u)
  return T
```


Data Structure for Kruskal Algortihm

- The algorithm maintains a forest of trees
- An edge is accepted it if connects distinct trees
- We need a data structure that maintains a **partition**, i.e., a collection of disjoint sets, with the operations:
  - **find**(u): return the set storing u
  - **union**(u,v): replace the sets storing u and v with their union



Representation of a Partition



- Each set is stored in a sequence
- Each element has a reference back to the set
  - operation find(u) takes $O(1)$ time, and returns the set of which u is a member.
  - in operation union(u,v), we move the elements of the smaller set to the sequence of the larger set and update their references
  - the time for operation union(u,v) is $\min(n_u,n_v)$, where n_u and n_v are the sizes of the sets storing u and v
- Whenever an element is processed, it goes into a set of size at least double, hence each element is processed at most log n times

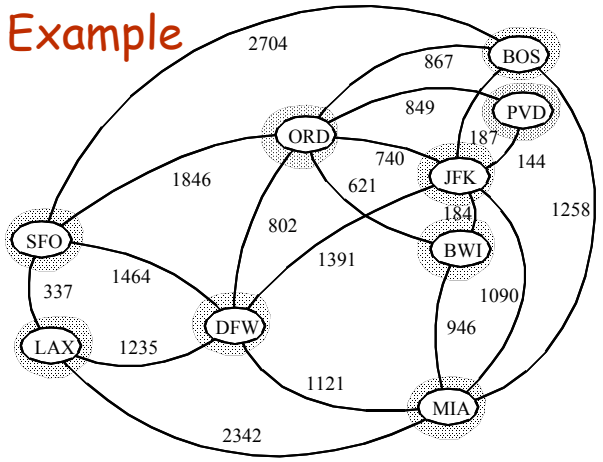
Partition-Based Implementation

- A partition-based version of Kruskal's Algorithm performs cloud merges as unions and tests as finds.

Algorithm Kruskal(*G*):
  Input: A weighted graph *G*.
  Output: An MST *T* for *G*.
Let *P* be a partition of the vertices of *G*, where each vertex forms a separate set.
Let *Q* be a priority queue storing the edges of *G*, sorted by their weights
Let *T* be an initially-empty tree
while *Q* is not empty **do**
  (*u*,*v*) ← *Q*.removeMin()
  if *P*.find(*u*) != *P*.find(*v*) **then**
    Add (*u*,*v*) to *T*
    P.union(*u*,*v*)
return *T*

Running time: O((n+m)log n)

Kruskal Example



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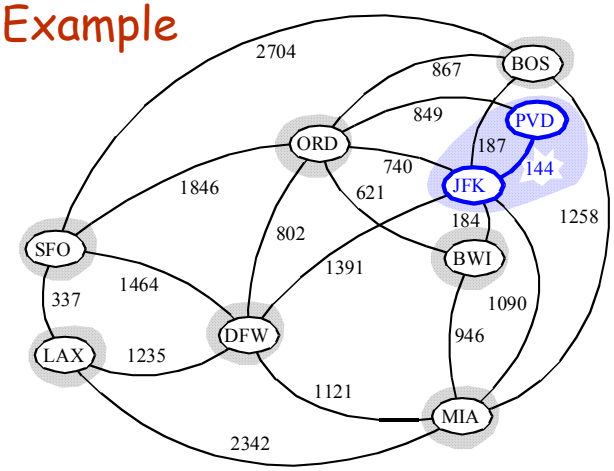

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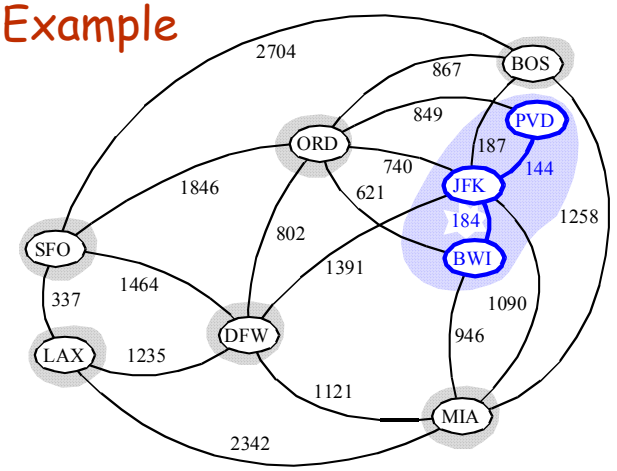

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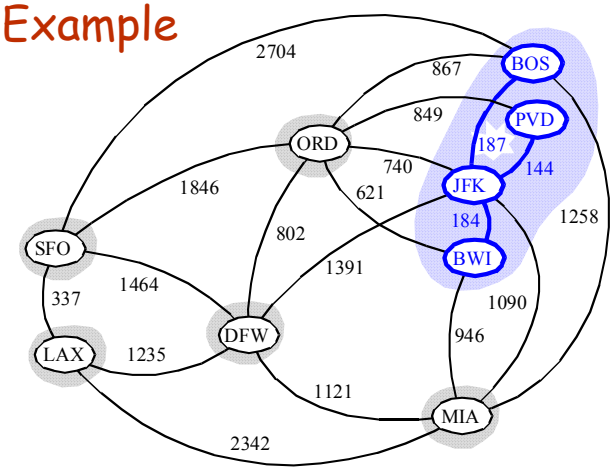

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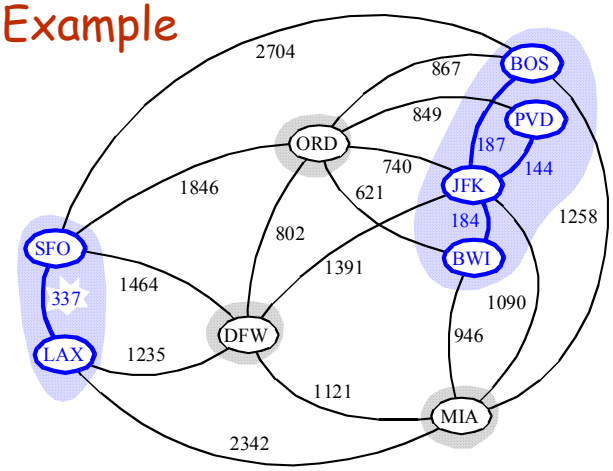

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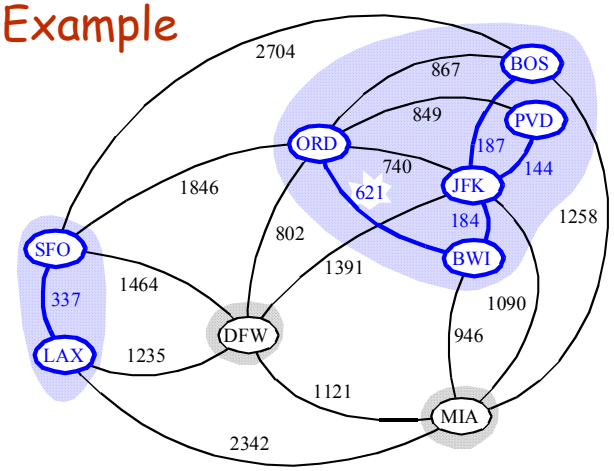

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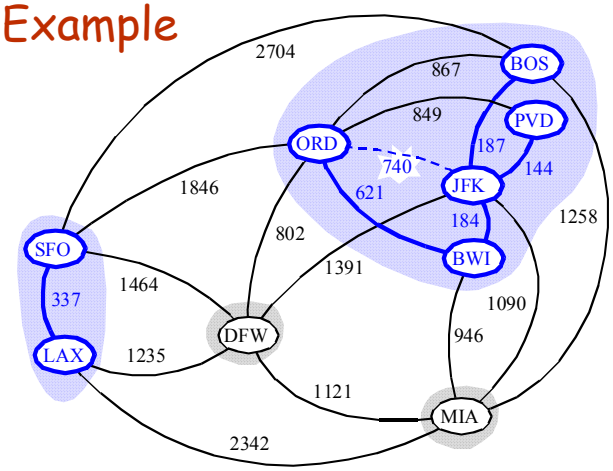

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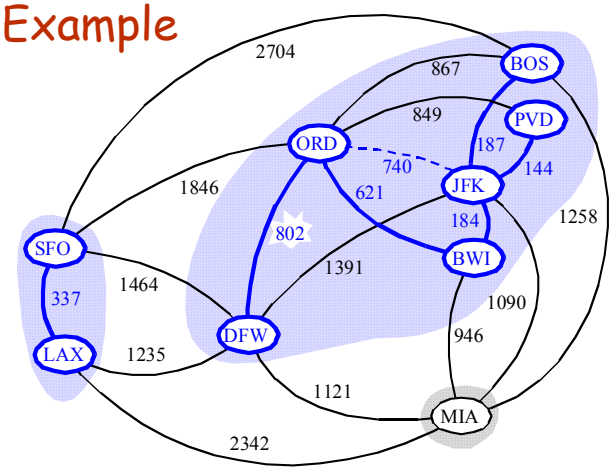

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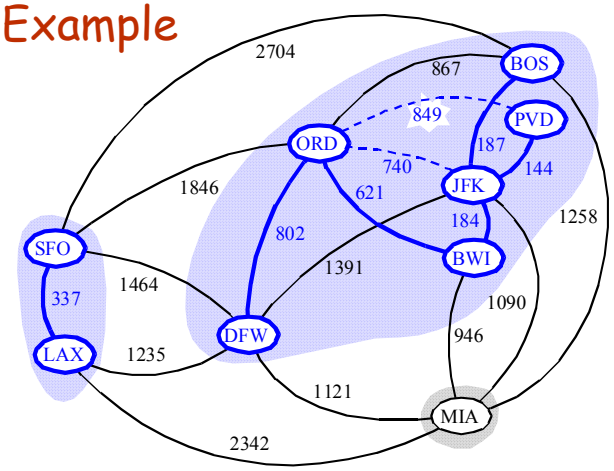

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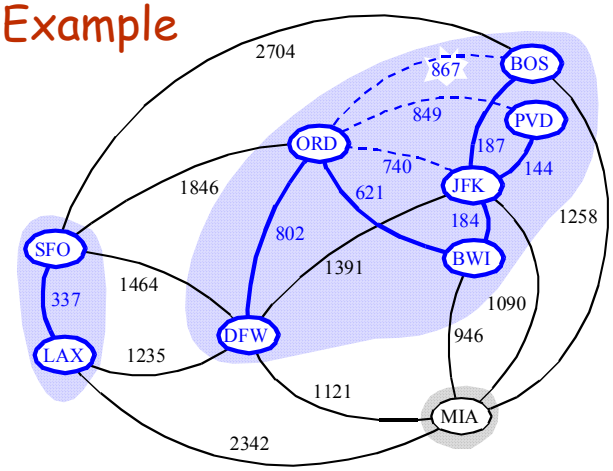

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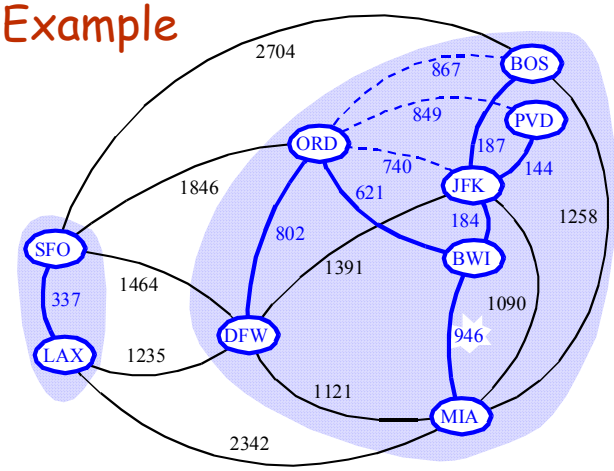

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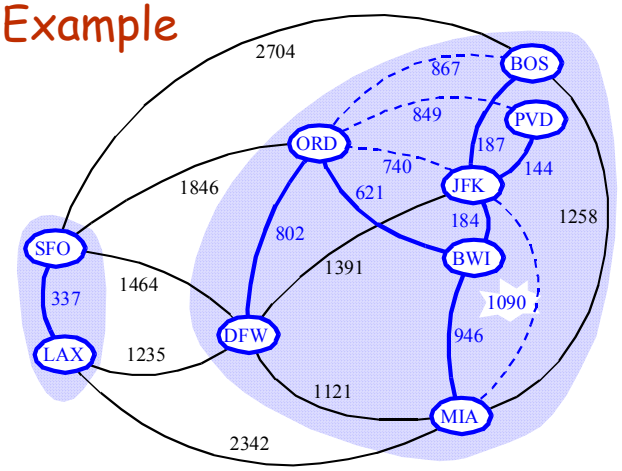

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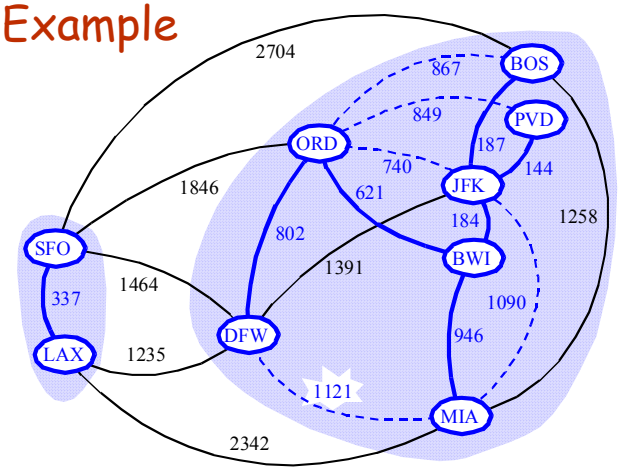

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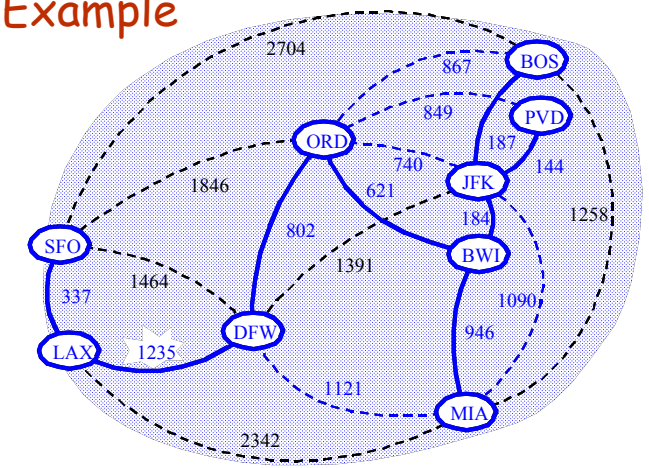

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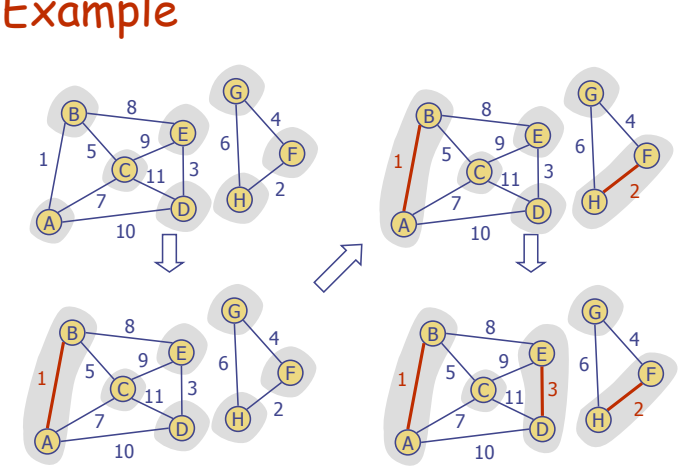

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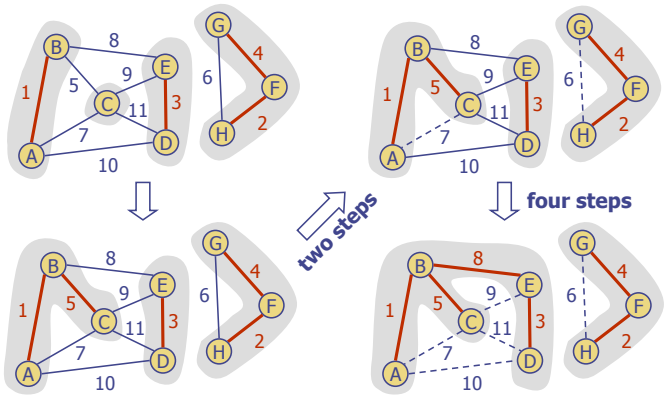
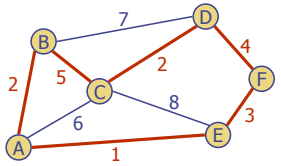


Traveling Salesperson Problem

- A tour of a graph is a spanning cycle (e.g., a cycle that goes through all the vertices)
- A traveling salesperson tour of a weighted graph is a tour that is simple (i.e., no repeated vertices or edges) and has has minimum weight
- No polynomial-time algorithms are known for computing traveling salesperson tours
- The traveling salesperson problem (TSP) is a major open problem in computer science
  - Find a polynomial-time algorithm computing a traveling salesperson tour or prove that none exists



Example of traveling salesperson tour (with weight 17)

TSP Approximation

- We can approximate a TSP tour with a tour of at most twice the weight for the case of Euclidean graphs
  - Vertices are points in the plane
  - Every pair of vertices is connected by an edge
  - The weight of an edge is the length of the segment joining the points
- Approximation algorithm
  - Compute a minimum spanning tree
  - Form an Eulerian circuit around the MST
  - Transform the circuit into a tour