Radial View: Observing Fuzzy Cellular Automata with a New Visualization Method *

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Abstract

Fuzzy Cellular Automata (FCA) are special couple map lattices that generalize Boolean cellular automata by allowing continuous values for the cell states. The local transition rule of a FCA is the “fuzzification" of the disjunctive normal form that describes the local function of the corresponding Boolean cellular automata.

Recently there has been some interest in elementary Fuzzy cellular automata and, in particular, in the understanding of their dynamics.

The classical observation medium for these systems - whether Boolean or Fuzzy - has always been the space-time diagram. In this paper we consider a new way to visualize the evolution of Fuzzy Cellular Automata that reveals interesting dynamics not observable with the space-time diagram.

We then experimentally group the elementary circular Fuzzy Cellular Automata on the basis of their behavior as observed with this new visualization tool. We discover that all circular FCA have a periodic behavior. Interestingly, the periods have only lengths 1,2,4,n (where n is the size of a configuration).

Keywords Fuzzy Cellular Automata, Visualization of dynamics, Evolution, Fixed Points, Asymptotic behavior

1 Introduction

Boolean cellular automata have been introduced by Von Neumann as models of self-organizing and reproducing behaviors [20] and their applications range from ecology to theoretical computer science (e.g., [1, 9, 13, 23]). A one dimensional Boolean cellular automaton (CA) is constituted

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by a collection of cells arranged in an array. Each cell has a state in $\{0, 1\}$ which changes at successive discrete steps by the iteration of a local transition function that depends on the states of the “neighbouring" cells. The global evolution of a CA is defined by the synchronous update of all states according to the local function applied to each cell. A configuration of the automaton is a description of all cell values. Cellular Automata can be bi-infinite or finite with defined boundary conditions. Typical boundary conditions consists of “ wrapping around" a finite array (circular CA) or fixing a certain initial size for the array and assuming all the cells beyond the boundaries to be quiescent (zero background).

While Cellular Automata are discrete both in time and in space, Coupled Map Lattices (CML) are discrete in time, but continuous in space. They have have been introduced by Kaneko as simple models with the features of spatiotemporal chaos, and have now applications in many different areas like fluid dynamics, biology, chemistry, etc. (e.g., [11, 12, 21]).

Fuzzy Cellular Automata (FCA) are particular couple map lattices where the local transition rule is obtained by “fuzzifying" the disjunctive normal form that describes the local function of the corresponding Boolean CA. In other words, a FCA is a generalization of a CA in a continuous environment; vice-versa, a boolean CA can be seen as a discretized version of its continuous counterpart.

FCA have been introduced in [3, 2] as a framework for understanding complex behaviors. They have been employed for studying the impact that perturbations (e.g. noisy sources, computation errors, mutations, etc.) can have on the evolution of Boolean CA (e.g., see [8]), and they have also been studied in relation to pattern recognition (e.g., see [14, 15]) and as a model to generate interesting images mimicking nature (e.g. [18, 4]. Moreover, dynamical properties of Fuzzy CA rules have been analytically studied (e.g., in [7, 16, 17]) to understand the relationship with Boolean cellular automata and the impact of discretization.

Since the analytical study of these systems is generally very complex, visualization plays an important role in understanding their dynamics. Space-time diagrams have been traditionally used to display the dynamics of one dimensional CA: the top-most row corresponding to the initial configuration at time $t = 0$ and rows corresponding to configurations at successive time steps. When visualizing the space-time diagram of Fuzzy rules, the interval [0,1] must be discretized, since only a finite number of states can obviously be represented. The interval is divided in k ranges and each is assigned a different colour. It has been shown in [7] that this discretization process could mislead the observer by showing a totally incorrect dynamics. This is the case, for example, of Rule 90 where depending on the parity of k, totally different behaviors are displayed. In fact, if the fixed point ($\frac{1}{2}$) happens to be the extreme of a range, the diagram shows an alternation between the two different values around the fixed point; on the other hand, if the fixed point belongs to a range, the diagram shows a quick convergence.

There is another aspect of space-time diagrams that could mislead the observer. Colours can show nice patterns, which allow the observer to differentiate between an extremely simple dynamics and a more complex one. They however do not give insights into the way a rule towards its attractor, especially if the number of iterations displayed is not very large. This is the case, for example, of Rule 30, which appears to be quite complex when observed with a space-time diagram, and has been shown analytically to have a rather simple behavior, converging to a fixed point [16].

In this paper we propose a new way of visualizing FCA (*Radial Representation*) and we show that by observing their dynamics in this different fashion we can detect properties of their behavior that were totally hidden in the space-time diagram. Based on this new observational point of view, we classify the observable behavior of circular fuzzy cellular automata.

We discover that all FCA have a periodic behavior and that the length of their periods is only 1,2,4, or n. We pay special attention to the most interesting dynamics, and in those cases we also verify analytically the observed periodic behaviors.

The paper is organized as follows: in Section 2 we give definitions and introduce some terminology, Section 3 briefly outlines some of the existing classifications of cellular automata. In Section 4 we introduce a new visualization method; in Section 5 we propose a classification based on the lengths of the periods of the various rules.

2 Definitions

A one dimensional bi-infinite Boolean Cellular Automaton is a collection of cells arranged in a linear a bi-infinite lattice. Cells have Boolean values and they synchronously update their values according to a local rule applied to their neighborhood. A configuration of the automaton is a description of all cell values. In the case of *elementary cellular automata* the neighborhood of a cell consists of the cell itself and its left and right neighbours, thus the local rule has the form: $g : \{0,1\}^3 \to \{0,1\}$. The global dynamics of an elementary one dimensional cellular automata is then defined by:

$$f : \{0,1\}^Z \to \{0,1\}^Z \quad s.t. \quad \forall i \in Z, f(x)_i = g(x_{i-1}, x_i, x_{i+1})$$

In the following we concentrate only on elementary Cellular Automata. The *local rule* of an elementary CA is given by 8 binary tuples, corresponding to the 8 possible local configurations a cell can detect in its direct neighborhood:

$$(000, 001, 010, 011, 100, 101, 110, 111) \to (r_0, \cdots, r_7).$$

In general, the binary representation $(r_0, \cdots, r_7)$ is converted into the decimal representation $\sum_{i=1:8} 2^{i-1} r_i$; this value is used as the name of the rule.

A local rule can also be canonically expressed as a *disjunctive normal form*:

$$f(x_1, x_2, x_3) = \bigvee_{i|r_i=1} \bigwedge_{j=1:3} x_j^{d_{ij}}$$

where d_{ij} is the j-th digit, from left to right, of the binary expression of i, and x^0 (resp. x^1) stands for $\neg x$ (resp. x).

A Fuzzy cellular automaton is a particular Coupled Map Lattice [11], obtained by "fuzzification" of the local rule of a classical Boolean CA. The "fuzzification" consists of a fuzzy extension of the boolean operators AND OR and NOT. Depending on which fuzzy operators are used a different type of Fuzzy cellular automata can be defined. Among the various possible choice for the fuzzy operators, we consider the following: $(a \vee b)$ is replaced by $(a + b)$; $(a \wedge b)$ by (ab), and $(\neg a)$ by $(1 - a)$. The resulting local rule becomes the following real function that generalizes the original function on $\{0, 1\}^3$:

$$f : [0,1]^3 \to [0,1] \;\; s.t.\,, \;\; f(x_1, x_2, x_3) = \sum_{i=0\ldots7} r_i \prod l(x_j, d_{i,j})$$

with $l(a, 0) = 1 - a$ and $l(a, 1) = a$. The usual fuzzification of the expression $(a \vee b)$ would be $max\{1, (a + b)\}$ so as to ensure that the result is not larger than 1. Note however, that taking $(a + b)$ for the CA fuzzification does not lead to values greater than 1 since the maximum possible sum is 1 and occurs for rule 255 which contains the sum of all the expressions $(000, 001, 010, 011, 100, 101, 110, 111) \to (1, 1, 1, 1, 1, 1, 1, 1)$; any other CA rule is a partial sum and must be bounded by 1.

Example of fuzzification. Consider, for example, rule $14 = 2 + 4 + 8$:

$$(000, 001, 010, 011, 100, 101, 110, 111) \to (0, 1, 1, 1, 0, 0, 0, 0).$$

The canonical expression of rule 14:

$$f_{14}(x_1, x_2, x_3) = (\neg x_1 \wedge \neg x_2 \wedge x_3) \vee (\neg x_1 \wedge x_2 \wedge \neg x_3) \vee (\neg x_1 \wedge x_2 \wedge x_3).$$

The fuzzification process, after simplification, yields:

$$f_{14}(x_1, x_2, x_3) = (1 - x_1) \cdot (x_2 + x_3 - x_2 \cdot x_3).$$

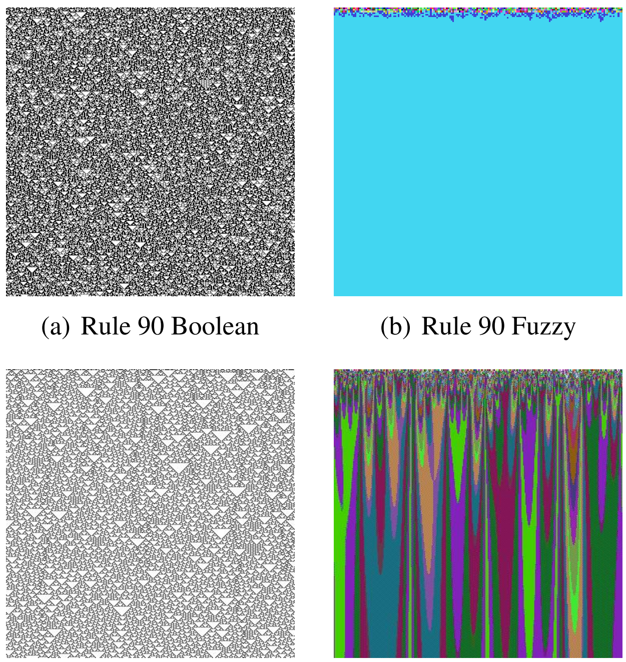
A circular Fuzzy CA is defined as a FCA where the last cell is a neighbour of the first; a CA in quiescent background has a finite initial configuration in an infinite support of cells in the state zero. In case of finite configurations, we indicate with $X^0 = (x_0^0, x_1^0, \ldots, x_{n-1}^0)$ the initial configuration, and with X^t the configuration at time t. When a configuration at time t is spatially periodic we shall indicate it by $X^t = (\alpha)^m$, where α is the smallest period and m is its size.

3 Classifications of CA behaviors

Classification of cellular automata has always been considered a fundamental problem. The first attempt to classify cellular automata has been done by Wolfram in [22] where cellular automata are grouped according to their observed behavior of their space-time diagram as follows:
- Class 1: automata that evolve to a unique, homogeneous state, after a finite transient,
- Class 2: automata whose evolution leads to a set of separated simple stable or periodic structures (space-time patterns),
- Class 3: automata whose evolution leads to aperiodic ("chaotic") space-time patterns,
- Class 4: automata that evolve to complex patterns with propagative localized structures, sometimes long-lived.

Although not formally precise, this classification captures important distinctions among cellular automata. Several other criteria for grouping CA have followed: some based on observable behaviors, some on intrinsic properties of CA rules (e.g., see [5, 6, 10, 19]).

Fuzzy Cellular Automata have been observed using the classical space-time diagram, where colours represent intervals of real values. Fuzzy cellular automata in boolean backgrounds have been grouped according to their observed behavior in [2] and, in particular, according to the level of “spread" of fuzziness. Essentially the possible observed behaviors are three: 1) Boolean values are destroyed by Fuzzy values; 2) Fuzzy values are destroyed by Boolean values; 3) Fuzzy and Boolean values co-exist forming various patterns in the space-time diagram. Circular Fuzzy cellular automata have never been classified; from the observation of their space-time diagram it is clear that some evolve to a fixed-point, some present a shifting behavior, while some display “nice" patterns (for some examples of patterns see Figures 1 and 10).



(a) Rule 90 Boolean (b) Rule 90 Fuzzy

(c) Rule 18 Boolean (d) Rule 18 Fuzzy

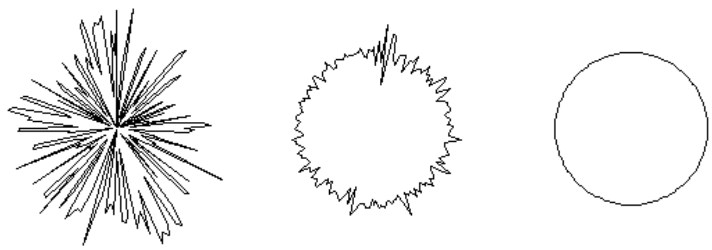
Figure 1: Two rules that display similar behavior in the Boolean setting and very different in the Fuzzy setting.

Interestingly, the behavior of Fuzzy cellular automata does not follow the behavior of their boolean counter-part: in fact, rules belonging to the same Wolfram-class can have, in some cases, an observable dynamics very different in the Fuzzy version (see Figure 1). In [7] the reasons for such differences are analytically explained for the case of rule 90.

Because of the complex patterns generated by some Fuzzy Cellular Automata in the space-time diagram, it was conjectured that some rules were more complex than others, possibly having chaotic asymptotic dynamics. The only analytical study of Fuzzy cellular automata has been done by Mingarelli, who conducted a comprehensive analysis of FCA in quiescent backgrounds showing that none of them has a chaotic dynamics ([16, 17]), thus disproving the above conjecture for FCA in quiescent backgrounds. It is left open what happens when the background is not quiescent or when the CA has circular boundary conditions. In the following we concentrate on circular FCA with circular boundary conditions and we propose a first classification.

4 Radial Representation

The visualization of a complex dynamical process is always very important. Different observation methods could give very different insights into the dynamics of the process. In the following we propose a new way to visualize circular Fuzzy Cellular Automata.



(a) Initially (b) After 3 iterations (c) After 20 iterations

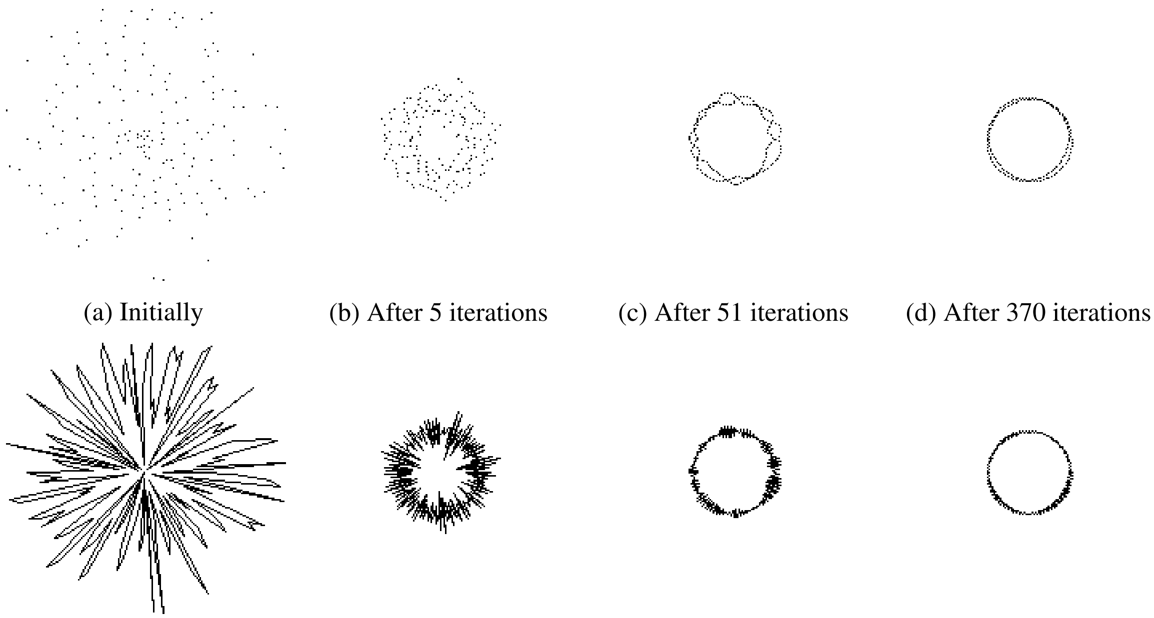
Figure 2: Some configuration in the evolution of rules R_{110} with the Polygon representation

Let k be the number of cells. Consider a unitary circle with center C divided in k equal sectors corresponding to radii $r_0, \ldots r_{k-1}$. Each cell of the CA corresponds to a radius r_i, its value x_i corresponds to distance x_i from C in r_i and is represented by a dot in the corresponding position. For example, cells with value 1 will have a dot in correspondence of the circumference, cells with value 0 will have a dot in the center, and any value between 0 and 1 will have its representational dot plotted inside the circle. A configuration of the CA then corresponds to a plotting on the circle of the various distances.

As opposed to the space-time diagram, with a radial representation we cannot observe the “life" of a CA in a 2-dimensional picture. In fact, the evolution of a CA corresponds to the time sequence of the plottings of the configurations. In other words, a radial representation provides a dynamic diagram where the observer can see the evolution through time in frames where each frame corresponds to a global configuration in an instant.

Two variants of the radial visulization are also employed: the *Polygon*, and the *Bar*. In the *Polygon* the dot corresponding to cell r_i is connected to the one corresponding to the neighbouring cells $r_{i\pm1}$ (the operations on indices are modulo n, the size of the configuration). In the *Bar* instead the values are arranges linearly and a vertical bar correspond to the cell’s value.

Examples. An example of Radial representation is shown in Figure 3 where rule R_{18} is displayed both in the Radial and in the Polygon representation at different times in the evolution. It is interesting to observe the formation of two curves during the evolution revealing a spatial correlation between the cells. Another example of the Polygon representation is given in Figure 2, where a sequence of configurations of Rule R_{110} are displayed. In this case the rule converges quickly to an homogeneous fixed point and already after 20 iteration it is clear that all the dots corresponding to the FCA's states are placed on a circle. Finally, an example of the three representations (Radial, Polygon, and Bar) is shown in Figure 6 in two moments during the evolution of rule R_{78}.



(a) Initially (b) After 5 iterations (c) After 51 iterations (d) After 370 iterations

Figure 3: Some configurations in the evolution of rules R_{18} with the Radial and the Polygon representations

5 Experimental Classification

In this section we group Fuzzy elementary cellular automata according to their behavior when observed with the Radial/Polygon/Bar representations.

We investigate in detail the case of circular boundary conditions, which correspond to considering infinite spatially periodic initial conditions. In all cases, we have observed all 256 elementary CAs; the results below refer to the 90 that are not equivalent under simple transformations (conjugation, reflection, and combination of conjugation and reflection). Since parity seems to be an important factor in the asymptotic behavior of Fuzzy Cellular Automata, each CA has been run starting from 100 different random initial condition of odd size and 100 of even size.

With circular boundary conditions, all cellular automata display a periodic behavior. Interestingly, the lengths of the periods that we have observed are only: 1,2,4, n (where n is the size of the configuration). In most cases the transient to the period is very short, in some cases it takes considerably longer time for the automata to converge. In the case of fixed points, we have observed homogeneous fixed points (Circle, Dot); non-homogeneous ones (Stars). In the case of periodic behavior of length 2 or 4 we have to distinguish between periodicities due to alternating behaviors and periodicities due to convergence to shifting behavior of spatially periodic configurations. The periods of length n instead, are all due to shifting behaviors.

Period 1 (zero)	R_0, R_8, R_{32}, R_{40}, R_{72}, R_{104}, R_{128}, R_{136} R_{160}, R_{168}, R_{24}, R_{36}, R_{152}, R_{164} , R_{44}, R_{56} R_{74}, R_{200}
Period 1 (Homogeneous)	R_6, R_9, R_{22}, R_{25}, R_{26}, R_{33}, R_{35}, R_{37}, R_{38}, R_{41}, R_{73}, R_{134}, R_{30}, R_{45}, R_{54},R_{57}, R_{60} R_{90}, R_{105}, R_{106}, R_{150}, R_{154}, R_{61},R_{62},R_{110},R_{122}, R_{188}, R_{126}, R_{172}
Period 1 (Non-Homo)	R_4, R_{12}, R_{13}, R_{76}, R_{77}, R_{132}, R_{140}, R_{204}, R_{232} R_{28} R_{108}, R_{156}
Period 1 non-homo for n even	R_{78}, R_{94}
Period 2 for all n	$R_1, R_5, R_{19}, R_{23}, R_{50}, R_{51}R_{178}$
Period 2 for n even	R_{18}, R_{27}, R_{29}, R_{58}, R_{146}, R_{184}
Period 4 for n multiple of 4	R_{46}
Period n (Shifts)	R_2, R_{10}, R_{15}, R_{34}, R_{42}, R_{130}, R_{138}, R_{162},R_{170} $R_3, R_7, R_{11}, R_{14}, R_{43}, R_{142}$

Table 1: Dynamics of Circular Elementary Cellular Automata

5.1 Periods of Length 1: Fixed Points

5.1.1 Homogeneous Convergence: Dot

After a small number of iterations, the CA converge to zero and in the Polygon/Radial representation we can observe a single dot in the center of the circle. Notice that in all cases, zero is the solution of the equation $f(x, x, x) = x$, it.e., it is the fixed point of the corresponding FCA from homogeneous configurations. For some of these rules (R_0, R_8, R_{32}, R_{40}, R_{72}, R_{104}, R_{128}, R_{136}, R_{160}, R_{168}, R_{200}) convergence happens in less then 100 iterations whereas in others (R_{24}, R_{36}, R_{152}, R_{164}) it will take considerable more than this to stabilize. Finally, for rules R_{44} R_{56} and R_{74} the convergence is extremely slow.

5.1.2 Homogeneous Convergence: Circles

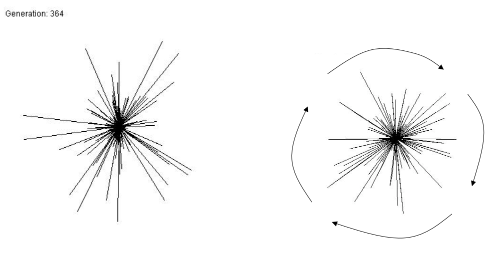
In our experiments, 26 rules quickly converge to a circle in the polygon diagram (see an example in Figure 2). In fact, after a small number of iterations, all the values in the lattice converge to the fixed point of the corresponding rule when starting from homogeneous configurations (i.e., to the solution to $f(x, x, x) = x$). This is the case of Rules R_{30}, R_{45}, R_{54}, R_{57}, R_{60}, R_{90}, R_{105}, R_{106}, R_{150}, R_{154} R_6, R_9, R_{22}, R_{25}, R_{26}, R_{33}, R_{35}, R_{37}, R_{38}, R_{41}, R_{73}, R_{134}, R_{61}, R_{62}, R_{110}, R_{122}.

Also Rules R_{172} and R_{126} converge to an homogeneous fixed point, but their convergence take a much longer time. Rule R_{172} ($f(x, y, z) = y - xy + xz$) is one of those rules for which any homogeneous configuration is a fixed point ($f(a, a, a) = a$, for all a). When starting from a random initial configuration, the rule always converges to an homogeneous fixed point, but the value is variable and depends on the initial configuration. Rule R_{126} ($f(x, y, z) = x + y - xy + z - xz - yz$)

seems to converge to $(2/3)^n$, which is the fixed point for homogeneous initial configurations. The convergence towards $(2/3)^n$ is slow and the values of consecutive configurations clearly oscillate around 2/3.

5.1.3 Non Homogeneous Convergence

Rule R_{204} ($f(x, y, z) = y$) is the perfect fixed point; its initial configuration is preserved. Other rules that converge to a non-homogeneous fixed point after a short transient are: R_4, R_{12}, R_{13}, R_{76}, R_{77}, R_{132}, R_{140}, R_{232}, R_{28} R_{108}, R_{156}.



(a) Rule 4 Polygon (b) Rule 10 Polygon

Figure 4: (a) a fixed point for rule R_4 observed after 364 steps (b) a shifting (i.e. rotating) configuration for rule R_{10}, observed after 364 steps.

Finally, Rules R_{94} and R_{78} merit special attention and are separately described below: when the size of the configuration is even, they both converge to fixed points consisting of spatially periodic configurations, while they converge to an homogeneous fixed point when the size is odd. During the evolution, the Radial/Polygon diagrams show very quickly the formation of two co-existing curves with consecutive values belonging to different curves, eventually stabilizing in two concentric circles (see, for Example, two moments in the evolution of Rule R_{78} in Figure 6).

	Observed Fixed Points
R_{94}	$f_{94}(x, y, z) = x + y - xy + z - 2xz - yz + xyz$
n even	$(a, b)^{\frac{n}{2}}$ with $2a + 2b - ab = 2$
n odd	$(2 - \sqrt{2})^n$
R_{78}	$f_{78}(x, y, z) = y + z - xz - yz$
n even	$(a, b)^{\frac{n}{2}}$ with $a + b = 1$
n odd	$(\frac{1}{2})^n$

Rule R_{94}. Our observations show that CA converges to a *spatially periodic fixed point* of the form $(a, b)^{\frac{n}{2}}$ (with $a \neq b$): that is, there exists a t such that $\forall t' > t$, $X^{t'} = (a, b)^{\frac{n}{2}}$. The following Theorem shows that a configuration of this type can be a fixed point only when $2a + 2b - ab = 2$.

Theorem 1. *Consider rule R_{94} on a configuration of even size. Let $X^t = (a, b)^{\frac{n}{2}}$. Configuration X^t is a fixed point if and only if $2a + 2b - ab = 2$.*

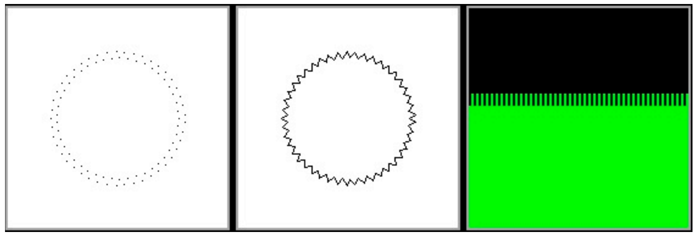
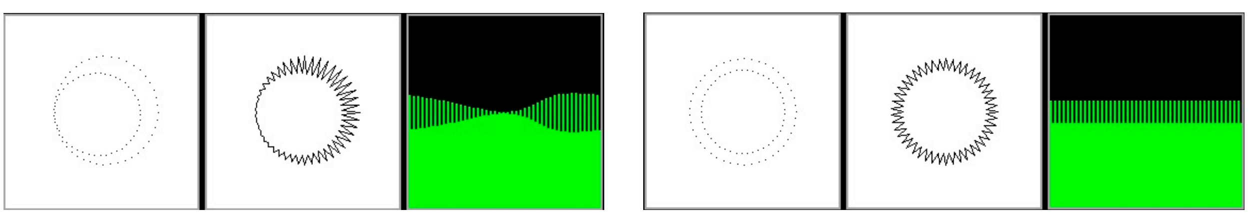


Figure 5: Rule R_{94} after 5000 iteration with 90 cells.

Proof. Rule R_{94} has the following analytical form: $f(x, y, z) = x + y - xy + z - 2xz - yz + xyz$. Let a, b, a be three consecutive values in configuration X^t; the only way to obtain a fixed point dynamic with configuration X^t is when the local function f satisfies the condition $f(a, b, a) = b$ and $f(b, a, b) = a$. We have that $f(a, b, a) = a+b-ab+a-2a^2-ab+a^2b = 2a+b-2ab-2a^2+a^2b$, thus $f(a, b, a) = b$ when either $a = 0$ or $2a + 2b - ab = 2$. When $2a + 2b - ab = 2$, we also have that $f(b, a, b) = b$ and thus this condition indeed guarantees that $(a, b)^{\frac{n}{2}}$ is a fixed point. When $a = 0$ it must be $f(b, 0, b) = 0$, which is verified only when $b \in \{0, 1\}$. It is easy to see that the homogeneous configuration $(0)^n$ is repelling, the remaining spatially 2-periodic fixed point $(0, 1)^{\frac{n}{2}}$ satisfy also condition $2a + 2b - ab = 2$. □

On the other hand, if the size of the CA is odd, the two curves converge to a single circle because the CA converges to $2 - \sqrt{2}$. Notice that $2 - \sqrt{2}$ is the attracting fixed point of rule R_{94} when starting from homogeneous configurations (and clearly satisfies also the condition $2a+2b-ab = 1$ of the above Theorem).

Rule R_{78}. During the evolution the behavior of the CA is shifting (i.e., the two curves rotate); however, the CA eventually converges to a spatially periodic fixed point of the form $(a, b)^{\frac{n}{2}}$ ($a \neq b$). We can easily verify that $(a, b)^{\frac{n}{2}}$ is indeed a fixed point for rule R_{78} with the condition $(a+b) = 1$.



(a) Rule 78 during the evolution

(b) Rule 78 at the convergence point

Figure 6: Rule R_{78} with 90 cells observed with Radial, Polygon, and Bar representation (a) after 2500 iterations (b) after 14,000 iterations.

Theorem 2. *Consider rule R_{78} on a configuration of even size. Let $X^t = (a, b)^{\frac{n}{2}}$ with $a \neq b$. Configuration X^t is a fixed point if and only if $a + b = 1$.*

Proof. The proof is analogous to the one of Theorem 1. Rule R_{78} has the following analytical form: $f(x,y,z) = y + z - xz - yz$. Let a, b, a be three consecutive values in configuration X^t We have that $f(a,b,a) = b$ for $b + a - a^2 - ab = b$, which implies that either $a + b = 1$ or $a = 0$. In the first case we also have $f(b,a,b) = a$, in the second case, we must have that $b \in \{0,1\}$. Configuration $(0)^n$ is repelling, the remaining spatially 2-periodic fixed point $(0,1)^{\frac{n}{2}}$ satisfy condition $a + b = 1$ as well. □

In the case of configurations of odd sizes, the CA converges to the homogeneous fixed point $(1/2)^n$, which is the attracting fixed point from homogeneous initial configurations (and clearly satisfies the condition $a + b = 1$).

5.2 Periods of Length Two

Some rules have a periodic behavior of length two for any value of n, while some other rules display this periodicity only when n is even. This is the case of rules where the configuration become spatially periodic in time and the temporal periodicity is actually given by a shifting behavior.

5.2.1 Periodicity for all sizes of the initial configuration.

The simplest rule with asymptotic periodic behavior is the complement rule R_{51} ($f(x,y,z) = 1 - y$). Other rules that very quickly stabilize in a periodic behavior of length two are R_1, R_5, R_{19}, R_{23}, R_{50}, R_{178}. Rule R_{27} converges to a periodic behavior of length two, like the previous rules, but with a much longer transient. Each configuration in the period is homogeneous and is the complement of the next (i.e., $X^t = (a)^n, X^{t+1} = (1-a)^n, \ldots$). This happens regardless of the parity of the initial configuration. It is indeed very simple to see that if $X^t = (a)^n$, the next configuration is its complement. In fact, it comes form the observation that, for rule R_{27}, whose analitycal expression is $f(x,y,z) = 1 - y - xz + yz$ we have that $f(a,a,a) = (1-a)$.

5.2.2 Periodicity for configurations of even size.

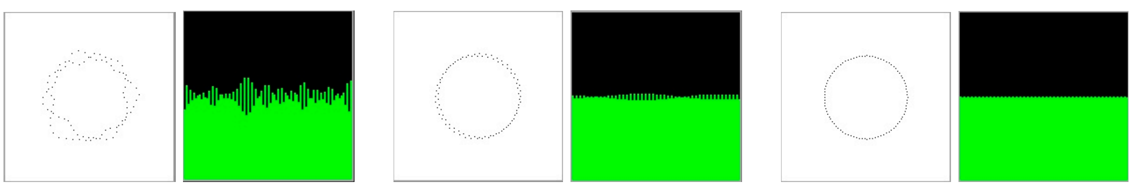
The periodic behavior of the rules described below (except for rule R_{29}) is really a shifting behavior occurring on spatially periodic configurations with period two. Thus, the asymptotic behavior is *spatially and temporally 2-periodic*. Due to this reason, this dynamics is visible only if the configuration is of even size. Interestingly, if it is of odd size, all the rules converge to an homogeneous fixed point. From a pictorial point of view, when the size of the configuration is even, these rules converge towards two concentric rotating circles. After a few time steps they all display two co-existing curves with consecutive values belonging to different curves (see, for an example, Figure 8).

	Spatially periodic Shifts		Spatially periodic Shifts
f_{18}	$x - xy + z - 2xz - yz + 2xyz$	f_{146}	$x - xy + z - 2xz - yz + 3xyz$
n even	$(a,b)^{\frac{n}{2}}$ with $a + b + ab = \frac{1}{2}$	n even	$(a,b)^{\frac{n}{2}}$ with $2a + 2b - 3ab = 1$
n odd	$(\frac{1}{3})^n$	n odd	$(\frac{1}{3})^n$
f_{58}	$x - xy + z - xz$	f_{184}	$x - xy + yz$
n even	$(a,b)^{\frac{n}{2}}$ with $a + b = 1$	n even	$(a,b)^{\frac{n}{2}}$, $\forall a, b$
n odd	$(\frac{1}{2})^n$	n odd	$(a)^n$, $\forall a$

	Shift of Complement
f_{29}	$1 - z + yz - xy$
n even	$(a,b)^{\frac{n}{2}}$ $\forall a, b$
n odd	$(a)^n$, $\forall a$

Rules R_{18}**,** R_{146}**,** R_{58} When the configuration size of is even, these cellular automata form two co-existing rotating curves that eventually converge to two concentric circles with consecutive values belonging to different curves. In other words, the rules seem to converge to a shifting configuration of the form $(a,b)^{\frac{n}{2}}$ (i.e., alternating with $(b,a)^{\frac{n}{2}}$) with $a \neq b$.

This behavior is not always obvious from the observation of the evolution (see, for example, three different moments in the evolution of Rule R_{58}), but we can prove that, for these rules, a configuration $X^t = (a,b)^{\frac{n}{2}}$ has indeed a shifting behavior (and thus a periodic behavior of length two) for a certain relationship between a and b, which depends on the rule.



(a) (b) (c) (d) (e) (f)

Figure 7: 58 Radial Bar.

The following Theorem shows that for rule R_{18} configuration $X^t = (a,b)^{\frac{n}{2}}$ has a shifting behavior (and thus a periodic behavior of length two) when $a + b + ab = \frac{1}{2}$ or for $X^t = (0,1)^{\frac{n}{2}}$.

Theorem 3. *Consider rule R_{18} on a configuration of even size. Let $X^t = (a,b)^{\frac{n}{2}}$ with $a \neq b$. Configuration $X^{t+1} = (b,a)^{\frac{n}{2}}$ (and thus $X^{t+2} = X^t$) if and only if $a + b + ab = \frac{1}{2}$ or $a = 0, b = 1$.*

Proof. Rule R_{18} has the following analytical form: $f(x,y,z) = x - xy + z - 2xz - yz + 2xyz$. Let a, b, a be three consecutive values in configuration X^t; in order for the global dynamics to be shifting, the local rule f must verify the condition $f(a,b,a) = a$ (and $f(b,a,b) = b$). We have that $f(a,b,a) = 2a - 2ab - 2a^2 + 2a^2 b$, which means that it must be either $a + b + ab = \frac{1}{2}$ or $a = 0$. Condition $a + b + ab = \frac{1}{2}$ verifies also $f(b,a,b) = b$. If $a = 0$, solving $f(b,0,b)$ we have that b must be either 0 or 1. Configuration $(0)^n$ is a repelling homogeneous fixed point, so the only two spatially 2-periodic shifting configurations are $(a,b)^{\frac{n}{2}}$ with $a + b + ab = \frac{1}{2}$, and $(0,1)^{\frac{n}{2}}$. □

Although configuration $(0,1)^{\frac{n}{2}}$ is also a fixed point for rule R_{18}, we never observe it in our experiments from random initial configurations, which always converge to $(a,b)^{\frac{n}{2}}$ with $a+b+ab=\frac{1}{2}$.

Analogously to the case of rule R_{18}, it can be shown that a shifting configuration of the form $(a,b)^{\frac{n}{2}}$ must verify condition $2a+2b-3ab=1$ for rule R_{146} (whose analytical form is $f_{146}(x,y,z)=x-xy+z-2xz-yz+3xyz$), and condition $a+b=1$ for rule R_{58} (whose analytical form is $f_{58}(x,y,z)=x-xy+z-xz$).

When the configuration size is odd, the radial diagram of these rules still show two co-existing curves with consecutive values belonging to different curves in all places except for one. The two curves eventually merge into one and the CA converge to an homogeneous configuration; in fact, from our experiments starting from random configurations, there exists a t such that $\forall t'>t$, $X^{t'}=(\frac{1}{3})^n$ in the case of rule R_{18} and R_{146}, and to $(\frac{1}{2})^n$ in the case of rule R_{58}. Notice that these are the attracting fixed points of the corresponding rules from homogeneous configurations.

Rule R_{184}. Rule R_{184} has the following analytical form: $f(x,y,z)=x-xy+yz$. This rule is special in many ways. First of all, when starting from homogeneous configurations, any configuration is a fixed point (this is due to the fact that $f(a,a,a)=a,\forall a$). Furthermore, any 2-periodic configuration gives rise to a periodic behavior (this is due to the fact that $f(a,b,a)=a$, $\forall a,b$.

As before, when the size of the CA is even, the radial diagram shows 2 co-existing curves with consecutive values belonging to different curves thus displaying a convergence to a *spatially and temporally 2-periodic* configuration $(a,b)^{\frac{n}{2}}$ (i.e., to two rotating concentric circles, as shown in Figure 8). However, the values of a and b are not linked to each other. From the experiments it seems that the CA can be attracted, depending on the initial configuration to various configuration of the type $(a,b)^{\frac{n}{2}}$, which will then shift.

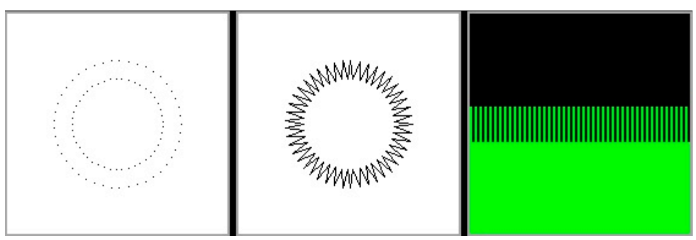


Figure 8: Rule R_{184} after 5500 iteration with a configuration of 90 cells.

When the size is odd the CA converges to an homogeneous configuration. The convergence value is variable and depends on the initial configuration.

Rule R_{29}. Also this rule has a unique behavior. When the size of the CA is even, the CA converges to a spatially 2-periodic configuration of the type $(a,b)^{\frac{n}{2}}$ and then alternates with the shift of its complement $(1-b,1-a)^{\frac{n}{2}}$ still giving rise to a period of length 2. Rule R_{29} has the following analytical form: $f(x,y,z)=1-xy-z+yz$, which means that $f(a,b,a)=1-a$ for any values of a and b. As a consequence $(a,b)^{\frac{n}{2}}$ is clearly a spatial and temporal period-2 configuration.

When the size of the CA is odd the CA converges to a periodic behavior of the type $X^t=(a)^n, X^{t+1}=(1-a)^n,\ldots$. The value of a varies and depends on the initial configuration.

5.3 Periods of Length 4: Rule R_{46}

There is only one rule in this category, rule R_{46}. During the evolution, we can observe the formation of four co-existing rotating curves that eventually converge to four concentric circles (see Figure 9). As for the rules of the previous subsection, the periodic behavior is due to the convergence to spatially 4-periodic configurations that are shifting, thus the requirement of configurations with size multiple of 4.

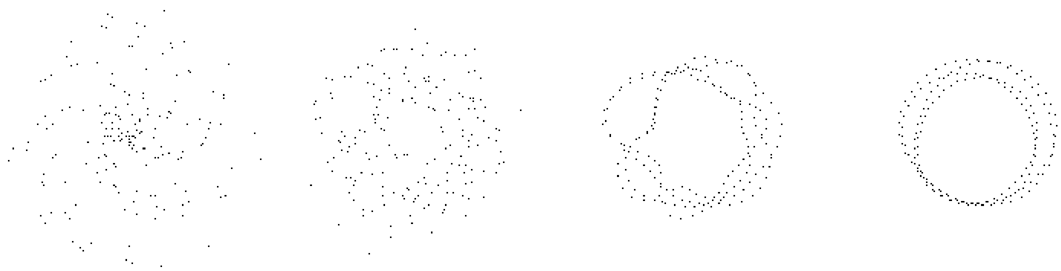


Figure 9: Some configurations in the evolution of rules R_{46} with the Radial representation.

This rule has an attracting fixed point $(\frac{1}{2})^{\frac{n}{2}}$ from homogeneous initial configurations; with configurations of even size it can be shown that it does not have any asymptotically shifting configuration of the type $(a, b)^{\frac{n}{2}}$ with $a \neq b$. On the other hand, with configurations that are multiple of four, the four-periodic configuration $(a, b, c, d)^{\frac{n}{4}}$ shifts resulting in a periodic behavior of length four when $c = 1 - a$ and $d = 1 - b$).

	Spatially periodic Shift
$f_{46}(x, y, z)$	$y - xy + z - yz$
n multiple of four	$(a, b, c, d)^{\frac{n}{4}}$ with $c = 1 - a$, $d = 1 - b$
n not multiple of four	$(\frac{1}{2})^n$

Theorem 4. *Consider rule R_{46} on a configuration of size $n = 4m$. Let $X^t = (a, b, c, d)^m$ with $a \neq b$, then $X^{t+4} = X^t$ iff $c = 1 - a$ and $d = 1 - b$.*

Proof. The analytical form of rule R_{46} is $f(x, y, z) = y - xy + z - yz$. In order for the configuration to be of the form $(a, b, c, d)^m$, we must have that: $f(a, b, c) = c$, $f(b, c, d) = d$ and $f(c, d, a) = a$. Solving the equations, we have that $b - ab + c - bc = c$ for $b = 0$ or $a + c = 1$; $c - bc + d - cd = d$ for $c = 0$ or $b + d - 1$; $d - cd + a - da = a$ for $d = 0$ or $c + a = 1$. Thus, the only spatially 4-periodic shifting configuration (with $a \neq b$) is $X^t = (a, b, 1 - a, 1 - b)^m$. □

When the configuration size is even but not a multiple of four or when it is odd the CA converges (very slowly) to an homogeneous configuration $(1/2)^n$, which is its fixed point from homogeneous initial configurations.

5.4 Periods of Length n: Shifts

Rules R_{170} ($f(x, y, z) = z$) is the perfect shift and the radial representation shows a perfect rotation of the initial configuration. A simple shifting behavior after a short transient is displayed also by rules R_2, R_{10}, R_{34}, R_{42}, R_{130}, R_{138}, R_{162} (Stars). An example of shifting configuration for rule R_{10} is shown in Figure 4, where the configuration rotates as indicated by the arrows. Rule R_{15} ($f(x, y, z) = (1 - z)$) is the complement of the perfect shift: since every configuration is the complement of the shift of the previous, we observe what we call a shifting periodic behavior. Every other configuration is shifted by two positions. The radial representation clearly shows both the periodic and the shifting nature of the rule. Similar behavior have rules $R_3, R_7, R_{11}, R_{14}, R_{43}, R_{142}$.

6 Concluding Remarks

In this paper we have described an experimental classification of circular fuzzy cellular automata based on a new visualization method. The different visualization method has allowed us to observe dynamics that were not visible with the classical space-time diagram (compare, for example, snapshots of configurations during the evolution of some rules observed with the classical space-time diagram, with the Radial and with the Polygon representation in Figure 10).

We have shown through experiments that all circular elementary FCA from random initial configurations have a periodic behavior, and we have grouped them on the basis of the length of their periods. Surprisingly, the only observed periods lengths are 1,2,4, or n. We have analytically verified that the configurations reached by the FCA in our experiments are indeed periodic points.

Several problems are now under investigation. First of all we are studying the reasons for FCA to have these particular period lengths and we are working on analytically showing that the periodic points are indeed attractors for these FCA. The classification of infinite configurations and configurations in zero backgrounds are also under investigation.

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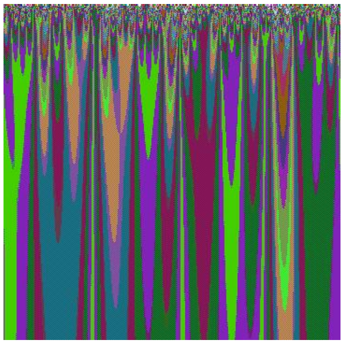
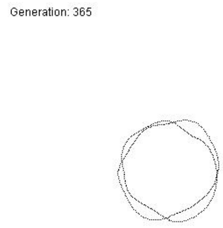
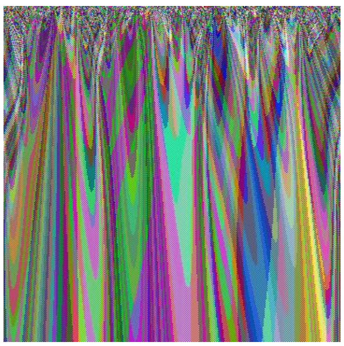


(a) Rule 18 Space-Time

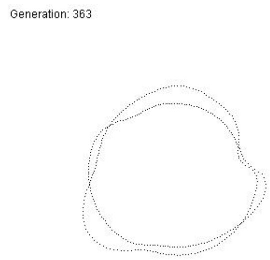


(b) Rule 18 Radial

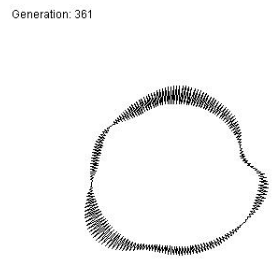
(c) Rule 18 Polygon



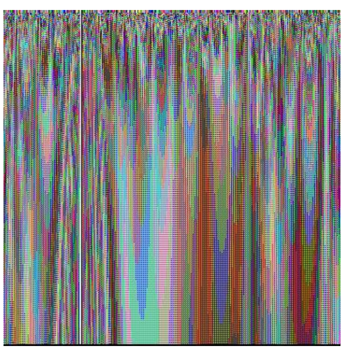
(d) Rule 184 Space-Time



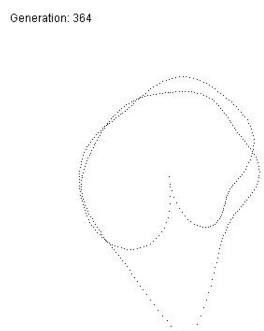
(e) Rule 184 Radial



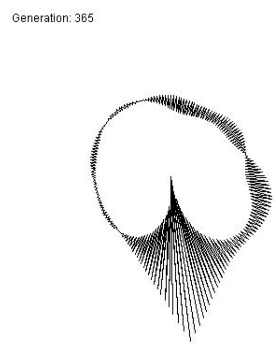
(f) Rule 184 Polygon



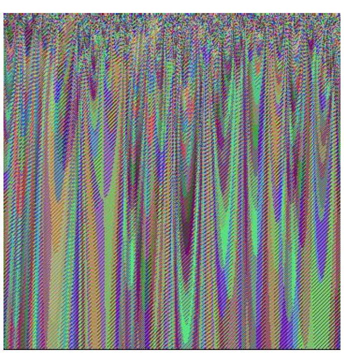
(g) Rule 29 Space-Time



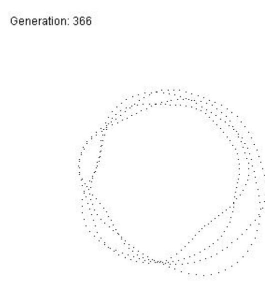
(h) Rule 29 Radial



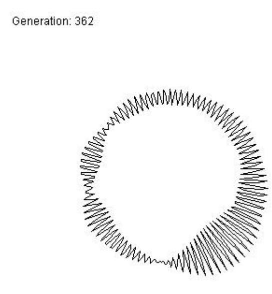
(i) Rule 29 Polygon



(j) Rule 46 Space-Time



(k) Rule 46 Radial



(l) Rule 46 Polygon

Figure 10: Examples of configurations during the evolution of Rules R_{18}, R_{184}, R_{29}, and R_{46} with the classical space-time diagram, with the Radial and with the Polygon representation. In all cases, the configuration has 180 cells.