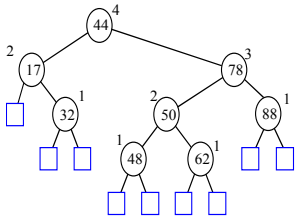
AVL Trees

- Height of an AVL Tree
- Insertion and restructuring
- Removal and restructuring
- Costs

AVL Tree

- **AVL trees are balanced.**
- An AVL Tree is a ***binary search tree*** such that for every internal node v of T, the *heights of the children of v can differ by at most 1.*

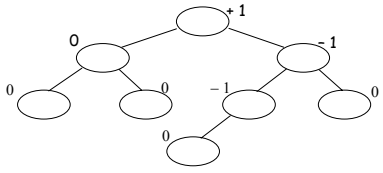


An example of an AVL tree where the heights are shown next to the nodes:

Balancing Factor

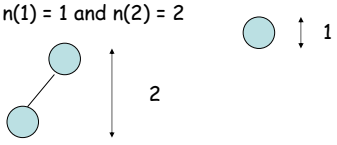
= height(right s.t.) – height(left s.t.)

$\in$ {-1, 0, 1} for AVL tree



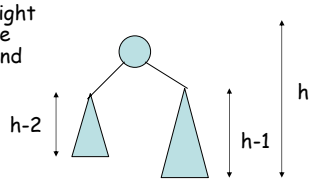
Height of an AVL Tree

- **Proposition**: The ***height*** of an AVL tree T storing n keys is O(log n).
- **Justification**: The easiest way to approach this problem is to find **n(h): the *minimum number of internal nodes* of an AVL tree of height h.**
- We see that n(1) = 1 and n(2) = 2



n(h): the *minimum number of internal nodes* of an AVL tree of height h.

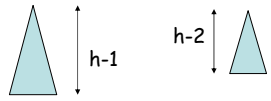
For n ≥ 3, an AVL tree of height h contains the root node, one AVL subtree of height h-1 and the other AVL subtree of height h-2.



i.e., n(h) = 1 + n(h-1) + n(h-2)

Height of an AVL Tree

n(h) = 1 + n(h-1) + n(h-2)



Clearly: n(h-1) > n(h-2),

that is: n(h) > 1+ n(h-2) +n(h-2)
which is > 2n(h-2)

so n(h) > 2n(h-2)

Height of an AVL Tree

So, now we know: n(h) > 2n(h-2)
but then also: n(h-2) > 2n(h-4)

} n(h) > 2n(h-4)= 2 2n(h-4)

n(h) > 4n(h-4)

but then also: n(h-4) > 2n(h-6)

n(h) > 8n(h-6)

We can continue:

n(h) > 2n(h-2)

n(h) > 4n(h-4)

...

n(h) > 2^in(h-2i)

n(h) > 2^in(h-2i)

with
n(1) = 1
n(2) = 2

h-2i = 2

for i = h/2 - 1

n(h) > $2^{h/2 - 1}$ n(2)

n(h) > $2^{h/2}$

log n(h) > log $2^{h/2}$

log n(h) > h/2

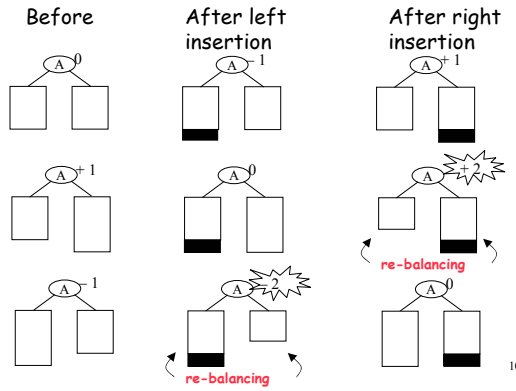
h < 2 log n(h)

which means that h is O(log n)

Insertion

- A binary search tree T is called ***balanced*** if for every node v, the height of v's children differ by at most one.
- Inserting a node into an AVL tree involves performing an insertAtExternal() on T, which changes the heights of some of the nodes in T.
- If an insertion causes T to become unbalanced we have to rebalance...

Insertion



Rebalancing after insertion

We are going to identify 3 nodes which form a grandparent, parent, child triplet and the 4 subtrees attached to them. We will rearrange these elements to create a new balanced tree.

Rebalancing

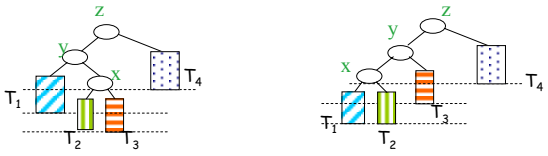
Step 1: Trace the path back from the point of insertion to the first node whose grandparent is unbalanced. Label this node x, its parent y, and grandparent z.

Examples .....

Rebalancing

Step 2: These nodes will have 4 subtrees connected to them. Label them T_1, T_2, T_3, T_4 from left to right.

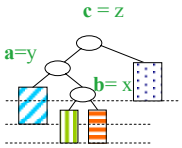
Examples .....



Rebalancing

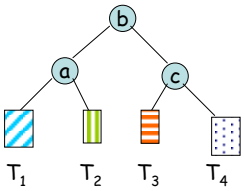
Step 3: Rename x, y, z to a, b, c according to their inorder traversal i.e. if y< x < z then label y 'a', x 'b' and z 'c'.

Example



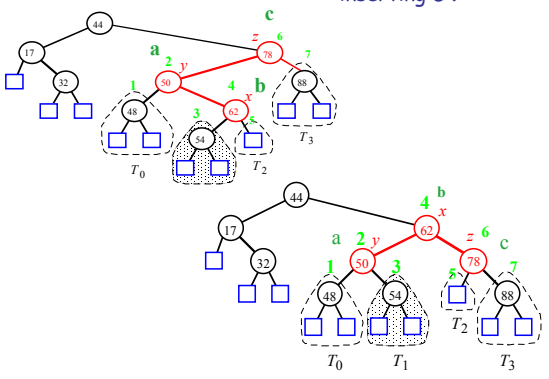
Rebalancing

Step 4: Replace the tree rooted at z with the following tree:



Rebalance done!

Example: after inserting **54**



Does this really work?

We need to see that the new tree is :

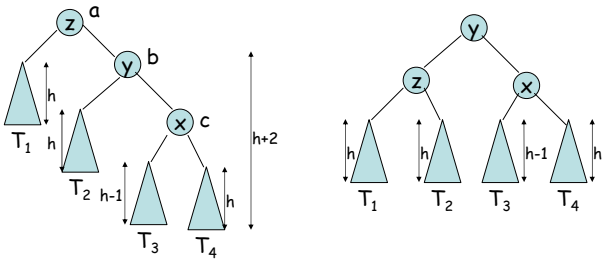
a) A Binary search tree - the inorder traversal of our new tree should be the same as that of the old tree

Inorder traversal: by definition is T1 a T2 b T3 c T4

b) Balanced: have we fixed the problem?

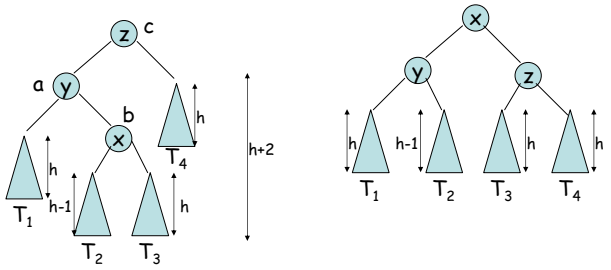
We consider 2 types of examples

Example 1



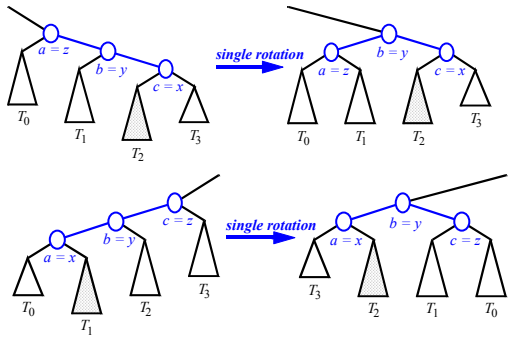
Inorder: T1 z T2 y T3 x T4

Example 2

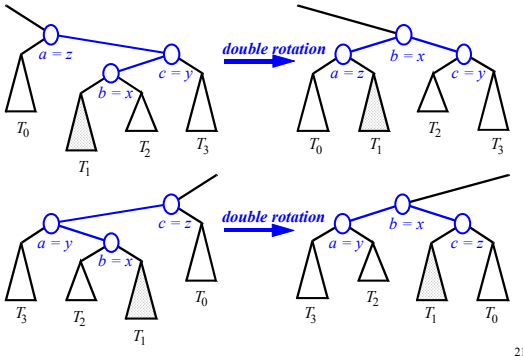


Inorder: T1 y T2 x T3 z T4

Single Rotation



Double Rotation



An Observation…

Notice that in both cases, the new tree rooted at b has the same height as the old tree rooted at z had before insertion.

So.. once we have done one rebalancing act, we are done.


```
restructure (v)
x <- v; Y <- x.parent; z <- y.parent
while (z.isBalanced and not(z.isRoot) )
   x <- y; y <- z; z <- z.parent
if ( not z.isBalanced)
   if ( x = y.left)  { x<=y}
      if (y = z.left) {x<=y<=z}
           a <- x; b <- y; c<- z;
           T2 <- x.right; T3 <- y.right;
      else            { z<=x<=y}
           a <- z; b <- x; c <- y;
           T2 <- x.left; T3 <- x.right;
 else                 {y<=x}
      if (y = z.left)  {y<=x<=z}
           a <- y; b <- x; c <- z;
           T2 <- x.left; T3 <- x.right

      else            { z<=y<=x}
           a <- z; b <- y; c <- x;
           T2 <- y.left; T3 <- x.left
```

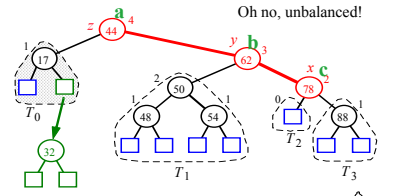
```
T1 <- a.left; T4 <- c.right
b.left <- a; b.right <- c
a.left <- T1; a.right <- T2
c.left <- T3; c.right <- T4
T1.parent <- a; T2.parent <-a
T3.parent <- b; T3.parent <- c

if (z.isRoot) then
     root <- b
     b.parent <- NULL
else if (z.isLeftChild)
        z.parent.left<-b
     else z.parent.right <- b
b.parent <- z.parent
a.parent <- b; c.parent <- b
```

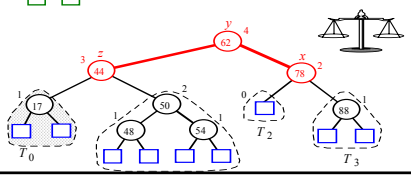

Removal

- We can easily see that performing a removeExternal(w) can cause T to become unbalanced.
- Let *z* be the first unbalanced node encountered while travelling up the tree from w. Also, **let y be the child of z with the larger height, and let x be the child of y with the larger height.**
- We can perform operation restructure(x) to restore balance at the subtree rooted at z.
- As this restructuring may upset the balance of another node higher in the tree, we must continue checking for balance until the root of T is reached

Removal (contd.)

the choice of x is not unique !!!

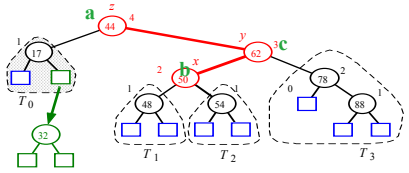




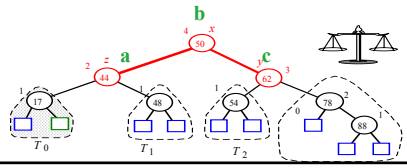
Whew, balanced!

Removal (contd.)

- we could choose a different x:



Oh no, unbalanced!



Whew, balanced!

COMPLEXITY

Searching: find(k):

Inserting: insert(k, o):

Removing: remove(k):

O(log n)