On Density and Convergence of Fuzzy Cellular Automata *

Heather Betel and Paola Flocchini
School of Information Technology and Engineering
University of Ottawa, 800 King Edward,
Ottawa, Ontario, K1N 6N5, Canada
{hbetel,flocchin}@site.uottawa.ca

Abstract

Fuzzy cellular automata (FCA) are cellular automata where the local rule is defined as the “fuzzification" of the local rule of the corresponding Boolean cellular automaton in disjunctive normal form. In this paper we are interested in two separate issues: the relationship between Boolean and fuzzy models, and the asymptotic behavior of elementary fuzzy cellular automata. On the first issue, we show that the number conservation property, extensively studied in the Boolean domain, is preserved in the fuzzy domain: a Boolean CA is number conserving if and only if the corresponding FCA is sum preserving; a similar result is established for another density property. This is the first formal proof of a relationship between the fuzzy and the Boolean model. As for the second issue, we analyse the asymptotic behavior of circular elementary FCA for the first time, and provide a complete analytical study of Weighted Average FCA, a class that includes the density conserving rules and that encompasses most of the observable behaviours.

1 Introduction

Since the introduction of Boolean cellular automata (CA) by von Neumann [24], the study of their properties has interested various disciplines as diverse as ecology, biology, engineering and theoretical computer science (e.g., see [2, 9, 15, 26]). A fundamental problem in the study of CA has always been their classification. The first attempt to classify cellular automata was done by Wolfram in [25] where they were grouped according to the observed behaviour of their space-time diagrams. Since class membership is undecidable, observation of the evolution of a CA starting from (possibly all) initial configurations becomes crucial to understanding its dynamics. The evolution of CA is usually observed either for finite initial configurations in zero backgrounds (*quiescent background*), or for infinite but periodic initial configurations (*circular CA*). Several other criteria for grouping CA have followed: some based on observable behaviours, some on intrinsic properties of CA rules (e.g., see [7, 10, 14, 23]).

*Work partially supported by NSERC.

Fuzzy cellular automata (FCA) are a particular type of continuous cellular automata where the local transition rule is the “fuzzification" of the local rule of the corresponding Boolean cellular automaton in disjunctive normal form[1]. Fuzzy cellular automata were introduced in [5] and some of their properties have been studied in [12, 18, 19], especially when considering finite configurations in quiescent backgrounds. Recently, they have been shown to be useful tools for pattern recognition purposes (e.g., see [16, 17]), and good models for generating images mimicking nature (e.g. [6, 22]).

Besides being interesting in their own right and providing a good model for certain applications, an important reason to study fuzzy CA is to understand their relationship with Boolean CA. In fact, the dynamics of fuzzy CA might shed some light on their Boolean counter-parts, and properties of Boolean CA could be interpreted differently in light of those of fuzzy CA. If clear links between the two systems can be established, properties of Boolean CA not previously observed might be revealed by their presence in FCA. Unfortunately, until now, no such light has been shed and no such results exist.

To date, little is know about the dynamics of FCA, except that, in quiescent backgrounds, none of them has chaotic dynamics [12, 18, 19]. The case of circular FCA has been studied experimentally from random initial configurations; an empirical classification has been proposed based on these studies [11], suggesting that all elementary rules have, as expected, asymptotic periodic behavior but, surprisingly, with periods of only certain lengths: 1,2,4, and n (where n is the size of the circular lattice). Unfortunately, to this date, no analytical proofs exist.

In this paper, we prove the existence of a relationship between fuzzy and Boolean CA, and we analytically derive the asymptotic behavior of a class of circular FCA.

First we study the discrete and the continuous model with regard to density-conservation. More precisely, we consider two types of density conservations: a temporal one, which is the classical notion of *number conservation* and has been studied extensively in the Boolean domain (e.g., see [3, 4, 8, 9]), and a spatial one that has not been studied before. We prove that our fuzzification preserves both: in other words, a one-dimensional Boolean circular cellular automaton (i.e., with periodic initial configuration) is density-conserving if and only if its corresponding circular fuzzy cellular automaton is. As a simple corollary of our result, we re-discover the number conservation property of elementary rule 184 (already well known in the Boolean domain) and we find an interesting spatial density conservation property of another elementary rule (rule 46) that can be translated into the Boolean domain: for any configuration of even size at time $t > 0$, the density of the odd cells is equal to the density of the even cells.

After showing the result on the link between the two models, we concentrate on the asymptotic behavior of elementary circular FCA and, in particular, of the two conserving rules discussed above. We consider a class of FCA, Weighted Average rules, which includes both conserving rules 184 and 46, as well as other rules displaying most of the observed dynamics: fixed points, periods of length 2, and periods of length 4. We analytically study the asymptotic behaviour of the rules belonging to this class and we prove that they all have spatial and temporal periodic behaviour from arbitrary initial configurations.

[1] These are not to be confused with a variant of cellular automata, also called fuzzy cellular automata, where the fuzziness refers to the choice of a deterministic local rule (e.g., see [1])

2 Notation and Definitions

A one-dimensional circular Boolean cellular automaton is a collection of cells arranged in a circular linear array. Cells have Boolean values and they synchronously update their values according to a local rule applied to their neighbourhood. A configuration $\mathbf{X}^t = (x_0^t, x_,^t \ldots, x_{n-1}^t)$ is a description of all cell values at a given time t. (Alternatively, one can think of an infinite array containing a periodic configuration.) The neighbourhood of a cell consists of the cell itself and its q left and right neighbours, thus the local rule has the form: $g : \{0,1\}^{2q+1} \rightarrow \{0,1\}$. The global dynamics of a one-dimensional cellular automaton composed of n cells is then defined by the global rule (or transition function): $B : \{0,1\}^n \rightarrow \{0,1\}^n \quad s.t. \quad \forall \mathbf{X} \in \{0,1\}^n, \forall i \in \{0,\ldots,n-1\}$, the i-th component $B(X)_i$ of $B(X)$ is $B(X)_i = g(x_{i-q} \ldots, x_i, \ldots, x_{i+q})$, where all operation on indices are modulo n. In this paper we focus only on circular cellular automata and we will omit the term circular.

The local rule $g : \{0,1\}^{2q+1} \rightarrow \{0,1\}$ of a Boolean CA is typically given in tabular form by listing the 2^{2q+1} binary tuples corresponding to the 2^{2q+1} possible local configurations a cell can detect in its direct neighbourhood, and mapping each tuple to a binary value r_i $(0 \leq i \leq 2^{2q+1}-1)$: $(00\cdots00, 00\cdots01, \ldots\ldots, 11\cdots01, 11\cdots11) \rightarrow (r_0, \cdots, r_{2^{2q+1}})$. Let us denote by d_i the tuple mapping to r_i, and by $\mathcal{T}_1$ the set of tuples mapping to one. The local rule can also be canonically expressed in *disjunctive normal form* (DNF) as follows:

$$g(v_0, \cdots, v_{2q}) = \bigvee_{i<2^{2q+1}} r_i \bigwedge_{j=0:2q} v_j^{d_{ij}}$$

where d_{ij} is the j-th digit, from left to right of d_i (counting from zero) and v_j^0 (resp. v_j^1) stands for $\neg v_j$ (resp. v_j).

A *fuzzy cellular automaton* (FCA) is a particular continuous cellular automaton where the local rule is obtained by *DNF-fuzzification* of the local rule of a classical Boolean CA. The fuzzification consists of a fuzzy extension of the boolean operators AND, OR, and NOT in the DNF expression of the Boolean rule. Depending on which fuzzy operators are used, different types of fuzzy cellular automata can be defined. Among the various possible choices, we consider the following: $(a \vee b)$ is replaced by $max\{1, (a+b)\}$[2], $(a \wedge b)$ by (ab), and $(\neg a)$ by $(1-a)$. Whenever we talk about fuzzification, we are referring to the *DNF-fuzzification* defined above. The resulting local rule $f : [0,1]^{2q+1} \rightarrow [0,1]$ becomes a real function that generalizes the canonical representation of the corresponding Boolean CA:

$$f(v_0, \cdots, v_{2q}) = \sum_{i<2^{2q+1}} r_i \prod_{j=0:2q} l(v_j, d_{i,j}) \tag{1}$$

where $l(a, 0) = 1 - a$ and $l(a, 1) = a$.

We define a *fixed point* $\mathbf{P} \in [0,1]^n$ for a FCA with global transition rule $F : [0,1]^n \rightarrow [0,1]^n$ as a configuration $\mathbf{P}$ such that $F(\mathbf{P}) = \mathbf{P}$. Also, a rule is said to be *Temporally Periodic* with period τ if $\exists T$ such that $\forall t > T$: $F(\mathbf{X}^t) = F(\mathbf{X}^{t+\tau})$. Similarly, a rule is *Spatially Periodic* with period ω if $\exists T$ such that $\forall t > T$, $\forall i$: $x_i^t = x_{i+\omega}^t$

Definition 1. *A rule is* Asymptotically Periodic in Time *(or Asymptotically Temporally Periodic) with period τ if $\forall \epsilon > 0$ $\exists T$ such that $\forall t > T$ and $\forall i$: $|x_i^t - x_i^{t+\tau}| < \epsilon$*

[2] note that, in the case of FCA, $max\{1, (a+b)\} = (a+b)$

Definition 2. *A rule is* Asymptotically Periodic in Space *(or Asymptotically Spatially Periodic) with period* ω *if* $\forall\epsilon > 0\ \exists T$ *such that* $\forall t > T$ *and* $\forall i$: $|x_i^t - x_{i+\omega}^t| < \epsilon$

A rule that is asymptotically periodic in space with period 1 will be called *asymptotically homogeneous*. A rule which is asymptotically periodic in time with period 1 will be said to be *convergent to a fixed point*.

3 Density Conservation in Boolean and Fuzzy CAs

We begin exploring the link between Boolean and fuzzy CA proving that there are density properties that are preserved through the fuzzification process.

Throughout this section, we will denote local rules of Boolean CA by g where $g : \{0,1\}^{2q+1} \to \{0,1\}$ and their fuzzifications for the corresponding FCA by f, $f : [0,1]^{2q+1} \to [0,1]$. For ease of notation, we will denote $g(y_{i-q}, \cdots, y_i, \cdots, y_{i+q})$ by $g[y_i]$ and $f(x_{i-q}, \cdots, x_i, \cdots, x_{i+q})$ by $f[x_i]$.

To study density conservation properties we will need to use random variables and some properties of their expectation. In the following, we denote random variables by uppercase letters.

3.1 Preliminaries

Given a random variable Z, let $E(Z)$ denote its expected value; note that when Z is a binary random variable, then $E(Z)$ is the probability $P(Z = 1)$. Furthermore, if given a linear function $\Psi : \Re^n \to \Re$, and by abuse of notation we denote the corresponding map on n random variables also by Ψ, the following is a simple property of expectation.

Lemma 1. *For* n *random variables* Z_i*, and any linear function* Ψ *we have:* $E(\Psi(Z_0, \cdots, Z_{n-1})) = \Psi(E(Z_0), \cdots, E(Z_{n-1}))$.

Next, we will need the following lemma relating the expectation of a Boolean local function to the fuzzification of an expectation. Essentially, we show that applying the fuzzification f of g to the expected values of a cell Y_i and its $2q$ neighbouring cells, we obtain the expected value of $g[Y_i]$, the cell at the next step.

Lemma 2. *Let* $(Y_0, \ldots, Y_{n-1})$ *be independent binary random variables. Then*

$$\forall i = 0, \ldots n-1, \quad E(g[Y_i]) = f[E(Y_i)]$$

Proof. Recall that $\mathcal{T}_1$ indicates the set of Boolean tuples mapping to one by the local rule g. If $d_j \in \mathcal{T}_1$ then $r_j = 1$, otherwise $r_j = 0$.

$$E(g[Y_i]) = P(g[Y_i] = 1) = P((Y_{i-q}, \cdots, Y_{i+q}) \in \mathcal{T}_1) = \sum_{j=0}^{2^{2q+1}-1} r_j \cdot P((Y_{i-q}, \ldots, Y_{i+q}) = d_j)$$

Since the variables are independent we have: $P((Y_{i-q}, \ldots, Y_{i+q}) = d_j) = \prod_{k=-q}^{+q} P(Y_{i+k} = d_{j,k+q})$, and thus:

$$\sum_{j=0}^{2^{2q+1}-1} r_j \cdot P((Y_{i-q}, \ldots, Y_{i+q}) = d_j) = \sum_{j=0}^{2^{2q+1}-1} r_j \prod_{k=-q}^{+q} P(Y_{i+k} = d_{j,k+q})$$

Recall the function l used to define the fuzzy local rule in equation (1): $l(a, 0) = 1 - a$ while $l(a, 1) = a$. If $d_{i,k+q} = 1$ we have that $P(Y_{i+k} = d_{j,k+q}) = E(Y_{i+k}) = l(E(Y_{i+k}), d_{i,k+q})$. Similarly, if $d_{j,k+q} = 0$ we have that $P(Y_{i+k} = d_{j,k+q}) = P(Y_{i+k} = 0) = 1 - P(Y_{i+k} = 1) = 1 - E(Y_{i+k}) = l(E(Y_{i+k}), d_{j,k+q})$.

So we can write:

$$\sum_{j=0}^{2^{2q+1}-1} r_j \prod_{k=-q}^{+q} P(Y_{i+k} = d_{j,k+q}) = \sum_{j=0}^{2^{2q+1}-1} r_j \prod_{k=-q}^{q} l(E(Y_{i+k}), d_{j,k+q}) = f[E(Y_i)]$$

□

Let $\mathcal{C}$ be the universe of all possible configurations for a CA (resp. FCA) of size n with local rule f and corresponding global transition function F.

Definition 3. *We call a property $\mathcal{P}$ of a CA (resp. FCA) a* global property *of the transition function if it holds for all configurations: i.e., such that $\mathcal{P}(F(\mathbf{X}))$ is true for all $\mathbf{X} \in \mathcal{C}$.*

Note that to verify that a property is global for a CA (FCA) it suffices to prove it for all initial configurations $\mathbf{X} \in \mathcal{C}^0$ (because $\mathcal{C} = \mathcal{C}^0$).

3.2 Number Conservation

Number conservation is a global property that has been extensively investigated (e.g., see [3, 4, 8, 9, 13, 20]) since its introduction in [21], a main focus being the study of linear time decision algorithms for the property of number conservation for finite or periodic configurations.

A Boolean CA is number conserving if the number of ones in the initial configuration is preserved at each subsequent iteration (we will also say that a rule is number conserving). The analogous property in fuzzy CA is that the sum of values of the initial configuration is preserved. In this section, we wish to show that using DNF-fuzzification, a Boolean CA with local function g is number conserving if and only if the fuzzification f of the corresponding FCA is sum conserving. We will actually first prove the following more general result.

Theorem 1. *Let Ψ be a real linear function. Then:*

$$\Psi(g[y_0], \cdots, g[y_{n-1}]) = \Psi(y_0, \cdots, y_{n-1}) \;\; \forall (y_0, \ldots, y_{n-1}) \in \{0, 1\}^n$$

if and only if

$$\Psi(f[x_0], \cdots, f[x_{n-1}]) = \Psi(x_0, \cdots, x_{n-1}) \;\; \forall (x_0, \ldots, x_{n-1}) \in [0, 1]^n$$

Proof. $\Rightarrow$: Let $\Psi(g[y_0], \cdots, g[y_{n-1}]) = \Psi(y_0, \cdots, y_{n-1})$ be a global property of the CA with local rule g; we need to show that the property $\Psi(f[x_0], \cdots, f[x_{n-1}]) = \Psi(x_0, \cdots, x_{n-1})$ is also global (i.e., it holds for all possible configurations). Since all possible configurations can be initial (i.e., $\mathcal{C} = \mathcal{C}^0$), it suffices to verify the property for all initial configurations.

First we note that if the property holds for all $(y_0, \ldots, y_{n-1}) \in \{0, 1\}^n$, then given binary random variables Y_i, we must have: $\Psi(g[Y_0], \cdots, g[Y_{n-1}]) = \Psi(Y_0, \cdots, Y_{n-1})$.

Let $(x_0, \dots, x_{n-1}) \in [0,1]^n$ be a randomly chosen initial configuration for the FCA with rule f. Let $(Y_0, \dots, Y_{n-1})$ be binary random variables such that $E(Y_i) = x_i$. We have:

$$\begin{aligned}
\Psi(f[x_0], \cdots, f[x_{n-1}]) &= \Psi(f[E(Y_0)], \cdots, f[E(Y_{n-1})]) \\
&= \Psi(E(g[Y_0]), \cdots, E(g[Y_{n-1}])) \text{ by Lemma2} \\
&= E\left(\Psi(g[Y_0], \cdots, g[Y_{n-1}])\right) \text{ by Lemma 1} \\
&= E(\Psi(Y_0, \cdots, Y_{n-1})) \text{ by hypothesis} \\
&= \Psi(E(Y_0), \cdots, E(Y_{n-1})) \text{ by Lemma 1} \\
&= \Psi(x_0, \cdots, x_{n-1})
\end{aligned}$$

$\Leftarrow$: Since the property applies to all values in $[0,1]$, it must apply to $\{0,1\}$ as well and the implication follows from the construction of f.

□

Note that, when Ψ is the summation of all values we have: $\sum_{i=0}^{n-1} g[y_i] = \sum_{i=0}^{n-1} y_i \forall (y_0, \dots, y_{n-1})$ if and only if $\sum_{i=0}^{n-1} f[x_i] = \sum_{i=0}^{n-1} x_i \forall (x_0, \dots, x_{n-1})$, that is:

Theorem 2. *A Boolean CA is number conserving if and only if the corresponding FCA is sum conserving.*

Example: Rule 184 is an example of a number conserving rule. The proof for the fuzzy case is quite simple:

Theorem 3. *Let f_{184} be fuzzy local rule 184. We have:*

$$\forall (x_0, \dots, x_n) \in [0,1]^n \ \ \sum_{i=0}^{n-1} f_{184}[x_i] = \sum_{i=0}^{n-1} x_i$$

Proof. Fuzzy rule 184 has the following form: $x_i^{t+1} = x_{i-1}^t - x_{i-1}^t x_i^t + x_i^t x_{i+1}^t$ Then we have:

$$\sum_{i=0}^{n-1} x_i^{t+1} = \sum_{i=0}^{n-1} x_{i-1}^t - \sum_{i=0}^{n-1} x_i^t x_{i-1}^t + \sum_{i=0}^{n-1} x_i^t x_{i+1}^t$$

Since we are in a circular FCA we have that $\sum_{i=0}^{n-1} x_{i-1}^t = \sum_{i=0}^{n-1} x_i^t$ and $\sum_{i=0}^{n-1} x_i^t x_{i-1}^t = \sum_{i=0}^{n-1} x_i^t x_{i+1}^t$, which implies:

$$\sum_{i=0}^{n-1} x_i^{t+1} = \sum_{i=0}^{n-1} x_i^t$$

□

The result for the Boolean case (which is already known) follows as a corollary, applying Theorem 1.

Corollary 1. *Let g_{184} be elementary Boolean local rule 184. We have:*
$\forall (y_0, \dots, y_n) \in \{0,1\}^n \ \ \sum_{i=0}^{n-1} g_{184}[y_i] = \sum_{i=0}^{n-1} y_i$

3.3 Spatial Number Conservation

We now describe another global property that is preserved with fuzzification. This property also deals with the density of configurations. In the following, we show that in a CA, linear properties hold for the Boolean function if and only if they hold for the corresponding FCA. The proof follows the same reasoning as that of Theorem 1 and is reported in the appendix.

Theorem 4. *Let* $g : \{0,1\}^{2q+1} \to \{0,1\}$ *be the local rule of a Boolean CA and let* $f : [0,1]^{2q+1} \to [0,1]$ *be its fuzzification. Let* Ψ *be a real linear function.*

$$\Psi(g[y_0], \cdots, g[y_{n-1}]) = 0 \;\; \forall (y_0, \ldots, y_{n-1}) \in \{0,1\}^n$$

if and only if

$$\Psi(f[x_0], \cdots, f[x_{n-1}]) = 0 \;\; \forall (x_0, \ldots, x_{n-1}) \in [0,1]^n$$

Proof. $\Rightarrow$: Let $\Psi(g[y_0], \cdots, g[y_{n-1}]) = 0$ be a global property of the CA with local rule g; we need to show that the property $\Psi(f[x_0], \cdots, f[x_{n-1}]) = 0$ is also global. Again, it follows from the hypothesis that $\Psi(g[Y_0], \cdots, g[Y_{n-1}]) = 0$ for all binary random variables Y_i. Since all possible configurations can be initial (i.e., $\mathcal{C} = \mathcal{C}^0$), it suffices to verify the property for all initial configurations.

Let $(x_0, \ldots, x_{n-1}) \in [0,1]^n$ be a randomly chosen initial configuration for the FCA with rule f. Let $(Y_0, \ldots, Y_{n-1})$ be the binary random variables such that $E(Y_i) = x_i$. We have:

$$\Psi(f[x_0], \cdots, f[x_{n-1}]) = \Psi(f[E(Y_0)] \cdots, f[E(Y_{n-1})])$$
$$= \Psi(E(g[Y_i]), \cdots, E(g[Y_{n-1}]))(\text{ by Lemma2}) = E\left(\Psi(g[Y_0], \cdots, g[Y_{n-1}])\right)(\text{ by Lemma } 1)$$
$$= E(0)(\text{ by hypothesis }) = 0$$

$\Leftarrow$: Since the property applies to all values in $[0,1]$, it must apply to $\{0,1\}$ as well and the implication follows from the construction of f.

□

Note that, when $\Psi(z_0, \ldots, z_{n-1}) = \sum_{i=0}^{n-1} (-1)^i z_i$ and n is even, we obtain the preservation through fuzzyfication of a spatial conservation property where the sum of the even numbered cells (x_{2i}) is equal to the sum of the odd numbered cells (x_{2i+1}) at any time after the initial configuration:

Theorem 5. *Let* n *be even.* $\sum_{i=0}^{n-1} (-1)^i g[y_i] = 0 \;\; \forall (y_0, \ldots, y_{n-1}) \in \{0,1\}^n$

if and only if

$\sum_{i=0}^{n-1} (-1)^i f[x_i] = 0 \;\; \forall (x_0, \ldots, x_{n-1}) \in [0,1]^n$.

Example: Rule 46 is an example of a spatial number conserving rule, where the sum of the even numbered cells (x_{2i}) is equal to the sum of the odd numbered cells (x_{2i+1}) at any time after the initial configuration. The proof is quite simple, and can be obtained from the fuzzy rule:

Theorem 6. *Let* f_{46} *be fuzzy local rule 46 in a FCA of even size. We have for all* $t > 0$*:*

$$\forall (x_0, \ldots, x_{n-1}) \in [0,1]^n \;\; \sum_{i=0}^{n-1} (-1)^i f_{46}[x_i] = 0$$

Proof. Rule 46 is given by: $x_i^{t+1} = x_i^t + x_{i+1}^t - x_{i-1}^t x_i^t - x_i^t x_{i+1}^t$. We have:

$$\sum_{i=0}^{n-1}(-1)^i x_i^{t+1} = \sum_{i=0}^{n-1}(-1)^i x_i^t + \sum_{i=0}^{n-1}(-1)^i x_{i+1}^t - \sum_{i=0}^{n-1}(-1)^i x_{i-1}^t x_i^t - \sum_{i=0}^{n-1}(-1)^i x_i^t x_{i+1}^t.$$

By a change of variables, due to circularity we have: $\sum_{i=0}^{n-1}(-1)^i x_{i+1}^t = -(\sum_{i=0}^{n-1}(-1)^i x_i^t)$, and $\sum_{i=0}^{n-1}(-1)^i x_i^t x_{i+1}^t = -(\sum_{i=0}^{n-1}(-1)^i x_{i-1}^t x_i^t)$. So we can conclude:

$$\sum_{i=0}^{n-1}(-1)^i x_i^{t+1} = \sum_{i=0}^{n-1}(-1)^i x_i^t - \sum_{i=0}^{n-1}(-1)^i x_i^t - \sum_{i=0}^{n-1}(-1)^i x_{i-1}^t x_i^t + \sum_{i=0}^{n-1}(-1)^i x_{i-1}^t x_i^t = 0$$

□

The result for the Boolean case now follows as a corollary of Theorem 5.

Corollary 2. *Let g_{46} be elementary Boolean local rule 46. We have:*
$\forall (y_0, \ldots, y_{n-1}) \in \{0,1\}^n \quad \sum_{i=0}^{n-1}(-1)^i g_{46}[y_i] = 0$

4 Asymptotic Behavior of Weighted Average FCA Rules

In this section, we focus on *elementary* FCA, where $q = 1$ and the binary representation $(r_0, \cdots, r_7)$ is converted into the decimal representation $\sum_{i=1:8} 2^{i-1} r_i$, and this value is used as the name of the rule.

Observation suggest that elementary FCA have simpler dynamics than elementary Boolean CA. In fact, a recent classification based on observations from random initial configurations has grouped elementary FCA into four categories [11]: FCA converging to a fixed point, periods of length two, four and n. We now start the analytical study of FCA by focusing on a class of rules called Weighted Average rules, which includes the two conserving rules discussed in the previous section: rules 184 and 46, and rules in most categories above.

4.1 Preliminaries

We first introduce the notion of a weighted average and of a type of generalized fuzzy CA.

A weighted average of two numbers α and β is an equation of the form $\mu = (1-\gamma)\alpha + \gamma\beta$ where $\gamma \in [0,1]$, (the usual mean occurs when $\gamma = \frac{1}{2}$). Weighted averages have the following useful properties.

Property 1. *If* $\mu = (1-\gamma)\alpha + \gamma\beta$ then $(1-\mu) = (1-\gamma)(1-\alpha) + \gamma(1-\beta)$

Property 2. *If* $\gamma_1 < \gamma_2$ *and* $\alpha < \beta$ then $(1-\gamma_1)\alpha + \gamma_1\beta < (1-\gamma_2)\alpha + \gamma_2\beta$

A *generalized fuzzy CA* is a lattice where every cell updates its state according to a different local rule. In this paper, we will consider generalized fuzzy CA with neighbourhood 1 ($q = 1$) and with rules of the type defined below.

Definition 4. $GWCA$
A generalized fuzzy cellular automata with weighted average rules *($GWCA$) is a generalized fuzzy CA where the local rule has the following form:*

$$x_i^{t+1} = \gamma_i^t x_i^t + (1 - \gamma_i^t) x_{i+1}^t \tag{2}$$

$$(or \ \ x_i^{t+1} = \gamma_i^t x_i^t + (1 - \gamma_i^t) x_{i-1}^t) \tag{3}$$

with bounded weights. That is, there exists $0 < \gamma < \frac{1}{2}$ such that $\gamma_i^t \in (\gamma, 1 - \gamma)$ for all i and for all t.

In other words, in a $GWCA$ the state of a cell i at time $t + 1$ takes the average of the state of the cell itself and of one of its neighbours at time t weighted by a value in $(0, 1)$ that varies from cell to cell. In this section, we consider only $GWCA$ of form (2); the proofs for averaging with the neighbour on the left as in form (3) are analogous.

4.2 A General Convergence Theorem

We now prove a convergence result for $GWCA$ when the initial configurations are in the open interval $(0, 1)^n$. We wish to show that repeated weighted averaging of a circular array results in the values in the array converging to a fixed point, as stated in the general theorem below.

Theorem 7. General Theorem
Consider a GWCA starting from configuration $\mathbf{X}^0 = (x_0^0, \cdots, x_{n-1}^0)$. Then for some $p \in [0, 1]$, $x_i^t \to p$ for all i as $t \to \infty$.

We will prove the theorem by a sequence of lemmas.

Lemma 3. *Consider the sequence $\min_i\{x_i^0\}, \min_i\{x_i^1\}, \ldots, \min_i\{x_i^t\}, \ldots$ Such a sequence converges to some value l_m.*

Proof. Since each value is averaged with its neighbour, the result is always between these two values. Assume that $x_i^t \leq x_{i+1}^t$. Then,

$$x_i^{t+1} = \gamma_i x_i^t + (1 - \gamma_i) x_{i+1}^t \geq \gamma_i x_i^t + (1 - \gamma_i) x_i^t \geq x_i^t$$

with equality if and only if $x_i^t = x_{i+1}^t$. In particular, the previous holds when x_i^t is the minimum value at time t. Since all values at time $t + 1$ are the weighted averages of values greater than or equal to the minimum value at time t, all values at time $t + 1$ must be greater than or equal to the minimum at time t. Thus the sequence is increasing and bounded above by 1 and therefore convergent. □

Analogously we have:

Lemma 4. *Consider the sequence $\max_i\{x_i^0\}, \max_i\{x_i^1\}, \ldots, \max_i\{x_i^t\}, \ldots.$ Such a sequence converges to some value l_M.*

We wish to show that $l_m = l_M$. We will proceed by showing that if there is a difference in the maximum and minimum values in the configuration at some time t, then in $n - 1$ iterations (i.e., at time $t + n - 1$), the minimum must increase by at least an amount proportional to this difference.

Lemma 5. *Given a configuration of length n, if at any time t, $\max_i\{x_i^t\} - \min_i\{x_i^t\} \geq \delta$ then $\min_i\{x_i^{t+n-1}\} - \min_i\{x_i^t\} \geq \gamma^{n-1}\delta$ where $\gamma < \gamma_i^t < 1-\gamma$.*

Proof. Let $m = \min_i \{x_i^t\}$ and $M = \max_i \{x_i^t\}$ so that $M - m \geq \delta$. Renumbering if necessary, we may assume that the maximum occurs at x_{n-1}^t, so that:

$$x_i^t \geq \begin{cases} m & \text{for } 0 \leq i < n-1 \\ M \geq m + \gamma^0\delta & \text{for } i = n-1 \end{cases} \tag{4}$$

with $x_i^t = m$ for at least one i and $x_{n-1}^t = M$.

We wish to show by induction that for $0 \leq s \leq n-1$,

$$x_i^{t+s} \geq \begin{cases} m & \text{for } 0 \leq i < n-s-1 \\ m + \gamma^s\delta & \text{for } n-s-1 \leq i < n \end{cases}$$

This is true when $s = 0$ by equation (4). Now assume it is true for x_i^{t+s} with $s \leq n-2$, and consider x_i^{t+s+1}.

We prove this separately for $i = n-s-2$, for $i = n-1$, and for $0 \leq n-s-1 < i < n-1$. In each case, we obtain a new lower bound by averaging the current lower bounds, giving the highest weight, $(1-\gamma)$, to the lower of the two values being averaged, as in Property 2. For $i = n-s-2$, we have that:

$$\begin{aligned} x_{n-s-2}^{t+s+1} &= (1-\gamma_{n-s-2}^{t+s})x_{n-s-2}^{t+s} + \gamma_{n-s-2}^{t+s}x_{n-s-1}^{t+s} \text{ by definition} \\ &\geq (1-\gamma_{n-s-2}^{t+s})m + \gamma_{n-s-2}^{t+s}(m+\gamma^s\delta) \text{ by the induction hypothesis} \\ &\geq (1-\gamma)m + \gamma(m+\gamma^s\delta) \text{ by Property 2 and by the fact that } \forall i \forall t, \gamma < \gamma_i^t < 1-\gamma \\ &\geq m + \gamma(m+\gamma^s\delta - m) \\ &\geq m + \gamma^{s+1}\delta \end{aligned}$$

In the case of $i = n-1$, we are averaging m and $m+\gamma^s\delta$, hence $x_{n-1}^{t+s+1} \geq (1-\gamma)m + \gamma\gamma^s\delta \geq m + \gamma^{s+1}\delta$. For $0 < i < n-s-2$, we are averaging values greater than or equal to m, and thus the result will be greater than or equal to m. Finally, for $n-s-1 < i < n-1$, we have weighted averages of values greater than $m+\gamma^s\delta$ and therefore, $x_i^{t+s+1} \geq m+\gamma^s\delta > m+\gamma^{s+1}\delta$.

As a consequence, when $s = n-1$, that is at time $t+n-1$, we have that $x_i^{t+n-1} \geq m+\gamma^{n-1}\delta$ for all i. Hence $\min_i\{x_i^{n-1}\} - \min_i\{x_i^0\} \geq (m+\gamma^{n-1}\delta) - m \geq \gamma^{n-1}\delta$.

□

We are now ready to prove the General Theorem.

Proof. (of the General Theorem) We will show by contradiction that $l_m = l_M$. Let $l_M - l_m = \delta > 0$. Fix ϵ such that $0 < \epsilon < \gamma^{n-1}\delta$ and let T be such that for all $t > T$, if $m = \min\{x_1^t, \ldots, x_n^t\}$ and $M = \max\{x_1^t, \ldots, x_n^t\}$ then $l_m - m < \epsilon$ and $M - l_M < \epsilon$. Note that $M - m > \delta$.

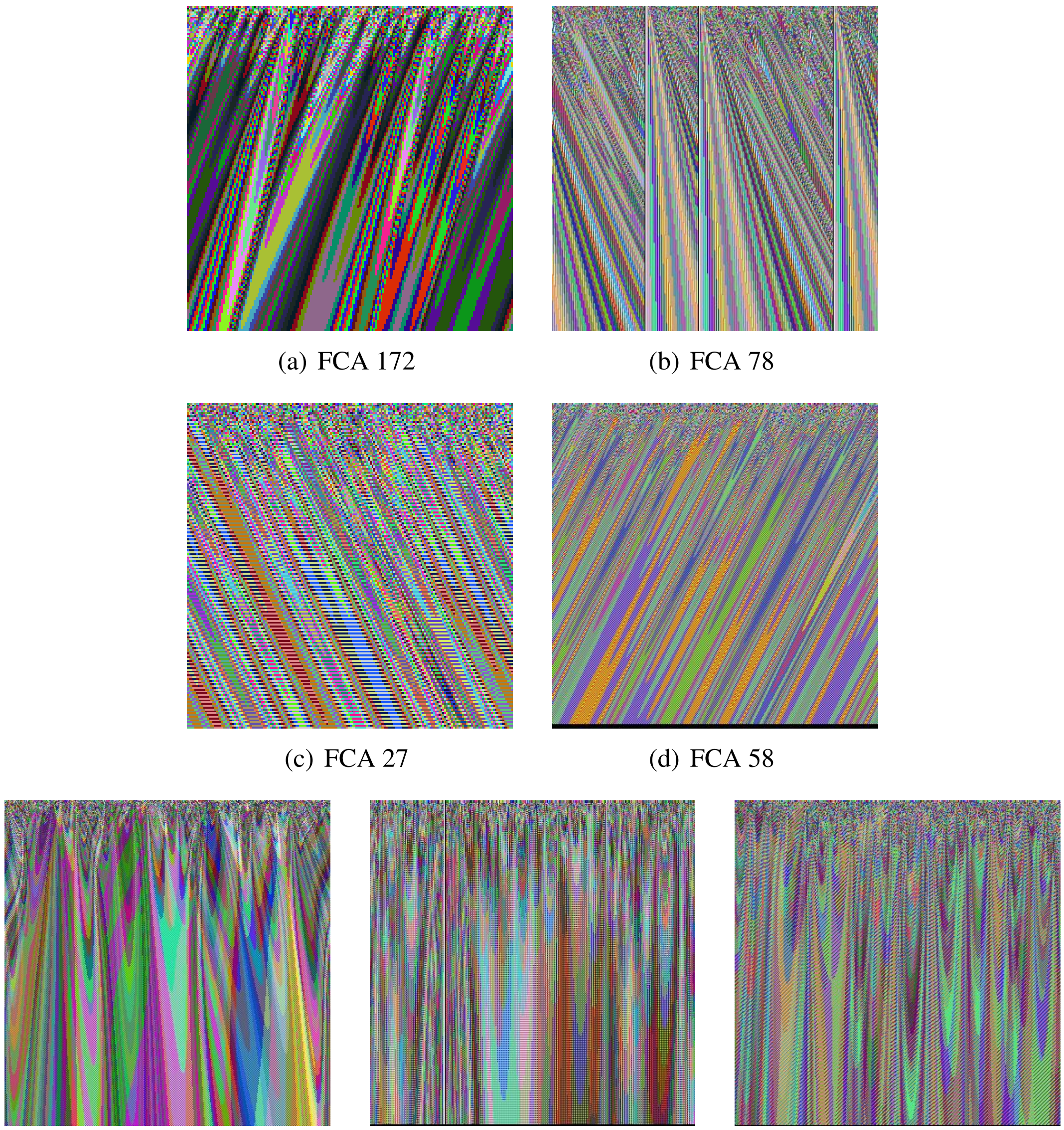
Let $m^t = \min_i\{x_i^t\}$. Fix $t > T$. By Lemma 5,

$$m^{t+n-1} > m^t + \gamma^{n-1}\delta > m + \epsilon > l_m$$

which is a contradiction. Hence $l_M - l_m = 0$. Let us call this limit p.

For all x_i^t, we have $m^t \leq x_i^t \leq M^t$. Since as $t \to \infty$, $m^t \to p$ and $M^t \to p$, $x_i^t \to p$ also. □

———————————————————-



(a) FCA 172 (b) FCA 78

(c) FCA 27 (d) FCA 58

(e) FCA 184 (f) FCA 29 (g) FCA 46

Figure 1: The space-time diagrams of the weigthed average rules (300 iteration from a random initial configuration). Different colours correspond to different ranges of values.

4.3 Weighted Average Rules

Elementary FCA rules 184 and 46, as well as some others, can be viewed as weighted averages of two values in a neighbourhood (or their negations) weighted by the third value (or its negation), closely resembling $GWCA$. Table 1 lists the elementary FCA with this feature and Figure 1 shows their space-time diagrams. Note that we will use $(1-x)$ and $\bar{x}$ interchangeably.

Rule	Equation	Averaged	Weight
$\mathbf{R_{184}}$ $(= R_{226})$	$(1-x_i)x_{i-1} + x_i x_{i+1}$	x_{i-1}, x_{i+1}	x_i
$\mathbf{R_{46}}$ $(= R_{116}, R_{139}, R_{209})$	$(1-x_i)x_{i+1} + x_i(1-x_{i-1})$	$x_{i+1}, \bar{x}_{i-1}$	x_i
R_{27} $(= R_{39}, R_{53}, R_{83})$	$(1-x_{i+1})(1-x_i) + x_{i+1}(1-x_{i-1})$	$\bar{x}_i, \bar{x}_{i-1}$	x_{i+1}
R_{29} $(= R_{71})$	$(1-x_i)(1-x_{i+1}) + x_i(1-x_{i-1})$	$\bar{x}_{i+1}\bar{x}_{i-1}$	x_i
R_{58} $(= R_{114}, R_{163}, R_{177})$	$(1-x_{i-1})x_{i+1} + x_{i-1}(1-x_i)$	$x_{i+1}, \bar{x}_i$	x_{i-1}
R_{78} $(= R_{92}, R_{141}, R_{197})$	$(1-x_{i+1})x_i + x_{i+1}(1-x_{i-1})$	$x_i, \bar{x}_{i-1}$	x_{i+1}
R_{172} $(= R_{212}, R_{202}, R_{228})$	$(1-x_{i-1})x_i + x_{i-1}x_{i+1}$	x_i, x_{i+1}	x_{i-1}

Table 1: Weighted Average Rules. Rules equivalent under conjugation, reflection and both are indicated in parenthesis.

Although Theorem 7 does not apply directly to these rules (except for the case of rule R_{172}), we can determine their asymptotic behavior by constructing, for each of them, a topologically conjugate system for which the theorem does apply. In this section we assume that the initial configurations are in the open interval $(0,1)^n$. We show convergence beginning with the simplest and progressing to the most complex.

4.3.1 Rule 172

The simplest rule to analyze is rule 172:

$$x_i^{t+1} = (1-x_{i-1}^t)x_i^t + x_{i-1}^t x_{i+1}^t$$

Theorem 8. *Rule 172 converges spatially and temporally to a homogeneous configuration.*

Proof. Rule 172 is the weighted average of x_i^t and x_{i+1}^t, so, to apply Theorem 7 we only need to show that there exists a value $0 < \gamma < \frac{1}{2}$ such that the weights are bounded by γ and $(1-\gamma)$. Let $\gamma = min\left\{x_0^0, \cdots, x_{n-1}^0, 1-x_0^0, \cdots, 1-x_{n-1}^0\right\}$ then $\forall t$,

$$\begin{aligned} \gamma &\leq min\left\{x_0^t, \cdots, x_{n-1}^t, 1-x_0^t, \cdots, 1-x_{n-1}^t\right\} \\ 1-\gamma &\geq max\left\{x_0^t, \cdots, x_{n-1}^t, 1-x_0^t, \cdots, 1-x_{n-1}^t\right\} \end{aligned}$$

Since the weights used by rule 172 are taken from these sets, it follows that $\gamma \leq \gamma_i^t \leq 1-\gamma$ and the conditions of Theorem 7 hold. Hence, there exists a value p such that $x_i^t \to p$ for all i as $t \to \infty$. □

In all subsequent rules, the lowest weight occuring in any equation at the first iteration provides a lower bound γ on the weights.

4.3.2 Rule 78

Consider rule R_{78} as a weighted average:

$$x_i^{t+1} = (1 - x_{i+1}^t)x_i^t + x_{i+1}^t(1 - x_{i-1})^t$$

Theorem 9. *When n is even, rule 78 converges temporally to a homogeneous configuration. Furthermore, if x_0^t converges to a value p, then for all i, x_{2i}^t converges to p, and x_{2i+1}^t converges to $(1-p)$. When n is odd, rule 78 converges to the homogeneous configuration $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$.*

Proof. Let A be a fuzzy cellular automaton following rule 78 and let $f : (0,1)^3 \to (0,1)$ be its local rule and F its global rule.

Assume that n is even. Consider a $GWCA$ A' with global rule F' given by two local rules $f_1', f_2' : (0,1)^3 \to (0,1)$ defined as follows:

$$\begin{aligned} f_1'(x,y,z) &= zy + (1-z)x \\ f_2'(x,y,z) &= (1-z)y + zx \end{aligned}$$

which are applied in alternation to a configuration $(x_0, \cdots, x_{n-1})$ as follows:

$$F'(x_0, \cdots, x_{n-1}) = (f_1'(x_{n-1}, x_0, x_1), f_2'(x_0, x_1, x_2), \cdots, f_2'(x_{n-2}, x_{n-1}, x_0))$$

Let $h : (0,1)^n \to (0,1)^n$ be a homeomorphism defined by:

$$h(x_0, x_1, x_2, \ldots, x_{n-1}) = (x_0, 1 - x_1, x_2, \ldots, 1 - x_{n-1})$$

We want to show that F and F' are conjugates, that is that $h \circ F = F' \circ h$. Let $\mathbf{X} = (x_0, x_1, \ldots, x_{n-1})$ be an arbitrary configuration for A.

$$\begin{aligned} h \circ\ & F(x_0, x_1, \cdots, x_{n-1}) \\ =\ & h((1-x_1)x_0 + x_1(1-x_{n-1}), (1-x_2)x_1 + x_2(1-x_0), \cdots, (1-x_0)x_{n-1} + x_0(1-x_{n-2})) \\ =\ & ((1-x_1)x_0 + x_1(1-x_{n-1}), \overline{(1-x_2)x_1 + x_2(1-x_0)}, \cdots, \overline{(1-x_0)x_{n-1} + x_0(1-x_{n-2})}) \\ =\ & ((1-x_1)x_0 + x_1(1-x_{n-1}), (1-x_2)(1-x_1) + x_2x_0, \cdots, (1-x_0)(1-x_{n-1}) + x_0x_{n-2}) \\ F' \circ\ & h(x_0, x_1, \cdots, x_{n-1}) \\ =\ & F'(x_0, (1-x_1), \cdots, (1-x_{n-1})) \\ =\ & ((1-x_1)x_0 + x_1(1-x_{n-1}), (1-x_2)(1-x_1) + x_2x_0, \cdots, (1-x_0)(1-x_{n-1}) + x_0x_{n-2}) \end{aligned}$$

By Theorem 7, the $GWCA$ A' converges to some point $(p, p, \cdots p)$. Then $h^{-1}(p, p, \cdots, p) = (p, 1-p, p, \cdots, 1-p)$ is the point of convergence A.

$$\mathbf{X}^t \to (p, 1-p, \cdots, p, 1-p)$$

When n is odd, consider a GWCA B of length $2n$ with global rule F' as above, and a homeomorphism $h : A \to B$ defined as:

$$h(x_0, x_1, \cdots, x_{n-1}) = (x_0, \bar{x}_1, \cdots, x_{n-1}, \bar{x}_0, x_1, \cdots, \bar{x}_{n-1}).$$

Functions F and F' are conjugate since $h \circ F = F' \circ h$. In fact:

$$\begin{aligned} F' \quad \circ & \quad h(x_0, x_1, \cdots, x_{n-1}) \\ = & \quad F'(x_0, \bar{x}_1, \cdots, x_{n-1}, \bar{x}_0, x_1, \cdots, \bar{x}_{n-1}) \\ = & \quad (\bar{x}_1 x_0 + x_1 \bar{x}_{n-1}, \bar{x}_2 \bar{x}_1 + x_2 x_0, \cdots, \bar{x}_0 x_{n-1} + x_0 \bar{x}_{n-2}, \bar{x}_1 \bar{x}_0 + x_1 x_{n-1}, \cdots, \bar{x}_0 \bar{x}_{n-1} + x_0 x_{n-2}), \end{aligned}$$

while

$$\begin{aligned} h \quad \circ & \quad F(x_0, x_1, \cdots, x_{n-1}) \\ = & \quad h(\bar{x}_1 x_0 + x_1 \bar{x}_{n-1}, \bar{x}_2 x_1 + x_2 \bar{x}_0, \cdots, \bar{x}_0 x_{n-1} + x_0 \bar{x}_{n-2}) \\ = & \quad (\bar{x}_1 x_0 + x_1 \bar{x}_{n-1}, \bar{x}_2 \bar{x}_1 + x_2 x_0, \cdots, \bar{x}_0 x_{n-1} + x_0 \bar{x}_{n-2}, \bar{x}_1 \bar{x}_0 + x_1 x_{n-1}, \cdots, \bar{x}_0 \bar{x}_{n-1} + x_0 x_{n-2}). \end{aligned}$$

Since f_1' and f_2' are weighted sums of neighbours, F' must converge to a single value, p. But we also notice that for all i, $x_i^t = (1 - x_i^t)$ as $t \to \infty$, since $x_{i+n}^t = (1 - x_i^t) \to p$ also. So we must have that $p = \frac{1}{2}$ and $h^{-1}(\frac{1}{2}, \cdots, \frac{1}{2}) = (\frac{1}{2}, \cdots, \frac{1}{2})$ is the point of convergence of A.

□

4.3.3 Rule 27

We restate R_{27} as a weighted average:

$$x_i^{t+1} = (1 - x_{i+1}^t)(1 - x_i^t) + x_{i+1}^t(1 - x_{i-1}^t)$$

Theorem 10. *Rule 27 converges spatially to a single value and temporally it has asymptotic periodicity with period 2. Furthermore, if for all i, as $t \to \infty$ $x_i^{2t} \to p$ then $x_i^{2t+1} \to 1 - p$.*

Proof. Let A be a fuzzy cellular automaton following rule 27 and let $f : (0,1)^3 \to (0,1)$ be its local rule and F its global rule.

Let B be a fuzzy cellular automaton with global rule $F \circ F$. Consider a $GWCA$ B' with global rule $F_2' \circ F_1'$ where F_1' is defined by local rule f_1' and F_2' by f_2'. Local rules $f_1', f_2' : (0,1)^3 \to (0,1)$ are defined as follows:

$$\begin{aligned} f_1'(x, y, z) &= (1 - z)y + zx \\ f_2'(x, y, z) &= zy + (1 - z)x \end{aligned}$$

We want to show that $F \circ F = F_2' \circ F_1'$. Let id be the identity homeomorphism and let inv be the negation homeomorphism: $inv(x_0, x_1, \cdots, x_{n-1}) = (\bar{x}_0, \bar{x}_1, \cdots, \bar{x}_{n-1})$.

We will prove the equality of $F \circ F$ and $F_2' \circ F_1'$ by showing that the following diagram commutes.

$$\begin{array}{ccc} & id & \\ A & \longrightarrow & B \\ F \downarrow & & \downarrow F_1' \\ & inv & \\ A & \longrightarrow & B \\ F \downarrow & & \downarrow F_2' \\ & id & \\ A & \longrightarrow & B \end{array}$$

That is, we will show that $F_1' = inv \circ F$ and $F = F_2' \circ inv$

$$\begin{aligned} inv \quad \circ \quad & F(\cdots, x_{i-1}, x_i, x_{i+1}, \cdots) \\ = \quad & inv(\cdots, \bar{x}_{i+1}\bar{x}_i + x_{i+1}\bar{x}_{i-1}, \cdots) \\ = \quad & (\cdots, \bar{x}_{i+1}x_i + x_{i+1}x_{i-1}, \cdots) \\ = \quad & F_1'(\cdots, x_{i-1}, x_i, x_{i+1}, \cdots) \end{aligned}$$

$$\begin{aligned} F_2' \quad \circ \quad & inv(\cdots, x_{i-1}, x_i, x_{i+1}, \cdots) \\ = \quad & F_2'(\cdots, \bar{x}_{i-1}, \bar{x}_i, \bar{x}_{i+1}, \cdots) \\ = \quad & (\cdots, \bar{x}_{i+1}\bar{x}_i + x_{i+1}\bar{x}_{i-1}, \cdots) \\ = \quad & F(\cdots, x_{i-1}, x_i, x_{i+1}, \cdots) \end{aligned}$$

Finally,

$$\begin{aligned} F \circ F &= F_2' \circ inv \circ F \\ &= F_2' \circ F_1' \end{aligned}$$

Thus $F \circ F$ and $F_2' \circ F_1'$ are equal. Since both F_1' and F_2' are weighted averages of neighbours with bounded weights, Theorem 7 applies, thus B' is converging to a homogeneous configuration. Moreover, every other iteration of A must also be converging.

If we let the limit of the even iterations be p, then

$$\begin{aligned} \lim_{t\to\infty}(x_0^{2t+1}, x_1^{2t+1}, \cdots, x_{n-1}^{2t+1}) &= \lim_{t\to\infty} F(x_0^{2t}, x_1^{2t}, \cdots, x_{n-1}^{2t}) \\ &= F(\lim_{t\to\infty} x_0^{2t}, \lim_{t\to\infty} x_1^{2t}, \cdots, \lim_{t\to\infty} x_{n-1}^{2t}) \\ &= F(p, p, \cdots, p) \\ &= (1-p, 1-p, \cdots, 1-p) \end{aligned}$$

So

$$\begin{aligned} \mathbf{X}^{2t} &\to (p, \cdots, p) \\ \mathbf{X}^{2t+1} &\to (1-p, \cdots, 1-p) \end{aligned}$$

□

4.3.4 Rule 58

To begin with, we recall the rule:

$$x_i^{t+1} = x_{i-1}^t \bar{x}_i^t + (1 - x_{i-1}^t) x_{i+1}^t$$

which can also be written:

$$\bar{x}_i^{t+1} = x_{i-1}^t x_i^t + (1 - x_{i-1}^t) \bar{x}_{i+1}^t$$

Theorem 11. *When n is even, rule 58 is spatially and temporally asymptotic with period 2. Furthermore, if $x_i^{2t} \to p$ then $x_i^{2t+1} \to 1-p$, $x_{i+1}^{2t} \to 1-p$ and $x_{i+1}^{2t+1} \to p$. When n is odd, rule 58 converges to the homogeneous configuration $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$.*

Proof. Let A be a fuzzy cellular automaton following rule 58 and let $f : (0,1)^3 \to (0,1)$ be its local rule and F its global rule.

Assume that n is even. Let B be a fuzzy cellular automaton with global rule $F \circ F$. Consider a $GWCA$ B' with global rule $F'_2 \circ F'_1$ where F'_1 and F'_2 are given by two local rules $f'_1, f'_2 : (0,1)^3 \to (0,1)$ defined as follows:

$$\begin{aligned} f'_1(x,y,z) &= (1-x)y + xz \\ f'_2(x,y,z) &= xy + (1-x)z \end{aligned}$$

and where F'_1 and F'_2 alternate f'_1 and f'_2 as shown:

$$\begin{aligned} F'_1(x_0, x_1, \cdots, x_{n-1}) &= (f'_1(x_{n-1}, x_0, x_1), f'_2(x_0, x_1, x_2), \cdots, f'_2(x_{n-2}, x_{n-1}, x_0)) \\ F'_2(x_0, x_1, \cdots, x_{n-1}) &= (f'_2(x_{n-1}, x_0, x_1), f'_1(x_0, x_1, x_2), \cdots, f'_1(x_{n-2}, x_{n-1}, x_0)) \end{aligned}$$

We want to show that $F \circ F$ and $F'_2 \circ F'_1$ are conjugate. Let h be the homeomorphism previously defined for rule 78.

$$h(x_0, x_1, x_2, \ldots, x_{n-1}) = (x_0, 1-x_1, x_2, \ldots, 1-x_{n-1})$$

Again we need to show that $h \circ F \circ F = F'_2 \circ F'_1 \circ h$. We will first show that $F'_1 \circ h = h' \circ F$, with $h' = h \circ inv$. Without loss of generality, assume that i is even.

$$\begin{aligned} F'_1 \quad \circ \quad & h(\cdots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \cdots) \\ = \quad & F'_1(\cdots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, x_{i+2}, \cdots) \\ & (\cdots, x_{i-1}x_i + \bar{x}_{i-1}\bar{x}_{i+1}, x_i\bar{x}_{i+1} + \bar{x}_i x_{i+2}, \cdots) \end{aligned}$$

$$\begin{aligned} h' \quad \circ \quad & F(\cdots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \cdots) \\ = \quad & h'(\cdots, x_{i-1}\bar{x}_i + \bar{x}_{i-1}x_{i+1}, x_i\bar{x}_{i+1} + \bar{x}_i x_{i+2}, \cdots) \\ = \quad & (\cdots, x_{i-1}x_i + \bar{x}_{i-1}\bar{x}_{i+1}, x_i\bar{x}_{i+1} + \bar{x}_i x_{i+2}, \cdots) \end{aligned}$$

Similarly, $h \circ F = F'_2 \circ h'$.

Then

$$\begin{aligned} h \circ F \circ F &= F'_2 \circ h' \circ F \\ &= F'_2 \circ F'_1 \circ h \end{aligned}$$

Thus $h \circ F \circ F$ and $F'_2 \circ F'_1 \circ h$ are conjugate. By Theorem 7, B' is converging to a homogeneous configuration of the form $(p, p, \cdots, p)$. Every other iteration of B must also be converging to a configuration of the form $h^{-1}(p, p, \cdots, p) = (p, 1-p, p, \cdots, 1-p)$.

Now for odd time steps we have:

$$\begin{aligned} \lim_{t\to\infty}(x_0^{2t+1}, x_1^{2t+1}, \cdots, x_{n-1}^{2t+1}) &= \lim_{t\to\infty} F(x_0^{2t}, x_1^{2t}, \cdots, x_{n-1}^{2t}) \\ &= F(\lim_{t\to\infty} x_0^{2t}, \lim_{t\to\infty} x_1^{2t}, \cdots, \lim_{t\to\infty} x_{n-1}^{2t}) \\ &= F(p, 1-p, p, \cdots, 1-p) \\ &= (1-p, p, 1-p, \cdots, p) \end{aligned}$$

Summarizing,

$$\begin{array}{rcl} \mathbf{X}^{2t} & \rightarrow & (p, 1-p, \cdots, p, 1-p) \\ \mathbf{X}^{2t+1} & \rightarrow & (1-p, p, \cdots, 1-p, p) \end{array}$$

We now turn our attention to the case where n is odd. We consider the FCA B described in Theorem 9 and we will show that A with global rule $F \circ F$ is conjugate to B with global rule $F_2' \circ F_1'$ and homeomorphism h as in the proof of Theorem 9 when n is odd.

If we let $h' = h \circ inv$ as before, we again have that $F_1' \circ h = h' \circ F$ and $h \circ F = F_2' \circ h'$. We give the details for the second equality.

$$\begin{array}{rcl} h & \circ & F(x_0, x_1, \cdots, x_{n-1}) \\ & = & h(x_{n-1}\bar{x}_0 + \bar{x}_{n-1}x_1, x_0\bar{x}_1 + \bar{x}_0 x_2, \cdots, x_{n-2}\bar{x}_{n-1} + \bar{x}_{n-2}x_0) \\ & = & (x_{n-1}\bar{x}_0 + \bar{x}_{n-1}x_1, x_0 x_1 + \bar{x}_0\bar{x}_2, \cdots, x_{n-2}\bar{x}_{n-1} + \bar{x}_{n-2}x_0, x_{n-1}x_0 + \bar{x}_{n-1}\bar{x}_1, x_0\bar{x}_1 + \bar{x}_0 x_2, \cdots, \\ & & \cdots x_{n-2}x_{n-1} + \bar{x}_{n-2}\bar{x}_0) \end{array}$$

$$\begin{array}{rcl} F_2' & \circ & h'(x_0, x_1, \cdots, x_{n-1}) \\ & = & F_2'(\bar{x}_0, x_1, \cdots, \bar{x}_{n-1}, x_0, \bar{x}_1, \cdots, x_{n-1}) \\ & = & (x_{n-1}\bar{x}_0 + \bar{x}_{n-1}x_1, x_0 x_1 + \bar{x}_0\bar{x}_2, \cdots, x_{n-2}\bar{x}_{n-1} + \bar{x}_{n-2}x_0, x_{n-1}x_0 + \bar{x}_{n-1}\bar{x}_1, x_0\bar{x}_1 + \bar{x}_0 x_2, \cdots, \\ & & \cdots x_{n-2}x_{n-1} + \bar{x}_{n-2}\bar{x}_0) \end{array}$$

As in the case of rule 78 with n odd, B must converge to $\frac{1}{2}$ so even time steps of A must also converge to $\frac{1}{2}$. Odd time steps will converge to $\frac{1}{2}$ since

$$F(\frac{1}{2}, \cdots, \frac{1}{2}) = (\frac{1}{2}, \cdots, \frac{1}{2}).$$

□

4.3.5 Rule 184

Rules 184, 46, and 29 use x_i^t itself as the weighting factor in the average of its two neighbours. When n is even, we effectively have two separate weighted averages, one of the even indices, the other of the odd. The weight factors in each case come from the other set of values. We will exploit this structure to determine topologically conjugate FCA where we can apply Theorem 7.

We recall rule 184:

$$x_i^{t+1} = (1 - x_i^t)x_{i-1}^t + x_i^t x_{i+1}^t.$$

Theorem 12. *When n is even, rule 184 is asymptotically periodic with period 2 both spatially and temporally. Furthermore, if $\forall i$ as $t \to \infty$, $x_{2i}^{2t} \to p$ and $x_{2i+1}^{2t} \to q$, then $x_{2i+1}^{2t+1} \to p$, and $x_{2i}^{2t+1} \to q$ with $p + q = \frac{2}{n}\sum_{i=0}^{n-1} x_i^0$. When n is odd, rule 184 converges spatially and temporally to a homogeneous configuration. Moreover, if $\forall i$ as $t \to \infty$, $x_i^t \to p$, then $p = \frac{1}{n}\sum_{i=0}^{n-1} x_i^0$.*

Proof. Let A be a fuzzy cellular automaton following rule 184 and let $f : (0,1)^3 \to (0,1)$ by $f(x,y,z) = \bar{y}x + yz$ be its local rule and F its global rule.

When n is even, let B be the fuzzy cellular automaton where the global rule G is the left shift of F:

$$G(x_0, \cdots, x_{n-1}) = (f(x_0,x_1,x_2), f(x_1,x_2,x_3), \cdots, f(x_{n-1},x_0,x_1))$$

Now let S be the shift self-homeomorphism of B:

$$S(x_0, \cdots, x_{n-1}) = (x_{n-1}, x_0, \cdots, x_{n-2})$$

Notice that $S \circ G = G \circ S$, also that $F = S \circ G$. Now consider F^n: $F^n = (S \circ G)^n = S^n G^n = G^n$.

Thus every n-th iteration of A is equal to every n-th iteration of B. Furthermore, in-between steps can be determined by the appropriate number of shifts of B. We will now determine the asymptotic behaviour of B.

Let B' be the cross product of two $GWCA$ of length $m = n/2$ with global rule G' given by:

$$\begin{aligned} & G'((u_0, \cdots, u_{m-1}) \times (v_0, \cdots, v_{m-1})) \\ = \; & (f(u_0,v_0,u_1), f(u_1,v_1,u_2), \cdots, f(u_{m-1},v_{m-1},u_0)) \\ \times \; & (f(v_0,u_1,v_1), f(v_1,u_2,v_2), \cdots, f(v_{m-1},u_0,v_0)) \end{aligned}$$

with γ as in the proof of Theorem 8.

The topological space B is conjugate to B' under the homeomorphism h defined as follows:

$$h(x_0, \cdots, x_{n-1}) = (x_0, x_2, \cdots, x_{n-2}) \times (x_1, x_3, \cdots, x_{n-1})$$

To prove this we need to show that $h \circ G = G' \circ h$. The i-th position of $G(x_0, \cdots, x_{n-2})$ is $(1 - x_{i+1})x_i + x_{i+1}x_{i+2}$ so

$$\begin{aligned} h \; \circ \; & G(x_0, \cdots, x_{n-1}) \\ = \; & h((1-x_1)x_0 + x_1x_2, (1-x_2)x_1 + x_2x_3, \cdots, (1-x_0)x_{n-1} + x_0x_1) \\ = \; & ((1-x_1)x_0 + x_1x_2, \cdots, (1-x_{n-1})x_{n-2} + x_{n-1}x_0) \times ((1-x_2)x_1 + x_2x_3, \cdots, (1-x_0)x_{n-1} + x_0x_1) \\ = \; & G'((x_0, x_2, \cdots, x_{n-2}) \times (x_1, x_3, \cdots, x_{n-1})) \\ = \; & G' \circ h(x_0, \cdots, x_{n-1}) \end{aligned}$$

Both $GWCA$ will converge to fixed points by Theorem 7. If $(p, \cdots, p) \times (q, \cdots, q)$ is the point of convergence of B', then $h^{-1}((p, \cdots, p) \times (q, \cdots, q)) = (p, q, p, \cdots, q)$ is the point of convergence of B.

Since, by definition, A's global rule is given by the right shift of B's global rule, the theorem follows. In summary we have,

$$\begin{aligned} \mathbf{X}^{2t} & \to (p, q, \cdots, p, q) \\ \mathbf{X}^{2t+1} & \to (q, p, \cdots, q, p) \end{aligned}$$

Now the sum of all values at the point of convergence is $\frac{n}{2}(p+q)$. From Theorem 3, this must be equal to the sum of values in the initial configuration. Hence, $p+q = \frac{2}{n}\sum_{i=0}^{n-1} x_i^0$.

With n odd, we consider the system B described as above. However, this time we will consider a homeomorphism h given by:

$$h(x_0, x_1, \cdots, x_{n-1}) = (x_0, x_2, \cdots, x_{n-1}, x_1, x_3, \cdots, x_{n-2})$$

and a $GWCA$ B' of size $2n$ with a new global rule G'. Let $m = \frac{n+1}{2}$, then

$$G'(x_0, x_1, \cdots, x_{n-1}) = (f(x_0, x_m, x_1), f(x_1, x_{m+1}, x_2), \cdots, f(x_i, x_{m+i}, f_{i+1}), \cdots)$$

where f is local rule 184, and γ as in the proof of Theorem 8.

The system B' is conjugate to B using h. To prove this we show that $h \circ G = G' \circ h$

$$\begin{aligned} h \quad &\circ \quad G(x_0, x_1, \cdots, x_{n-1}) \\ &= \quad h((1-x_1)x_0 + x_1x_2, (1-x_2)x_1 + x_2x_3, \cdots, (1-x_0)x_{n-1} + x_0x_1) \\ &= \quad (\bar{x}_1x_0 + x_1x_2, \bar{x}_3x_2 + x_3x_4, \cdots, \bar{x}_0x_{n-1} + x_0x_1, \bar{x}_2x_1 + x_2x_3, \cdots, \bar{x}_{n-1}x_{n-2} + x_{n-1}x_0) \end{aligned}$$

$$\begin{aligned} G' \quad &\circ \quad h(x_0, x_1, \cdots, x_{n-1}) \\ &= \quad G'(x_0, x_2, \cdots, x_{n-1}, x_1, x_3, \cdots, x_{n-2}) \\ &= \quad (\bar{x}_1x_0 + x_1x_2, \bar{x}_3x_2 + x_3x_4, \cdots, \bar{x}_0x_{n-1} + x_0x_1, \bar{x}_2x_1 + x_2x_3, \cdots, \bar{x}_{n-1}x_{n-2} + x_{n-1}x_0) \end{aligned}$$

Now G' must converge to a single value and since h merely reorders those values, G must also converge to a single value. Finally, since the shift is the identity of the point of convergence, A must converge to a single value both spatially and temporally. Moreover, if $\forall i$ as $t \to \infty$, $x_i^t \to p$, then $p = \frac{1}{n}\sum_{i=0}^{n-1} x_i^0$ by Theorem 3.

□

4.3.6 Rule 29

We now examine rule 29:

$$x_i^{t+1} = x_i^t(1 - x_{i-1}^t) + (1 - x_i^t)(1 - x_{i+1}^t)$$

Theorem 13. *When n is even, rule 29 is asymptotically periodic with period 2 both spatially and temporally. Furthermore, if $\forall i$ as $t \to \infty$, $x_{2i}^{2t} \to p$, and $x_{2i+1}^{2t} \to q$ then $x_{2i}^{2t+1} \to 1-q$, and $x_{2i+1}^{2t+1} \to 1-p$. When n is odd, rule 29 converges spatially to a homogeneous configuration and is asymptotically periodic with period 2 temporally.*

Proof. We proceed as for rule 184. Let A be a fuzzy cellular automaton following rule 29 and let $f : (0,1)^3 \to (0,1)$ be its local rule and F its global rule.

Let B be a fuzzy cellular automaton with global rule G given by the shift of F:

$$G(x_0, \cdots, x_{n-1}) = (f(x_0, x_1, x_2), f(x_1, x_2, x_3), \cdots, f(x_{n-1}, x_0, x_1))$$

When n is even, we let B' be the cross product of two $GWCA$ of length $m = n/2$ with global rules G'_1 and G'_2 given by local rules f_1 and f_2 as follows:

$$\begin{aligned} f_1(x,y,z) &= yx + (1-y)z \\ f_2(x,y,z) &= (1-y)x + yz \end{aligned}$$

$$\begin{aligned} & G'_i((u_0,\cdots,u_{m-1}) \times (u_0,\cdots,u_{m-1})) \\ = \; & (f_i(u_0,v_0,u_1), f_i(u_1,v_1,u_2),\cdots,f_i(u_{m-1},v_{m-1},u_0)) \\ \times \; & (f_i(v_0,u_1,v_1), f_i(v_1,u_2,v_2),\cdots,f_i(v_{m-1},u_0,v_0)) \end{aligned}$$

for $i = 1, 2$.

The topological space B using rule $G \circ G$ is conjugate to B' using rule $G'_2 \circ G'_1$ under the homeomorphism h defined as follows:

$$h(x_0,\cdots,x_{n-1}) = (x_0,x_2,\cdots,x_{n-2}) \times (x_1,x_3,\cdots,x_{n-1})$$

To show this, we consider a second homeomorphism $h' = h \circ inv$ which maps B to B' by

$$h'(x_0,\cdots,x_{n-1}) = (\bar{x}_0,\bar{x_2},\cdots,\bar{x}_{n-2}) \times (\bar{x_1},\bar{x_3},\cdots,\bar{x}_{n-1})$$

We first show that $G'_2 \circ h' = h \circ G$

$$\begin{aligned} G'_2 \; & \circ \; h'(x_0,x_1,x_2,x_3,\cdots) \\ & = \; G'_2((\bar{x}_0,\bar{x_2},\cdots,\bar{x}_{n-2}) \times (\bar{x}_1,\bar{x}_3,\cdots,\bar{x}_{n-1}) \\ & = \; (f_2(\bar{x}_0,\bar{x}_1,\bar{x}_2), f_2(\bar{x}_2,\bar{x}_3,\bar{x}_4),\cdots) \\ & \times \; (f_2(\bar{x}_1,\bar{x}_2,\bar{x}_3), f_2(\bar{x}_3,\bar{x}_4,\bar{x}_5),\cdots) \\ & = \; ((x_1\bar{x}_0 + \bar{x}_1\bar{x}_2),(x_3\bar{x}_2 + \bar{x}_3\bar{x}_4),\cdots) \\ & \times \; ((x_2\bar{x}_1 + \bar{x}_2\bar{x}_3),(x_4\bar{x}_3 + \bar{x}_4\bar{x}_5),\cdots) \\ h \; & \circ \; G(x_0,x_1,x_2,x_3,\cdots) \\ & = \; h((x_1\bar{x}_0 + \bar{x}_1\bar{x}_2, x_2\bar{x}_1 + \bar{x}_2\bar{x}_3, x_3\bar{x}_2 + \bar{x}_3\bar{x}_4,\cdots)) \\ & = \; ((x_1\bar{x}_0 + \bar{x}_1\bar{x}_2),(x_3\bar{x}_2 + \bar{x}_3\bar{x}_4),\cdots) \\ & \times \; ((x_2\bar{x}_1 + \bar{x}_2\bar{x}_3),(x_4\bar{x}_3 + \bar{x}_4\bar{x}_5),\cdots) \end{aligned}$$

Furthermore, $G'_1 \circ h = h' \circ G$:

$$\begin{aligned} G'_1 \; & \circ \; h(x_0,x_1,x_2,x_3,\cdots) \\ & = \; G'_1((x_0,x_2,\cdots x_{n-2}) \times (x_1,x_3,\cdots x_{n-1})) \\ & = \; (x_1x_0 + \bar{x}_1x_2, x_3x_2 + \bar{x}_3x_4,\cdots) \times (x_2x_1 + \bar{x}_2x_3, x_4x_3 + \bar{x}_4x_5,\cdots) \\ & = \; h'((x_1\bar{x}_0 + \bar{x}_1\bar{x}_2, x_2\bar{x}_1 + \bar{x}_2\bar{x}_3, x_3\bar{x}_2 + \bar{x}_3\bar{x}_4, x_4\bar{x}_3 + \bar{x}_4\bar{x}_5,\cdots) \\ & = \; h' \circ G(x_0,x_1,x_2,x_3,\cdots) \end{aligned}$$

Finally we have,

$$\begin{aligned} G'_2 \circ (G'_1 \circ h) &= (G'_2 \circ h') \circ G \\ &= h \circ G \circ G \end{aligned}$$

as required.

By Theorem 7, both $GWCA$ in B' converge to homogeneous configurations. If $(p, \cdots, p) \times (q, \cdots, q)$ is the point of convergence of B', then $h^{-1}((p, \cdots, p) \times (q, \cdots, q)) = (p, q, p, \cdots, q)$ is the point of convergence of B.

Since, by definition, A's global rule is given by the right shift of B's global rule, the theorem follows and we have,

$$\mathbf{X}^{2t} \rightarrow (p, q, \cdots, p, q)$$

and

$$\begin{aligned}\mathbf{X}^{2t+1} &\rightarrow F(p, q, \cdots, p, q) \\ &\rightarrow (1-q, 1-p, \cdots, 1-q, 1-p)\end{aligned}$$

When n is odd, let B' and h be as in the proof of Theorem 12 the case of rule 184 when n is odd. But let B' transform under $G'_2 \circ G'_1$ where

$$G'_j(x_0, x_1, \cdots, x_{2n-1}) = (f_j(x_0, x_m, x_1), f_j(x_1, x_{m+1}, x_2), \cdots, f_j(x_i, x_{i+m}, x_{i+1}), \cdots)$$

for $j = 1, 2$ and f_j defined as for n even above.

Let h' be the negation of h, so that $h' = h \circ inv$

$$h'(x_0, x_1, \cdots, x_{n-1}) = (\bar{x}_0, \bar{x}_2, \cdots, \bar{x}_{n-1}, \bar{x}_1, \cdots, \bar{x}_{n-2})$$

then $h \circ G \circ G = G'_2 \circ G'_1 \circ h$. Again we prove this by showing that $h \circ G = G'_2 \circ h'$ and $h' \circ G = G'_1 \circ h$

$$\begin{aligned}h \quad &\circ \quad G(x_0, x_1, \cdots, x_{n-1}) \\ &= \quad h(x_1\bar{x}_0 + \bar{x}_1\bar{x}_2, x_2\bar{x}_1 + \bar{x}_2\bar{x}_3, \cdots, x_0\bar{x}_{n-1} + \bar{x}_0\bar{x}_1) \\ &= \quad (x_1\bar{x}_0 + \bar{x}_1\bar{x}_2, x_3\bar{x}_2 + \bar{x}_3\bar{x}_4, \cdots, x_0\bar{x}_{n-1} + \bar{x}_0\bar{x}_1, x_2\bar{x}_1 + \bar{x}_2\bar{x}_3, \cdots, x_{n-1}\bar{x}_{n-2} + \bar{x}_{n-1}\bar{x}_0)\end{aligned}$$

$$\begin{aligned}G'_2 \quad &\circ \quad h'(x_0, x_1, \cdots, x_{n-1}) \\ &= \quad G'_2(\bar{x}_0, \bar{x}_2, \cdots, \bar{x}_{n-1}, \bar{x}_1, \cdots, \bar{x}_{n-2}) \\ &= \quad (x_1\bar{x}_0 + \bar{x}_1\bar{x}_2, x_3\bar{x}_2 + \bar{x}_3\bar{x}_4, \cdots, x_0\bar{x}_{n-1} + \bar{x}_0\bar{x}_1, x_2\bar{x}_1 + \bar{x}_2\bar{x}_3, \cdots, x_{n-1}\bar{x}_{n-2} + \bar{x}_{n-1}\bar{x}_0) \\ &= \quad h \circ G(x_0, x_1, \cdots, x_{n-1})\end{aligned}$$

$$\begin{aligned}h' \quad &\circ \quad G(x_0, x_1, \cdots, x_{n-1}) \\ &= \quad h(x_1\bar{x}_0 + \bar{x}_1\bar{x}_2, x_2\bar{x}_1 + \bar{x}_2\bar{x}_3, \cdots, x_0\bar{x}_{n-1} + \bar{x}_0\bar{x}_1) \\ &= \quad (x_1x_0 + \bar{x}_1x_2, x_3x_2 + \bar{x}_3rx_4, \cdots, x_0x_{n-1} + \bar{x}_0x_1, x_2x_1 + \bar{x}_2x_3, \cdots, x_{n-1}x_{n-2} + \bar{x}_{n-1}x_0) \\ &= \quad G'_1(x_0, x_2, \cdots, x_{n-1}, x_1, \cdots, x_{n-2}) \\ &= \quad G'_1 \circ h(x_0, x_1, \cdots, x_{n-1})\end{aligned}$$

Since B' satisfies Theorem 7, it converges to a homogeneous configuration $(p, \cdots, p)$ and even time steps of B must also converge to a $(p, \cdots, p)$. At odd time interval,

$$\begin{aligned}
\lim_{t\to\infty} x_i^{2t+1} &= \lim_{t\to\infty} f(x_{i-1}^{2t}, x_i^{2t}, x_{i+1}^{2t}) \\
&= f(\lim_{t\to\infty} x_{i-1}^{2t}, \lim_{t\to\infty} x_i^{2t}, \lim_{t\to\infty} x_{i+1}^{2t}) \\
&= f(p, p, p) \\
&= p(1-p) + (1-p)(1-p) \\
&= 1-p
\end{aligned}$$

In other words,

$$\begin{aligned}
\mathbf{X}^{2t} &\rightarrow (p, \cdots, p) \\
\mathbf{X}^{2t+1} &\rightarrow (1-p, \cdots, 1-p).
\end{aligned}$$

□

4.3.7 Rule 46

Rule 46 yields the most interesting results of all FCA in this class and it actually appears to be unique among all elementary FCA. In fact, from the experimental observations of [11] it is the only rule with periodic behavior of length 4. It once again uses the central value as the weight factor, but the resulting rule individuates sub-automata where the rule uses one value directly while the other is inverted resulting in temporally periodic behaviour in each of the $GWCA$ in the cross product.

We restate the rule:

$$x_i^{t+1} = (1 - x_i^t)x_{i+1}^t + x_i^t \bar{x}_{i-1}^t$$

We have two cases: when n is a multiple of 4, and when it is not.

Theorem 14. *When n modulo 4 is equal to 0, rule 46 is asymptotically periodic with period 4 both spatially and temporally. Furthermore, if $\forall i$ as $t \rightarrow \infty$, $x_{4i}^{4t} \rightarrow p$, and $x_{4i+1}^{4t} \rightarrow q$ then $x_{4i+2}^{4t} \rightarrow 1-p$, and $x_{4i+3}^{4t} \rightarrow 1-q$. Also, $x_{4i}^{4t+1} \rightarrow q$, $x_{4i}^{4t+2} \rightarrow 1-p$, and $x_{4i}^{4t+3} \rightarrow 1-q$. When n modulo 4 is not equal to 0, rule 46 converges to the homogeneous configuration $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$.*

Proof. Let A be a fuzzy cellular automaton following rule 46 and let $f : (0,1)^3 \rightarrow (0,1)$ be its local rule and F its global rule.

Assume to begin with, that n modulo 4 is 0. Let B be a fuzzy CA with global rule G given by the left shift of F:

$$G(x_0, \cdots, x_{n-1}) = (f(x_0, x_1, x_2), f(x_1, x_2, x_3), \cdots, f(x_{n-1}, x_0, x_1))$$

Let B' be the usual cross product with global rule G' given by local rules f_1 and f_2 as follows:

$$\begin{aligned}
f_1(x, y, z) &= yx + (1-y)z \\
f_2(x, y, z) &= (1-y)x + yz
\end{aligned}$$

$$
\begin{aligned}
& G'((u_0,\cdots,u_{m-1})\times(u_0,\cdots,u_{m-1})) \\
= \; & (f_1(u_0,v_0,u_1), f_2(u_1,v_1,u_2), f_1(u_2,v_2,u_3),\cdots, f_2(u_{m-1},v_{m-1},u_0)) \\
\times \; & (f_2(v_0,u_1,v_1), f_1(v_1,u_2,v_2), f_2(v_2,u_3,v_3)\cdots, f_1(v_{m-1},u_0,v_0)).
\end{aligned}
$$

The topological space B using the rule $G\circ G$ is conjugate to B' using rule $G'\circ G'$ under the homeomorphism h:

$$h(x_0,\cdots,x_{n-1}) = (x_0,\bar{x}_2,x_4,\cdots,\bar{x}_{n-2})\times(x_1,\bar{x}_3,x_5,\cdots,\bar{x}_{n-1})$$

To show this we consider a second homeomorphism $h' = h\circ inv$ which maps B to B' by

$$h'(x_0,\cdots,x_{n-1}) = (\bar{x}_0,x_2,\bar{x}_4,\cdots,x_{n-2})\times(\bar{x}_1,x_3,\bar{x}_5,\cdots,x_{n-1})$$

We first show that $G'\circ h = h'\circ G$

$$
\begin{aligned}
G' \;\circ\; & h(x_0,x_1,x_2,x_3,\cdots) \\
= \; & G'(x_0,\bar{x}_2,x_4,\cdots,\bar{x}_{n-2})\times(x_1,\bar{x}_3,x_5,\cdots,\bar{x}_{n-1}) \\
= \; & (f_1(x_0,x_1,\bar{x}_2), f_2(\bar{x}_2,\bar{x}_3,x_4),\cdots) \\
\times \; & (f_2(x_1,\bar{x}_2,\bar{x}_3), f_1(\bar{x}_3,x_4,x_5),\cdots) \\
= \; & ((x_1x_0+\bar{x}_1\bar{x}_2),(x_3\bar{x}_2+\bar{x}_3x_4),\cdots) \\
\times \; & ((x_2x_1+\bar{x}_2\bar{x}_3),(x_4\bar{x}_3+\bar{x}_4x_5),\cdots) \\
h' \;\circ\; & G(x_0,x_1,x_2,x_3,\cdots) \\
= \; & h'((x_1\bar{x}_0+\bar{x}_1x_2, x_2\bar{x}_1+\bar{x}_2x_3, x_3\bar{x}_2+\bar{x}_3x_4, x_4\bar{x}_3+\bar{x}_4x_5,\cdots)) \\
= \; & ((x_1x_0+\bar{x}_1\bar{x}_2),(x_3\bar{x}_2+\bar{x}_3x_4),\cdots) \\
\times \; & ((x_2x_1+\bar{x}_2\bar{x}_3),(x_4\bar{x}_3+\bar{x}_4x_5),\cdots)
\end{aligned}
$$

Similarly, $h\circ G = G'\circ h'$.

Finally we have,

$$
\begin{aligned}
G'\circ(G'\circ h) &= (G'\circ h')\circ G \\
&= h\circ G\circ G
\end{aligned}
$$

as required.

Since both systems in the cross product of B' are GWCA with γ as in the proof of Theorem 8, they will both converge to fixed points by Theorem 7. If $(p,\cdots,p)\times(q,\cdots,q)$ is the point of convergence of B', then $h^{-1}((p,\cdots,p)\times(q,\cdots,q)) = (p,q,1-p,1-q,p,q,1-p,1-q\cdots,p,q,1-p,1-q)$ is the point of convergence of B.

Since A is the right shift of B.

$$
\begin{aligned}
\mathbf{X}^{4t} &\rightarrow (p,q,1-p,1-q,p,q,\cdots,1-p,1-q) \\
\mathbf{X}^{4t+1} &\rightarrow (q,1-p,1-q,p,\cdots,q,1-p,1-q,p) \\
\mathbf{X}^{4t+2} &\rightarrow (1-p,1-q,p,q,\cdots,1-p,1-q,p,q) \\
\mathbf{X}^{4t+3} &\rightarrow (1-q,p,q,1-p,\cdots,1-q,p,q,1-p)
\end{aligned}
$$

When n is even but not divisible by 4. Let B' be a cross product of 2 GWCAs of length n, and let h map B to B' as follows:

$$\begin{aligned} h(x_0, x_1, \cdots, x_{n-1}) &= (x_0, \bar{x}_2, x_4, \cdots, x_{n-2}, \bar{x}_0, x_2, \cdots, \bar{x}_{n-2}) \\ &\times (x_1, \bar{x_3}, \cdots, x_{n-1}, \bar{x}_1, x_2, \cdots, \bar{x}_{n-1}) \end{aligned}$$

Let $h' = h \circ inv$. Let G' be as in the proof of the previous theorem.

As before, we can show that $h \circ G \circ G = G' \circ G' \circ h$ by showing the intermediary steps using h'. This time, however, we see that if $x_i \to p$ then $1 - x_i \to p$ also and so $p = \frac{1}{2}$.

When n is odd, we consider one $GWCA$ of length $2n$, which alternates original values and negations. So for n modulo 4 equal to 1, we will have:

$$(x_0, \bar{x}_2, x_4, \cdots, x_{n-1}, \bar{x}_1, x_3, \cdots, x_{n-2}, \bar{x}_0, x_2, \cdots, \bar{x}_{n-1}, x_1, \bar{x}_3, \cdots, \bar{x}_{n-2}),$$

and for n modulo 4 equal to 3 we will have:

$$(x_0, \bar{x}_2, x_4, \cdots, \bar{x}_{n-1}, x_1, \bar{x}_3, \cdots, x_{n-2}, \bar{x}_0, x_2, \cdots, x_{n-1}, \bar{x}_1, x_3, \cdots, \bar{x}_{n-2}).$$

The rule G' uses appropriate weights to average neighbours in this array. Clearly, due to its structure, B' must converge to $\frac{1}{2}$. □

References

[1] A. I. Adamatzky. Hierarchy of fuzzy cellular automata. *Fuzzy Sets and Systems*, (62):167–174, 1994.

[2] N. Boccara and K. Cheong. Automata network epidemic models. In *Cellular Automata and Cooperative Systems*, volume 396, pages 29–44. Kluwer, 1993.

[3] N. Boccara and H. Fuks. Cellular automaton rules conserving the number of active sites. *J. of Physics A: math. gen.*, 31:6007–6018, 1998.

[4] N. Boccara and H. Fuks. Number-conserving cellular automaton rules. *Fundamenta Informaticae*, 52:1–13, 2001.

[5] G. Cattaneo, P. Flocchini, G. Mauri, C. Quaranta-Vogliotti, and N. Santoro. Cellular automata in fuzzy backgrounds. *Physica D*, 105:105–120, 1997.

[6] A.M. Coxe and C.A. Reiter. Fuzzy hexagonal automata and snowflakes. *Computers and Graphics*, 27:447–454, 2003.

[7] K. Culik II, L. P. Hurd, and S. Yu. On the limit sets of cellular automatas. *SIAM Journal on Computing*, 18:831–842, 1989.

[8] B. Durand, E. Formenti, A. Grange, and Z. Ròka. Number conserving cellular automata: new results on decidability and dynamics. *Discrete Mathematics and Theoretical Computer Science*, pages 129–140, 2003.

[9] B. Durand, E. Formenti, and Z. Ròka. Number-conserving cellular automata I: decidability. *Theoretical Computer Science*, 299(1-3):523–535, 2003.

[10] N. Fatès. Experimental study of elementary cellular automata dynamics using the density parameter. In *Discrete Models for Complex Systems*, pages 155–166, 2003.

[11] P. Flocchini and V. Cezar. Radial view of continuous cellular automata. *Fundamenta Informaticae*, 87:1–19, 2008.

[12] P. Flocchini, F. Geurts, A. Mingarelli, and N. Santoro. Convergence and aperiodicity in fuzzy cellular automata: revisiting rule 90. *Physica D*, 42:20–28, 2000.

[13] E. Formenti and A. Grange. Number conserving cellular automata II: dynamics. *Theoretical Computer Science*, 304(1-3):269–290, 2003.

[14] H. A. Gutowitz. A hierachical classification of cellular automata. *Physica D*, 45:136–156, 1990.

[15] C. G. Langton. Studying artificial life with cellular automata. In *Evolution, Games, and Learning*. North Holland, 1986.

[16] P. Maji and P. P. Chaudhuri. Fuzzy cellular automata for modeling pattern classifier. *IEICE Transactions on Information and Systems*, 88(4):691–702, 2005.

[17] P. Maji and P. P. Chaudhuri. RBFFCA: A hybrid pattern classifier using radial basis function and fuzzy cellular automata. *Fundamenta Informaticae*, 78(3):369–396, 2007.

[18] A. Mingarelli. The global evolution of general fuzzy automata. *J. of Cellular Automata*, 1(2):141–164, 2006.

[19] A. Mingarelli. A study of fuzzy and many-valued logics in cellular automata. *J. of Cellular Automata*, 1(3):233–252, 2006.

[20] A. Moreira. Universality and decidability of number-conserving cellular automata. *Theoretical Computer Science*, 292(3):711–721, 2003.

[21] K. Nagel and M. Schreckenberg. A cellular automaton for freeway traffic. *J. of Physics I*, 2:2221–2229, 1992.

[22] C. A. Reiter. Fuzzy automata and life. *Complexity*, 7(3):19–29, 2002.

[23] K. Sutner. Classifying circular cellular automata. *Physica D*, 45:386–395, 1990.

[24] J. Von Neumann. *Theory of Self-Reproducing Automata*. University of Illinois Press, Urbana, 1966.

[25] S. Wolfram. Universality and complexity in cellular automata. *Physica D*, 10:1–35, 1984.

[26] S. Wolfram. *Theory and Applications of Cellular Automata*. World Scientific, 1986.