# Constructing Minimum Cost 2-Edge-Connected Spanning Subgraphs of Metric Cost Functions

Sylvia Boyd Paul Elliott-Magwood

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## 1 Introduction

In the paper by Monma, Munson, and Pulleyblank [4], we saw that for every metric cost function, $c$ and every $n \geq 3$, there is a minimum cost 2-edge-connected spanning subgraph, $G$, of $K_n$ with respect to $c$ which has the following properties:

1. $G$ is 2-vertex-connected
2. $G$ is edge-minmally 2-edge-connected
3. Every vertex of $G$ has degree 2 or degree 3
4. Removing any pair of edges leaves a bridge in one of the resulting components

We will use $\mathcal{M}$ to denote the set of all the graphs which have the above properties.

Furthermore, the authors showed that for any graph, $G \in \mathcal{M}$, there is a metric cost function, $c$, for which $G$ is the unique minimum cost 2-edge-connected subgraph of $K_n$ with respect to $c$. In this paper, I want to show how to construct all the graphs in $\mathcal{M}$.

## 2 Ear Decompositions

Consider constructing a graph, $G = H_k$, as follows. Start with a cycle, $H_0$. Let $R_i$ be a non-trivial (possibly closed) path for each $1 \leq i \leq k$. Then for each $1 \leq i \leq k$ construct $H_i = H_{i-1} \cup R_i$ where $H_{i-1}$ and $R_i$ have exactly the endpoints of $R_i$ in common. Such a construction is called an *ear decomposition* of $G$ and the $R_1, \ldots, R_k$ are called *ears*. For the purposes of this paper, we will call $H_0, \ldots, H_k$, as defined above, the *subgraphs of the ear decomposition of* $G$. This definition of an ear decomposition leads to the following well-known theorem.

**Theorem 1.** *$G$ is 2-edge-connected if and only if $G$ has an ear decomposition.*

Now, every graph is $\mathcal{M}$ is 2-edge connected so they all have ear decompositions. We can use this ear decomposition construction to build all the graphs in $\mathcal{M}$. For now we will explore the properties of these ear decompositons.

**Lemma 2.** *Let $G \in \mathcal{M}$ have an ear decomposition with subgraphs $H_0, \ldots, H_k$ and ears $R_1, \ldots, R_k$. Let $R_i$ where $1 \leq i \leq k$ be a $\{u, v\}$-path. Then*

1. *$u$ and $v$ have degree two in $H_{i-1}$ and degree three in $H_i$,*
2. *$u \neq v$, and*
3. *$R_i$ contains at least two edges.*

*Proof.* Since $H_0 \subset H_1 \subset \cdots \subset H_{k-1} \subset H_k = G$, every vertex of $H_{i-1}$ must have degree two or three. Furthermore, since $u$ and $v$ have degree two or three in $G$, they must have degree two in $H_{i-1}$ and degree three in $H_i$. Now, since $u$ and $v$ have degree two in $H_{i-1}$ and degree three in $H_i$ it must be that $u \neq v$ (otherwise $u = v$ would have degree four in $H_i$). Lastly, suppose for a contradiction that $R_i$ consists of a single edge $uv$. Since $H_{i-1}$ has an ear decomposition, it is 2-edge-connected. Thus there are two edge-disjoint $\{u, v\}$-paths in $H_{i-1}$. Since $H_{i-1} \subset G$, these two edge-disjoint $\{u, v\}$-paths are in $G$ as well. Furthermore, neither of these paths contain the edge $uv$. Hence $G - uv$ is 2-edge connected which contradicts the edge-minimality of $G$. Therefore $R_i$ must contain at least two edges. □

**Lemma 3.** *If $G \in \mathcal{M}$ then any ear of an ear decomposition of $G$ must contain at least three edges.*

*Proof.* Let $G$ have an ear decomposition with subgraphs $H_0, \ldots, H_k$ and ears $R_1, \ldots, R_k$. We know from Lemma 2 that every ear must contain at least two edges. Suppose for a contradiction, that there is some $1 \le j \le k$ such that $R_j$ has exactly two edges, say $uw$ and $wv$.

If $w$ has degree two in $G$ then $G$ also has an ear decomposition with ears $R'_1, \ldots, R'_k$ and subgraphs $H'_0, \ldots, H'_k$ where

$$R'_i = \begin{cases} R_i & \text{for } 1 \le i \le j-1 \\ R_{i+1} & \text{for } j \le i \le k-1 \\ R_j & \text{for } i = k \end{cases}$$

and

$$H'_i = \begin{cases} H_i & \text{for } 1 \le i \le j-1 \\ H_{i+1} - w & \text{for } j \le i \le k-1 \\ G & \text{for } i = k \end{cases}.$$

This new ear decomposition follows from the old by simply adding the ear $R_j$ at the end of the construction. Since $w$ has degree two in $G$ we know that $w$ will not later be used as the endpoint of a later ear in our old ear decomposition. Hence, moving this ear to be the final addition in our construction will still yield $G$.

Now, $G - \{uw, wv\}$ has exactly two components, namely the isolated vertex $w$and $H'_{k-1}$. However, $H'_{k-1}$ itself has an ear decomposition (with ears $R'_1, \ldots, R'_{k-1}$ and subgraphs $H'_0, \ldots, H'_{k-1}$) and hence is 2-edge-connected. Therefore neither of the components of $G - uw, wv$ contains a bridge which contradicts the fact that $G \in \mathcal{M}$. Thus $w$ cannot have degree two in $G$.

If $w$ has degree three in $G$ then there is an ear, $R_l$, where $j+1 \le l \le k$ and $R_l$ has $w$ as one of its endpoints. Furthermore, note that this is the only ear which has $w$ as an endpoint. Again we have an alternative ear decomposition of $G$ with ears $R'_1, \ldots, R'_k$ and subgraphs $H'_0, \ldots, H'_k$ where

$$R'_i = \begin{cases} R_i & \text{for } 1 \le i \le j-1 \\ R_{i+1} & \text{for } j \le i \le l-2 \\ R_l \cup \{uw\} & \text{for } i = l-1 \\ \{wv\} & \text{for } i = l \\ R_i & \text{for } l+1 \le i \le k \end{cases}$$

and

$$H'_i = \begin{cases} H_i & \text{for } 1 \le i \le j-1 \\ H_{i+1} - w & \text{for } j \le i \le l-2 \\ H_l - vw & \text{for } i = l-1 \\ H_i & \text{for } l \le i \le k \end{cases} .$$

However, this new ear decomposition has an ear which is a path of length one. By Lemma 2, $G$ cannot have an ear decomposition with such an ear. Thus we have a contradiction and so $w$ cannot have degree three in $G$.

Unfortunately, $w$ must have degree two or three in $G$ since $G \in \mathcal{M}$ and hence $R_j$ cannot be a path of length two. Therefore, every ear must have length at least three. □

**Theorem 4.** *Let $G$ be a 2-edge-connected graph and let $H_0, \dots, H_k$ be the subgraphs in an ear decomposition of $G$. $G \in \mathcal{M}$ if and only if $H_0, \dots, H_k \in \mathcal{M}$.*

*Proof.* Since $G = H_k$, if $H_k \in \mathcal{M}$ then $G \in \mathcal{M}$. Hence we have proved the reverse direction of the theorem.

Now suppose $G \in \mathcal{M}$. For each $0 \le i \le k$, $H_i \subseteq G$. Hence since every vertex of $G$ has degree two or three, the same is true of $H_i$ for each $0 \le i \le k$. Now, consider a 2-edge-connected graph which has a cut vertex, $v$. In any such graph, $v$ must have degree at least four. Thus since $H_i$ is 2-edge-connected and every vertex of $H_i$ has degree two or three for each $0 \le i \le k$, $H_i$ cannot have any cut-vertex and so $H_i$ is 2-vertex-connected for each $0 \le i \le k$.

Suppose, for a contradiction, that for some $0 \le j \le k$ that $H_j$ is not edge-minimally 2-edge-connected. Hence $H_j$ contains an edge $uv$ and two edge-disjoint $\{u, v\}$-paths, neither of which contain the edge $uv$. Since $H_i \subseteq G$, $G$ also has these paths along with the edge $uv$. Thus $G - uv$ is 2-edge-connected which contradicts the fact that $G$ is edge-minimally 2-edge connected. Therefore, for each $0 \le i \le k$, $H_i$ is edge-miniamlly 2-edge connected.

Suppose, for a contradiction, that $H_j \notin \mathcal{M}$ for some $0 \le j \le k$ and furthermore, let $j$ be the maximum such index. Clearly $j \ne 0$ since any cycle is in $\mathcal{M}$ and $j \ne k$ since $G \in \mathcal{M}$. From the arguments above, we see that if $H_j \notin \mathcal{M}$ then there must be two edges, $uv$ and $wx$, of $H_j$ such that no

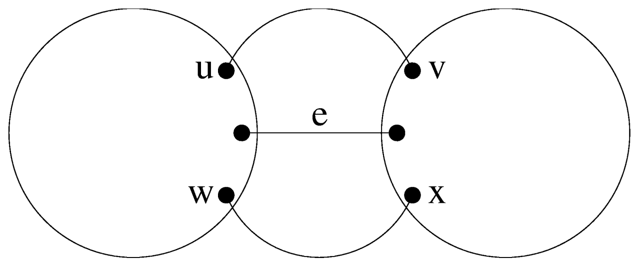


Figure 1: The first configuration

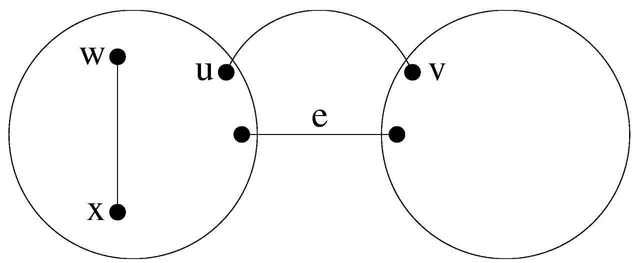


Figure 2: The second configuration

component of $H_j - \{uv, wx\}$ contains a bridge, that is the components of $H_j - \{uv, wx\}$ are either isolated vertices or are 2-edge-connected. Furthermore, since $H_j$ is 2-edge-connected, there are at most two components.

Now, by the maximality of $j$, $H_{j+1} \in \mathcal{M}$. Thus, one of the components of $H_{j+1} - \{uv, wx\}$ contains a bridge, call it $e$. Since $H_{j+1}$ is 2-edge-connected, the only possible configurations, up to relabelling $u$, $v$, $w$, or $x$, are shown in Figure 1, Figure 2, and Figure 3. Here the large circles are the node sets of the components of $H_{j+1} - \{uv, wx, e\}$.

Notice that in the second and third configurations, since $e$ is an edge of $R_{j+1}$, if we remove $R_{j+1}$ from $H_{j+1}$ to get $H_j$ then we see that $uv$ is a bridge of $H_j$. This contradicts the fact that $H_j$ is 2-edge-connected. Thus, we must have the first configuration as shown in Figure 1. Furthermore, since $uv$ and $wx$ are edges of $H_j$ and $e$ is an edge of $R_{j+1}$ then $H_{j+1}$ must have, up to relabelling the vertices, the structure shown in Figure 4 and where $V_1$ and $V_2$ as shown are the node sets of the two components of $H_j - \{uv, wx\}$ and $R_{j+1}$ is a $\{y, z\}$-path.

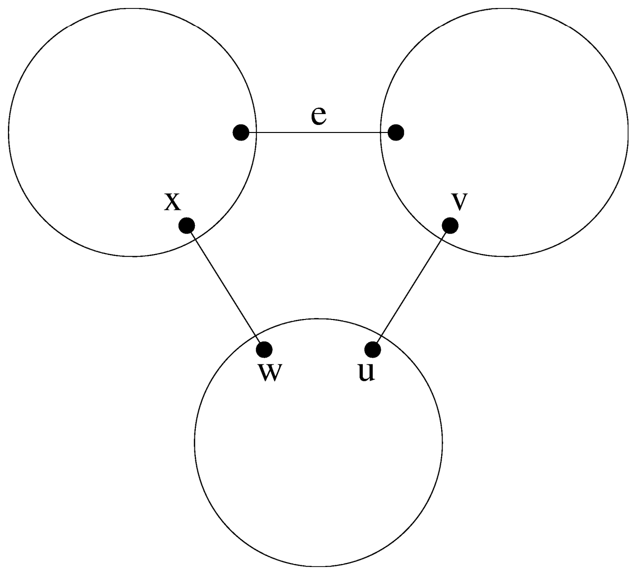


Figure 3: The third configuration

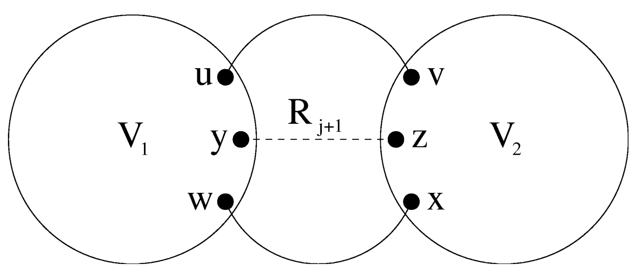


Figure 4: The structure of $H_{j+1}$

Now since the components of $H_j - \{uv, wx\}$ are bridgeless, $H_j[V_1]$ is either a single vertex or is 2-edge-connected. If $H_j[V_1]$ is not a single vertex, consider adding a new vertex, $t$, and the edges $yt$ and $wt$ to $H_j[V_1]$. Since this new graph is obtained by adding an ear to a 2-edge-connected graph, it must be 2-edge-connected. Thus there are two edge-disjoint $\{u, t\}$-paths. By removing $t$ from our new graph, we see that there is a $\{u, y\}$-path and a $\{u, w\}$-path in $H_j[V_1]$ which are edge-disjoint. By symmetry, we get the similar result that if $H_j[V_2]$ is not a single vertex then it contains a $\{v, z\}$-path and a $\{v, x\}$-path which are edge-disjoint. Hence the union of the $\{u, w\}$-path, the edge $wx$ and the $\{x, v\}$-path gives a $\{u, v\}$-path in $H_{j+1}$. Also, the union of the $\{u, y\}$-path, the path $R_{j+1}$, and the $\{z, v\}$-path gives a $\{u, v\}$-path in $H_{j+1}$. If $|V_1| = 1$ or $|V_2| = 1$ then we can replace the appropriate path above with an empty path. Furthermore, these two $\{u, v\}$-paths in $H_{j+1}$ are edge-disjoint and neither contains the edge $uv$. Hence $H_{j+1} - uv$ is 2-edge-connected which contradicts the fact that $H_{j+1}$ is edge-minimally 2-edge-connected.

Hence no such $j$ can exist and therefore $H_i \in \mathcal{M}$ for every $0 \leq i \leq k$. □

## 3 Necklaces and Beads

Let $G \in \mathcal{M}$ then we know that $G$ is edge-minimally 2-edge-connected. Hence if $e$ is an edge of $G$ then $G - e$ is connected but must have at least one bridge. Let $f$ be any bridge of $G - e$. Then $G - \{e, f\}$ is not connected and has exactly two components. Since $G \in \mathcal{M}$, one of these components must have a bridge, call it $g$. Thus we can partition the vertices of $G$ into three sets, $V_1$, $V_2$, and $V_3$ where $G[V_1]$, $G[V_2]$, and $G[V_3]$ are connected and $G$ has the structure as shown in Figure 5.

If $G[V_1]$ contains a bridge, say $h$, then since $G$ is 2-edge-connected we must have that $h$ is also a bridge of $G - e$. Conversely, if $G - e$ has a bridge, $h$, whose endpoints are both in $V_1$ then $h$ is also a bridge of $G[V_1]$. If such an edge $h$ exists then the structure of $G$ is as shown in Figure 6.

By continuing this process until we have found all the bridges of $G - e$, we create a partition, $V_1, \ldots, V_k$, of the vertices of $G$ such that

- $|\delta(V_i)| = 2$ for each $1 \leq i \leq k$,

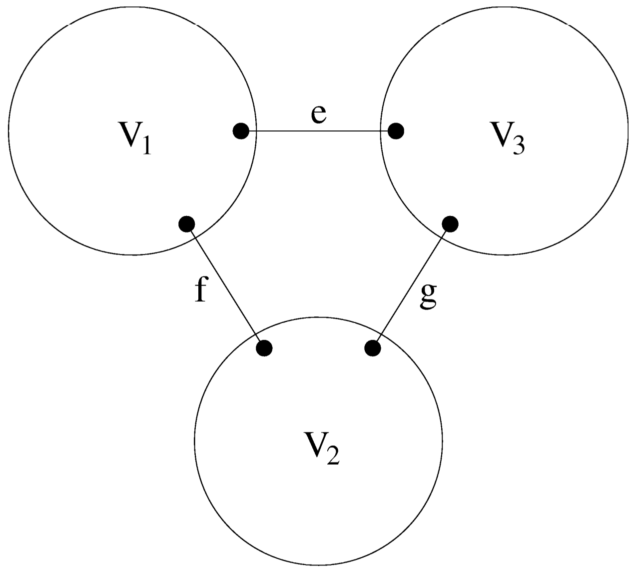


Figure 5: The structure of $G$

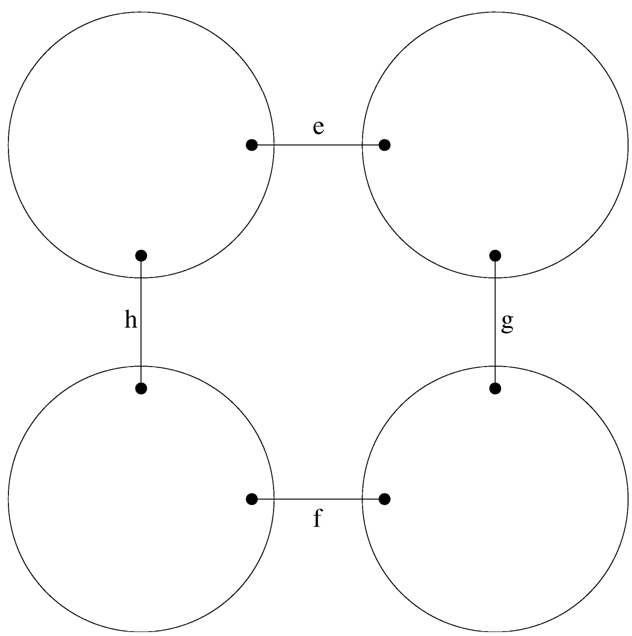


Figure 6: Another bridge

- $e$ is the unique edge with an endpoint in $V_1$ and the other in $V_k$,
- there is a unique edge with an endpoint in $V_i$ and the other in $V_{i+1}$ for each $1 \leq i \leq k-1$ and these are exactly the bridges of $G - e$,
- any remaining edge of $G$ has both its endpoints in $V_i$ for some $1 \leq i \leq k$, and
- for each $1 \leq i \leq k$, either $G[V_i]$ is an isolated vertex or $G[V_i]$ is 2-edge-connected.

Such a partition of the vertices of $G$ is called a *necklace* and the parts $V_1, \ldots, V_k$ are called *beads*. Our partition is considered to be circular (that is $V_1$ follows $V_k$). For the purposes of this paper, we will call any edge which has its endpoints in different (adjacent) beads an *inter-bead edge*. Notice that if we remove any inter-bead edge from this necklace and find all the resulting bridges, we get the same necklace. Thus an inter-bead edge is in a unique necklace. Furthermore, the minimum cuts of $G$ which contain an inter-edge bead of a necklace are exactly those which are induced by the union of consecutive beads of the necklace. For instance, in the previous example, any minimum cut containing $e$ is of the form $\delta(V_1 \cup V_2 \cup \cdots \cup V_j)$ where $j$ is some integer in $\{1, \ldots, k-1\}$. All of the above properties are well-known attributes of the necklaces of a graph with a minimum cut of size 2. Lemma 5 discusses the properties of the necklaces that are specific to the graphs in $\mathcal{M}$.

**Lemma 5.** *If $G \in \mathcal{M}$ then every necklace of $G$ contains at least three beads and every edge of $G$ is an inter-bead edge in a unique necklace.*

*Proof.* Since $G$ is edge-minimally 2-edge-connected, each edge of $G$ is in some minimum cut. Hence each edge of $G$ is an inter-bead edge in some necklace of $G$. As noted above, each inter-bead edge is in a unique necklace. Hence, each edge of $G$ is an inter-bead edge in a unique necklace.

Secondly, as noted in the previous example, $f$ and $g$ are distinct bridges of $G - e$. Since our choice of $e$ was arbitrary and $e$, $f$, and $g$ are inter-bead edges of the resulting necklace, there must be at least three beads in the necklace. □

We can say even more about the necklaces of the graphs in $\mathcal{M}$.

**Theorem 6.** *If $G \in \mathcal{M}$ and $(V_1, \dots, V_k)$ is a necklace of $G$ then for each $1 \leq i \leq k$ either $G[V_i]$ is an isolated vertex or $G[V_i] \in \mathcal{M}$.*

*Proof.* Since we can arbitrarily choose which bead is labelled $V_1$, relabelling if necessary, it is enough to prove that the theorem holds true for $G[V_1]$.

If $G[V_1]$ is an isolated vertex then the result follows. Otherwise, as noted above, $G[V_1]$ is 2-edge-connected. Since $G[V_1] \subset G$ and every vertex has degree two or three, every vertex of $G[V_1]$ must have degree at most three. Since $G[V_1]$ is 2-edge-connected, every vertex must have degree at least two. Now any cut vertex of a 2-edge-connected graph must have degree at least four, so $G[V_1]$ cannot contain any cut vertices. Hence $G[V_1]$ is 2-vertex-connected.

Suppose that $G[V_1]$ is not edge-minimally 2-edge-connected. Then there is an edge $uv$ of $G[V_1]$ such that $G[V_1] - uv$ is 2-edge-connected. Hence there are two edge-disjoint $\{u, v\}$-paths in $G[V_1]$, neither of which contain the edge $uv$. However, these two paths exist in $G$ so $G - uv$ is also 2-edge-connected, contradicting the fact that $G$ is edge-minimally 2-edge-connected. Therefore, $G[V_1]$ is edge-minimally 2-edge-connected.

Now consider any two edges, $e$ and $f$ of $G[V_1]$.

Case 1: $G[V_1] - \{e, f\}$ is connected.

Since $G[V_1]$ is edge-minimally 2-edge-connected, there exists and edge, $g$, of $G[V_1]$ such that $G[V_1] - \{e, g\}$ is not connected. Hence $G[V_1] - \{e, f, g\}$ is not connected and hence $g$ is a bridge of $G[V_1] - \{e, f\}$.

Case 2: $G[V_1] - \{e, f\}$ is not connected.

Assume that both of the components of $G[V_1] - \{e, f\}$ are bridgeless. Hence each component is either an isolated vertex or is 2-edge-connected. There are only two possible configurations for how these components can interact with the rest of $G$ as depicted in Figure 7 and Figure 8.

Since the components of $G[V_1] - \{e, f\}$ in Figure 7 are bridgeless (and hence each component is either an isolated vertex or is 2-edge-connected), notice then that the two components of $G - \{e, f\}$ are also bridgeless. This contradicts the fact that $G \in \mathcal{M}$.

As for Figure 8, since the components of $G[V_1] - \{e, f\}$ are bridgeless (and

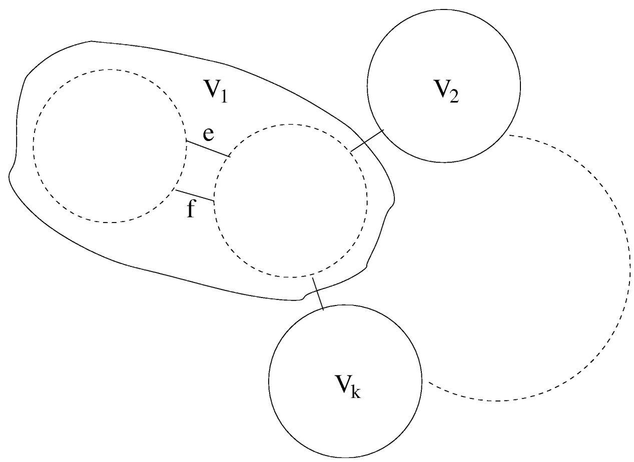


Figure 7: $G - \{e, f\}$ is disconnected

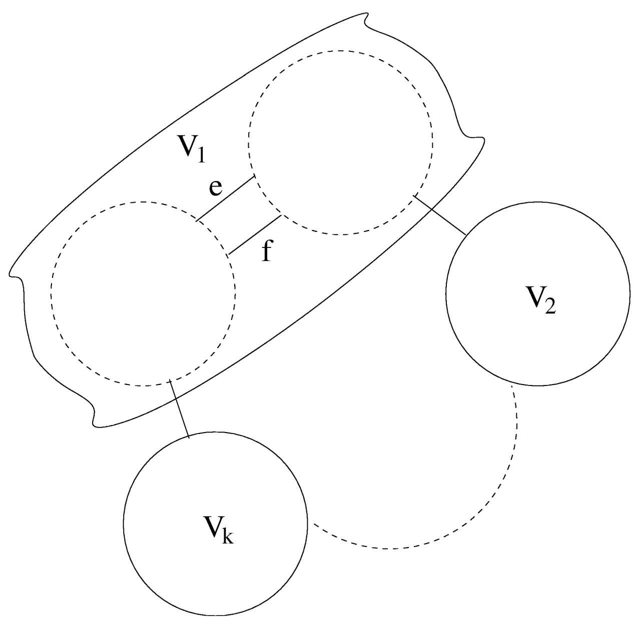


Figure 8: $G - \{e, f\}$ is connected

hence each component is either an isolated vertex or is 2-edge-connected), we have that both $G - e$ or $G - f$ is 2-edge-connected. This contradicts the edge-minimality of $G$.

Therefore, in both cases, one of the components of $G[V_1] - \{e, f\}$ contains a bridge. Thus $G[V_1] \in \mathcal{M}$.

□

We can say more about the necklaces of a graph from $\mathcal{M}$ and how they interact with the ear decompositions. This gives us a way of building $\mathcal{M}$ recursively.

Before we begin the next theorem, I want to introduce a new definition regarding necklaces. If we take any necklace with $k \geq 3$ beads and we identify all the vertices contained in each bead then we get a new graph, $C$, which is a cycle with $k$ vertices. Each of the vertices in $C$ corresponds to a unique bead. For the purposes of this paper, the *distance between two beads* is the distance between their corresponding vertices in $C$. Alternatively, it is the minimum number of inter-bead edges on a path whose endpoints are in the respective beads.

**Theorem 7.** *Let $H \in \mathcal{M}$ and let $u$ and $v$ be two distinct vertices of $H$, each of degree 2. Let $R$ be a new path of length at least 3 and let $G$ be the graph obtained by identifying the distinct endpoints of $R$ with vertices $u$ and $v$ in $H$. Then $G \in \mathcal{M}$ if and only if, for any necklace of $H$, either $u$ and $v$ are in the same bead or the distance between the bead containing $u$ and the bead containing $v$ is at least 3.*

*Proof.* Suppose that, for some necklace $(V_1, \ldots, V_k)$ of $H$ that $u$ and $v$ are contained in beads which are a distance 1 apart. Without loss of generality, we may assume that $u \in V_1$ and $v \in V_2$. Let $e$ be the inter-bead edge between $V_1$ and $V_2$. Consider the structure of $G - e$ as depicted in Figure 9. Since each of $H[V_1], \ldots, H[V_k]$ are either an isolated vertex or are 2-edge-connected and $R$ is a path from a vertex in $V_1$ to a vertex in $V_2$, we have that $G - e$ is 2-edge-connected. Thus $G$ is not edge-minimally 2-edge-connected and hence $G \notin \mathcal{M}$. Therefore, if $G \in \mathcal{M}$ then there cannot be any necklace of $H$ where $u$ and $v$ are contained in beads which are a distance 1 apart.

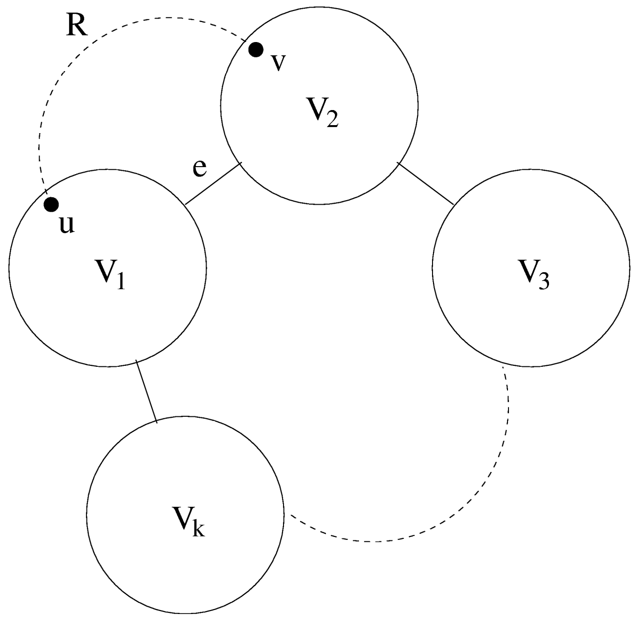


Figure 9: Beads which are a distance 1 apart

Suppose that, for some necklace $(V_1, \ldots, V_k)$ of $H$ that $u$ and $v$ are contained in beads which are a distance 2 apart. Without loss of generality, we may assume that $u \in V_1$ and $v \in V_3$. Let $e$ and $f$ be the inter-bead edges between $V_1$ and $V_2$ and between $V_2$ and $V_3$ respectively. Consider the structure of $G - \{e, f\}$ as depicted in Figure 10. Since each of $H[V_1], \ldots, H[V_k]$ are either an isolated vertex or are 2-edge-connected and $R$ is a path from a vertex in $V_1$ to a vertex in $V_3$, we have that $G - V_2$ is 2-edge-connected and $G[V_2]$ is either an isolated vertex or is 2-edge-connected. Hence $G - \{e, f\}$ has two components, neither of which contains a bridge. Thus $G \notin \mathcal{M}$. Therefore, if $G \in \mathcal{M}$ then there cannot be any necklace of $H$ where $u$ and $v$ are contained in beads which are a distance 2 apart.

We can conclude that if $G \in \mathcal{M}$ then for every necklace of $H$ either $u$ and $v$ are in the same bead or the distance between the beads containing $u$ and $v$ is at least 3.

Now, let $H \in \mathcal{M}$ and $R$ be as described above. Let $u$ and $v$ be two vertices of degree two of $H$ such that for every necklace either $u$ and $v$ are in the same bead or $u$ and $v$ are in beads which are a distance at least 3 apart.

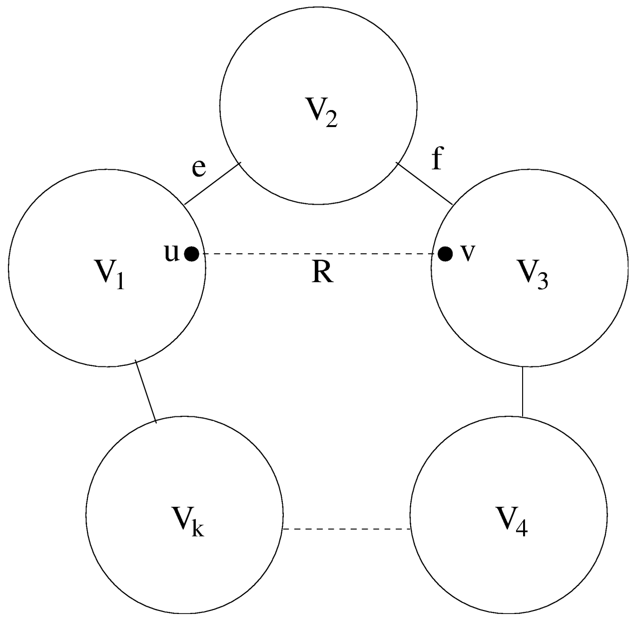


Figure 10: Beads which are a distance 2 apart

Let $G$ be the graph resulting from adding $R$ as an ear to $H$ at $u$ and $v$.

Clearly, every vertex of $G$ has degree 2 or 3. Futhermore, since $G$ has an ear decomposition, it is 2-edge-connected. Since $G$ has maximum degree 3 and is 2-edge-connected, it is also 2-vertex-connected.

Now, let $e$ be any edge of $G$. If $e$ is an edge of $R$ then the remaining edges of $R$ (there are at least two of them since $R$ has at least three edges) are bridges in $G - e$. If $e$ is an edge of $H$, consider the unique necklace of $H$ which has $e$ as an inter-bead edge. If $u$ and $v$ are in the same bead of this necklace, then all the inter-bead edges in this necklace, other than $e$, are bridges of $G - e$. Since the necklace has at least three beads, there must be at least two such bridges. If $u$ and $v$ are in beads which are a distance at least three apart then consider any $\{u, v\}$-path of $H$ which contains $e$. The inter-bead edges of any such path are unique, and there must be at least three of them. Furthermore, any one of these edges, apart from $e$, is a bridge in $G - e$. Hence, in all cases, $G - e$ has a bridge and thus $G$ is edge-minimally 2-edge-connected.

Let $e$ and $f$ be any two edge of $G$. If either or both of these edges is in $R$ then the remaining edges (there is at least 1) or $R$ are bridges in $G - \{e, f\}$. Hence $G - \{e, f\}$ contains a bridge. If $e$ and $f$ are edges of $H$ then consider the unique necklace of $G$ which has $e$ as an inter-bead edge. If $f$ is not an inter-bead edge of this necklace then, by the same reasoning which proves that $G$ is edge-minimally 2-edge-connected, $G - \{e, f\}$ contains a bridge. If $e$ and $f$ are inter-bead edges of the same necklace of $H$, and $u$ and $v$ are in the same bead, then all the remaining inter-bead edges apart from $e$ and $f$ (there must be at least 1) are bridges of $G - \{e, f\}$. If $e$ and $f$ are inter-bead edges of the same necklace of $H$, and $u$ and $v$ are in beads which are a distance at least 3 apart, then consider any $\{u, v\}$-path of $H$ which contains $e$. The inter-bead edges of any such path are unique, and there must be at least three of them. Furthermore, each of these edges, apart from $e$ and $f$, is a bridge of $G - \{e, f\}$. Hence in all cases, $G - \{e, f\}$ contains a bridge.

Therefore $G \in \mathcal{M}$.

□

Notice that Theorem 7 tells us exactly how to recursively construct $\mathcal{M}$. We simply start with all the cycles and successively add ears in the manner proscribed in Theorem 7 to build larger graphs of $\mathcal{M}$. Since we rely on the necklaces to decide whether or not adding a certain ear will yield a graph in $\mathcal{M}$ it would be nice to have a way to find the necklaces of the new graphs which are created.

**Proposition 8.** *Let $H \in \mathcal{M}$ and let $R$ be a path of length at least 3. Let $u$ and $v$ be vertices of $H$ of degree 2 such that in every necklace of $H$ either $u$ and $v$ are in the same bead or $u$ and $v$ are in beads which are a distance at least 3 apart. Let $G$ be the graph obtained by adding $R$ to $H$ by identifying the endpoints of $R$ with $u$ and $v$ respectively. Then the necklaces of $G$ can be obtained from the necklaces of $H$ as follows. Let $(V_1, \ldots, V_k)$ be a necklace of $H$ and let $W$ be the set of internal vertices of $R$. If $u$ and $v$ are in the same bead of the necklace, say $V_i$, then $(V_1, \ldots, V_{i-1}, V_i \cup W, V_{i+1}, \ldots, V_k)$ is a necklace of $G$. If $u$ and $v$ are in distinct beads, say $V_i$ and $V_j$ where $1 \le i < j \le k$ then both $(V_1, \ldots, V_{i-1}, V_i \cup V_{i+1} \cup \cdots \cup V_{j-1} \cup V_j \cup W, V_{j+1}, \ldots, V_k)$ and $(V_{i+1}, \ldots, V_{j-1}, V_j \cup V_{j+1} \cup \cdots \cup V_k \cup V_1 \cup \cdots \cup V_{i-1} \cup V_i \cup W, V_{j+1}, \ldots, V_k)$ are necklaces of $G$. As well, if $r_1, \ldots, r_s$ are the individual internal vertices of $R$, ordered as we follow $R$ from $u$ to $v$, and $V$ is the set of vertices of $H$*

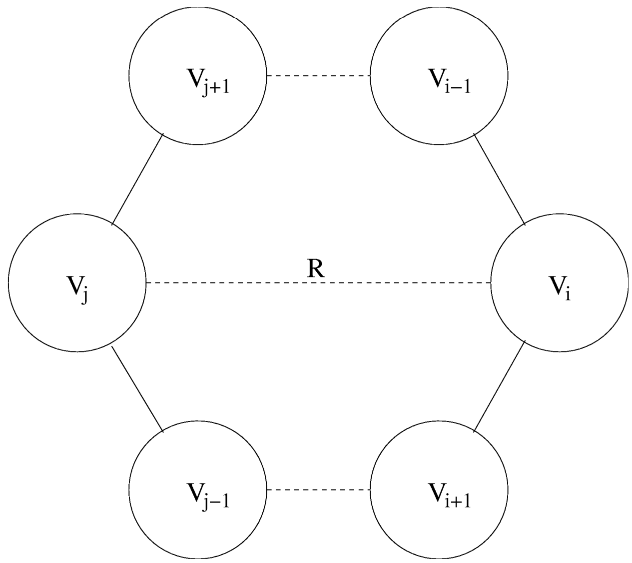


Figure 11: $u$ and $v$ are in distinct beads

*then* $(r_1, \ldots, r_s, V)$ *is also a necklace of* $G$*. Furthermore, these are the only necklaces of* $G$*.*

*Proof.* Suppose $u$ and $v$ are in the same bead, $V_i$, of the necklace. Let $e$ be any inter-bead edge of the necklace. Since $H[V_i]$ is 2-edge-connected, so is $G[V_i \cup W]$. Hence $G-e$ has the same bridges as $H-e$ and so $(V_1, \ldots, V_{i-1}, V_i \cup W, V_{i+1}, \ldots, V_k)$ is a necklace of $G$.

Now suppose $u$ and $v$ are in distinct beads, say $V_i$ and $V_j$ where $1 \leq i < j \leq k$. Then the structure of $G$, as it relates to $R$ and the necklace, is as shown in Figure 11.

Now let $e$ be any edge which is an inter-bead edge between two consecutive beads among $V_i, V_{i+1}, \ldots, V_{j-1}, V_j$. We can see that $G[V_j \cup V_{j+1} \cup \cdots \cup V_k \cup V_1 \cup \cdots V_{i-1} \cup V_i \cup W]$ is 2-edge-connected. Hence the bridges of $G - e$ are exactly the edges (apart from $e$ itself) between two consecutive beads among $V_i, V_{i+1}, \ldots, V_{j-1}, V_j$. Hence $(V_{i+1}, V_{i+2}, \ldots, V_{j-2}, V_{j-1}, V_j \cup V_{j+1} \cup \cdots \cup V_k \cup V_1 \cup \cdots V_{i-1} \cup V_i \cup W)$ is the necklace of $G$ which has $e$ as an inter-bead edge.

If $e$ is an edge which is an inter-bead edge between two consecutive beads among $V_j, V_{j+1}, \ldots, V_k, V_1, \ldots, V_{i-1}, V_i$ then notice that $G[V_i \cup V_{i+1} \cup \cdots \cup$

$V_{j-1} \cup V_j \cup W]$ is 2-edge-connected so the bridges of $G - e$ are exactly the inter-bead edges (apart from $e$ itself) between two consecutive beads among $V_j, V_{j+1}, \ldots, V_k, V_1, \ldots, V_{i-1}, V_i$. Thus $(V_1, \ldots, V_{i-2}, V_{i-1}, V_i \cup V_{i+1} \cup \cdots \cup V_{j-1} \cup V_j \cup W, V_{j+1}, V_{j+2}, \ldots, V_k)$ is the necklace of $G$ which has $e$ as an inter-bead edge.

Now let $e$ be an edge of $R$. Since $H$ is 2-edge-connected, $V$ is contained in some bead of the necklace of $G$ containing $e$ as an inter-bead edge. Furthermore, every edge of $R$ (apart from $e$) is a bridge of $G - e$. Thus the necklace of $G$ which has $e$ as an inter-bead edge is exactly $(r_1, \ldots, r_s, V)$.

We have considered all the edges of $G$ and we know that every edge of $G$ is in a unique necklace of $G$. Thus we have exhaustively considered all the necklaces of $G$. □

# 4 Min-cut Cacti

A *cactus* is a connected graph such that every edge is in a unique cycle. We allow parallel edges and we consider a pair of parallel edges to be a (degenerate) cycle. Cacti have certain very nice properties relevent to 2-edge-connected graphs. Let $G$ be a cactus and let $e$ be any edge of $G$. Firstly, since $e$ is in a cycle, $G$ is 2-edge-connected. Secondly, since $e$ is in a unique cycle, $G$ is edge-minimally 2-edge-connected and the bridges of $G - e$ are exactly the remaining edges in the cycle containing $e$. Hence the minimum cuts of a cactus are very easy to find and consist of any two edges of the same cycle.

While cacti on their own are interesting, Dinits, Karzanov, and Lomonosov [1] showed that a cactus can efficiently store information about the minimum cuts in a graph. The minimum cuts of the cactus correspond exactly to the minimum cuts of the original graph. Although, there is not always a unique min-cut cactus for every graph, there are a series of simple operations which can be applied to the a cactus to get the unique *canonical cactus*. Finding the canonical cactus is helpful since the cycles in a canonical min-cut cactus correspond exactly to the necklaces in the original graph.

A min-cut cactus of $G$ has nodes which are labelled by (possibly empty) disjoint subsets of the vertices of $G$. Furthermore, every vertex of $G$ must appear in exactly one node of the cactus. Let $e$ and $f$ be two edges of a cycle of the cactus. Then, by removing $e$ and $f$ from the cactus, we get exactly two components. Let $U$ and $\overline{U}$ be the union of all the subsets of vertices contained in all the nodes of the first and second components respectively. Then $\delta(U) = \delta(\overline{U})$ is a minimum cut of $G$. If we have the canonical min-cut cactus of $G$ (which we denote $\mathcal{H}(G)$) and we remove all the edges in a cycle of $\mathcal{H}(G)$ then, for each of the resulting components, the union of all the vertices contained in the nodes of the component is a bead in the corresponding necklace.

For any $G \in \mathcal{M}$ we can use $\mathcal{H}(G)$ to compactly store all the information about the necklaces of $G$. Furthermore, the canonical min-cut cacti of the graphs of $\mathcal{M}$ have certain useful properties described below.

**Proposition 9.** *If $G \in \mathcal{M}$ then $\mathcal{H}(G)$ has no pair of parallel edges.*

*Proof.* A pair of parallel edges in $\mathcal{H}(G)$ corresponds to a necklace of $G$ with exactly two beads. However, since $G \in \mathcal{M}$, every necklace of $G$ has at least three beads. Therefore $\mathcal{H}(G)$ has no pair of parallel edges. □

**Proposition 10.** *If $G \in \mathcal{M}$ then every node of $\mathcal{H}(G)$ contains a nonempty subset of vertices of $G$.*

*Proof.* Every edge of $\mathcal{H}(G)$ is contained in a unique cycle and thus every node is the intersection of edge-disjoint cycles. Hence every node has even degree.

Suppose, for a contradiction, that $a$ is a node of $\mathcal{H}(G)$ which contains no vertices of $G$.

If $a$ has degree 2, with neighbours $b$ and $c$, then we can remove the node $a$ and add the edge $bc$ without changing the information that the cactus is telling us about the minimum cuts of $G$. However, this is one of the operations used to construct the canonical min-cut cactus of $G$, contradicting the fact the $\mathcal{H}(G)$ is the canonical min-cut cactus of $G$.

If $a$ has degree at least 4, then let $U_1, \ldots, U_l$ be the vertex sets contained in the nodes of each of the components of $\mathcal{H}(G) - a$ respectively. Then $\delta(U_i)$ is a minimum cut of $G$ for each $1 \leq i \leq l$. Furthermore, since $U_1, \ldots, U_l$ is a partition of the vertices of $G$, we can relabel these subsets (if necessary) so that $(U_1, \dots, U_l)$ is a min-cut circular partition of $G$. Hence, there is a necklace of $G$ such that each of the subsets $U_1, \dots, U_l$ is the union of consecutive beads of this necklace. Thus there is a cycle of $\mathcal{H}(G)$ which corresponds to this necklace. On this cycle, there is an edge between a node containing vertices from $U_1$ and a node containing vertices from $U_2$. But then these nodes would not be in different components of $\mathcal{H}(G) - a$ which is a contradiction. □

**Proposition 11.** *If $G \in \mathcal{H}(G)$ then there is a bijection between the edges of $G$ and the edges of $\mathcal{H}(G)$. Furthermore, if $ab$ is an edge of $\mathcal{H}(G)$ then there exist vertices $u$ and $v$, contained in $a$ and $b$ respectively, such that $ab$ corresponds to $uv$ under the bijection.*

*Proof.* As noted before, there is a 1 to 1 correspondance between the cycles of $\mathcal{H}(G)$ and the necklaces of $G$. Further notice that every edge of $G$ is an inter-bead edge in a unique necklace and every edge of $\mathcal{H}(G)$ is in a unique cycle. If we remove a cycle of $\mathcal{H}(G)$, the components of the resulting graph tell us exactly the beads of the corresponding necklace, as well as their circular ordering. Hence there is an obvious bijection between the inter-bead edges in a necklace of $G$ and the edges of a cycle of $\mathcal{H}(G)$. Since every edge of $G$ is an inter-bead edge in a unique necklace and every edge of $\mathcal{H}(G)$ is in a unique cycle, we can extend our bijection to all edges of $G$ and all edges of $\mathcal{H}(G)$.

Now suppose that $ab$ is an edge of $\mathcal{H}(G)$ in cycle $C$ and the edge of $G$ corresponding to $ab$ is $uv$ where $u$ is contained in some node of the component of $\mathcal{H}(G) - C$ containing $a$ and $v$ is contained in some node of the component of $\mathcal{H}(G) - C$ containing $b$. Let $uv$ be an inter-bead edge in the necklace $(W_1, \dots, W_k)$. Then $uv$ is contained in exactly $k-1$ (the number of inter-bead edges apart from $uv$) minimum cuts of $G$. These minimum cuts correspond exactly to the minimum cuts of $\mathcal{H}(G)$ obtained by removing $ab$ along with another edge of $C$. There are exactly $k - 1$ such cuts too. Let $u \in W_1$ and suppose $u$ is not contained in $a$. Let $a'$ be the node of $\mathcal{H}(G)$ which contains $u$. Now the component of $\mathcal{H}(G) - C$ which contains both $a$ and $a'$, call it $A$, is itself a cactus. Furthermore, since no pair of nodes in a cactus can have three edge-disjoint paths between them, every pair of nodes in a cactus are separated by some minimum cut. Thus there is a cut in $A$ which separates $a$ and $a'$. Let $b'$ be the node of $\mathcal{H}(G)$ which contains $v$. Since $a$ is the only node of $\mathcal{H}(G)$ which is adjacent to nodes outside of $A$, this minimum cut separates $a'$ and $b'$, but not $a$ and $b$. Hence we have a corresponding cut of $G$ which contains the edge $uv$ but does not correspond to any cut obtained by removing a bridge of $G - uv$ which is a contradiction. Therefore, $u$ must be contained in $a$ and (by symmetry) $v$ must be contained in $b$. □

**Proposition 12.** *Let $G \in \mathcal{M}$ and $a$ a node of $\mathcal{H}(G)$. If $a$ contains two distinct vertices, $u$ and $v$, then $u$ and $v$ each have degree 3 in $G$. Furthermore, there exist three internally-vertex-disjoint $\{u, v\}$-paths in $G$ and the length of every $\{u, v\}$-path in $G$ is at least 3.*

*Proof.* If $u$ and $v$ are in the same node of $\mathcal{H}(G)$ then there is no minimum cut of $G$ that separates $u$ from $v$. Hence there are 3 internally-edge-disjoint $\{u, v\}$-paths in $G$. Thus $u$ and $v$ must have degree 3. Furthermore, since every vertex of $G$ has degree at most 3, the paths must be internally-vetex-disjoint. From Proposition 11 we know that any $\{u, v\}$-path in $G$ corresponds to a (closed) path in $\mathcal{H}(G)$ containing the node $a$. However, the cycles of $\mathcal{H}(G)$ have length at least 3 and therefore any $\{u, v\}$-path in $G$ must also have length at least 3. □

**Corollary 13.** *If $G \in \mathcal{H}(G)$ and $a$ is a node of $\mathcal{H}(G)$ containing more than one vertex of $G$ then $a$ must contain an even number of vertices of $G$ and the number of cycles of $\mathcal{H}(G)$ which contain $a$ is divisible by 3.*

*Proof.* Since, by Proposition 12 the vertices contained in $G$ form a stable set and all have degree 3 in $G$, the degree of $a$ must be divisible by 3. However, $a$ is the intersection of otherwise node-disjoint cycles. Hence $a$ must have even degree. Thus the degree of $a$ is divisible by 6 so the number of cycles of $\mathcal{H}(G)$ containing $a$ is divisible by 3. Furthermore, since the total degree of all the vertices contained in $a$ is even and all these vertices have degree 3 in $G$, there must be an even number of them. □

**Proposition 14.** *Let $G \in \mathcal{M}$, let $v$ be a vertex of $G$ and let $a$ be the node of $\mathcal{H}(G)$ containing $v$. The following are equivalent:*

- *$v$ has degree 2 in $G$,*
- *$v$ is the only vertex contained in $a$, and*
- *$a$ has degree 2 in $\mathcal{H}(G)$.*

*Proof.* Suppose that $v$ is a vertex of $G$ of degree 2. Then $\delta(\{v\})$ is a minimum cut of $G$ and this minimum cut separates $\{v\}$ from all other vertices of $G$. Hence, there cannot be any other vertex of $G$ contained in $a$.

Suppose that $v$ is the only vertex contained in $a$. From Proposition 11 we see that, since $a$ only contains $v$, $a$ must have the same degree as $v$. Thus the degree of $a$ is either 2 or 3. However, every node in a cactus is the intersection of otherwise node-disjoint cycles and hence every node has even degree. Therefore, $a$ has degree 2 in $\mathcal{H}(G)$.

Suppose that $a$ has degree 2 in $\mathcal{H}(G)$. By Proposition 12 we know that the vertices contained in $a$ must form a stable set in $G$ and hence the sum of the degrees of the vertices contained in $a$ must be 2. However, each vertex of $G$ has degree at least 2. Thus $a$ contains a single vertex, namely $v$, and $v$ must have degree 2. □

Now we have a good sense what the min-cut cacti of graphs in $\mathcal{M}$ look like. We can use Theorem 7 to tell us how to properly construct new graphs in $\mathcal{M}$ by using the min-cut cactus.

**Theorem 15.** *Let $H \in \mathcal{M}$ and let $u$ and $v$ be two vertices of $H$ of degree 2. Let $R$ be a path of length at least 3 and let $G$ be the graph obtained by identifying the endpoints of $R$ with $u$ and $v$ respectively. Let $a$ and $b$ be the nodes in $\mathcal{H}(H)$ containing $u$ and $v$ respectively and let $P$ be a shortest $\{a, b\}$-path in $\mathcal{H}(H)$. Then $G \in \mathcal{M}$ if and only if $P$ does not intersect any cycle of $\mathcal{H}(H)$ in exactly one or exactly two edges.*

*Proof.* Suppose that $P$ intersects some cycle of $\mathcal{H}(H)$ in exactly one or exactly two edges. Then this cycle corresponds to a necklace of $H$ where the beads containing $u$ and $v$ are a bead distance of one or two apart. Hence, by Theorem 7, $G \notin \mathcal{M}$.

Suppose $G \notin \mathcal{M}$. Then, by Theorem 7, there must be a necklace of $H$ where the beads containing $u$ and $v$ are a bead distance of one or two apart. Let $C$ be the cycle of $\mathcal{H}(H)$ corresponding to this necklace. Then the distinct components of $\mathcal{H}(H) - C$ containing nodes $a$ and $b$ are either incident to the same edge of $C$, or there is a subpath of $C$ of length two joining these two components. Hence $P \cap C$ is either one or two. □

Now we can use the min-cut cactus of a graph in $\mathcal{M}$ to build larger graphs in $\mathcal{M}$. All that remains is to update the min-cut cactus. Let $G \in \mathcal{M}$, let $P$ be a path in $\mathcal{H}(G)$ and let $a$ be a node of $P$. For the purposes of this paper, we will say that $a$ is a *transition node of* $P$ if $a$ is the intersection of two distinct cycles of $\mathcal{H}(G)$, say $C_1$ and $C_2$, such that $C_1 \cap P \neq \emptyset$ and $C_2 \cap P \neq \emptyset$.

**Theorem 16.** *Let $H \in \mathcal{M}$ and let $u$ and $v$ be two vertices of $H$ of degree 2. Let $R$ be a path of length at least 3 and internal vertices $r_1, \ldots, r_k$ (ordered as we encounter them travelling from one endpoint of $R$ to the other). Let $G$ be the graph obtained by identifying the endpoints of $R$ with $u$ and $v$ respectively. Let $a$ and $b$ be the nodes in $\mathcal{H}(H)$ containing $u$ and $v$ respectively and let $P$ be a shortest $\{a, b\}$-path in $\mathcal{H}(H)$. If $G \in \mathcal{M}$ then $\mathcal{H}(G)$ is obtained from $\mathcal{H}(H)$ by identifying all the transition nodes of $P$, along with the nodes $a$ and $b$, to a single node, $c$, and adding a cycle of length $k+1$ incident only to $c$. The nodes, apart from $c$, of this cycle are labelled $\{r_1\}, \ldots, \{r_k\}$ (in that order).*

*Proof.* Consider a necklace $(V_1, \ldots, V_l)$ of $H$ and let $C$ be the corresponding cycle of $\mathcal{H}(H)$.

If $u$ and $v$ are in the same bead, say $V_i$, of the necklace then $P \cap C = \emptyset$. Thus $C$ remains unchanged by the addition of $R$ apart from the nodes corresponding to the vertices of $R$ are added to the component of $\mathcal{H}(H)$ containing the vertices of $V_i$.

If $u$ and $v$ are in different beads, say $V_i$ and $V_j$ where $1 \leq i < j \leq l$, then $P \cap C \neq \emptyset$. Hence there must be exactly two transition nodes, say $a'$ and $b'$ of $P$, on $C$ such that $a'$ and $b'$ are in the components of $\mathcal{H}(H) - C$ containing $a$ and $b$ respectively. The beads $V_i$ and $V_j$ correspond to these components and by identifying $a'$ and $b'$ to a single node, we get two cycles which exactly describe the necklaces of $\mathcal{H}(G)$ obtained as described in Proposition 8.

Lastly, adding the cycle corresponds to the necklace $(V, r_1, \ldots, r_l)$, where $V$ is the set of vertices of $H$, as outlined in Proposition 8.

Thus, the min-cut cactus constructed in this theorem does in fact describe all the necklaces of $G$ and therefore it is $\mathcal{H}(G)$. $\square$

# 5 Constructing the Graphs of $\mathcal{M}$ with at most $n$ Vertices

In this section, we will gather together all the information we know about constructing the graphs of $\mathcal{M}$ and use an algorithm, designed by Brendan McKay [3] to actually generate all the graphs of $\mathcal{M}$ with at most $n$ vertices. For the purposes of this paper, let $o(G)$ denote the number of vertices of the graph $G$ and we say that $C$ is a *leaf cycle* of a cactus if $C$ has a single node of degree greater than two. Let $C_j$ denote the cycle of length $j$. For any $n \geq 3$, calling the procedures `scan`($C_3$, $C_3$, $n$), `scan`($C_4$, $C_4$, $n$), ..., `scan`($C_{n-1}$, $C_{n-1}$, $n$), and `scan`($C_n$, $C_n$, $n$) will generate all the graphs of $\mathcal{M}$ with at most $n$ vertices.

**procedure** `scan`($G$, $\mathcal{H}(G)$, $n$)
**output** $G$
**if** $o(G) \leq n - 2$ **then**
  Initialize $\mathcal{L} = \emptyset$
  **for** each pair, $\{u, v\}$ of vertices of $G$ of degree 2 **do**
    Let $a$ and $b$ be the nodes of $\mathcal{H}(G)$ containing $u$ and $v$ respectively
    Construct a shortest $\{a, b\}$-path, $P$, in $\mathcal{H}(G)$
    **if** $P$ does not intersect any cycle of $\mathcal{H}(G)$ in exactly 1 or 2 edges **then**
      Identify the transition nodes of $P$ along with $a$ and $b$ to a node $d$
        to obtain a cactus $H'$
      Let $l$ be the length of the smallest leaf cycle of $H'$
      Let $k = \min(l, n - o(G) + 1)$
      **for** $i = 3$ **to** $k$ **do**
        Construct $G'$ by adding an ear of length $i$ to $G$ with endpoints $u$ and $v$
        Construct $\mathcal{H}(G')$ by adding a cycle, $C'$, of length $i$ to $H'$ incident only to $d$
        Construct the canonicalization, $\phi(G')$, of $G'$
        **if** $\phi(G') \notin \mathcal{L}$ **then**
          **if** $i < l$ **then**
            Add $\phi(G')$ to $\mathcal{L}$
            `scan`($G'$, $\mathcal{H}(G')$, n)
          **if** $i = l$ **then**
            Let $C$ be the leaf cycle of $\mathcal{H}(G')$ of length $l$ such that $\phi(C)$ has the
              smallest lablelled node
            **if** there exists an automorphism of $G'$ mapping $C$ to $C'$ **then**

Add $\phi(G')$ to $\mathcal{L}$
$\texttt{scan}(G', \mathcal{H}(G'), n)$

# 6 Bounding the Number of Edges of Graphs in $\mathcal{M}$

In this section, we will prove an upper bound on the number of edges of a graph $G \in \mathcal{M}$. However, before we proceed to this theorem, we need to consider the structure of the canonical min-cut cactus $\mathcal{H}(G)$. The following lemma summarizes some of the important attributes, that we had previously noted, about $\mathcal{H}(G)$.

**Lemma 17.** *If $G \in \mathcal{M}$ then $\mathcal{H}(G)$ is a cactus with no tree-edges and such that every node which has degree more than 2 is the intersection of $3k$ cycles for some integer $k \geq 1$.*

Due to the tree-like nature of a cactus, any cactus with the properties described in Lemma 17 can be constructed recursively in the following manner. Let $H$ be a single cycle and apply a sequence of the following operations to $H$ to obtain $\mathcal{H}(G)$.

**O1** Choose a node, $a$, of $H$ which has degree 2 and add two new cycles to $H$ which are mutually node-disjoint except at $a$.

**O2** Choose a node, $a$, of $H$ which has degree more than 2 and add three new cycles to $H$ which are mutually node-disjoint except at $a$.

We can use these way of constructing the min-cut cactus to find an upper bound on the number of edges of $G$.

**Theorem 18.** *If $G \in \mathcal{M}$ has $n$ vertices and $m$ edges then $m < \frac{6}{5}n$.*

*Proof.* In such a construction of $\mathcal{H}(G)$, let $l_0$ be the length of the initial cycle and let $l_i$ be the number of edges added to the cactus at iteration $i \geq 1$. If operation O1 is performed at iteration $i$, then the number of edges of the cactus is increased by $l_i$ and the number of nodes of degree 2 is increased by

$l_i - 3$. If operation O2 is performed then the number of edges of the cactus is also increased by $l_i$ and the number of nodes of degree 2 is increased by $l_i - 3$. Thus, in either case, if we perform a sequence of $r$ operations to obtain $\mathcal{H}(G)$ then $\mathcal{H}(G)$ has $l_0 + l_1 + \ldots + l_r$ edges and $l_0 + l_1 + \ldots + l_r - 3r$ nodes of degree 2.

Now $G$ and $\mathcal{H}(G)$ have the same number of edges (according to Proposition 11) and the same number of vertices/nodes of degree 2 (according to Proposition 14). Thus $G$ has $m = l_0 + l_1 + \ldots + l_r$ edges and $G$ has $l_0 + l_1 + \ldots + l_r - 3r = m - 3r$ vertices of degree 2. Hence $G$ has $n - m + 3r$ vertices of degree 3. Thus, by summing up the degrees of all the vertices of $G$ we get

$$\begin{aligned} 2m &= 2(m - 3r) + 3(n - m + 3r) \\ 2m &= 2m - 6r + 3n - 3m + 9r \\ 3m &= 3n + 3r \\ m &= n + r. \end{aligned}$$

Returning to the cactus $\mathcal{H}(G)$, notice that we start with a cycle of length at least 3 and so $l_0 \geq 3$. At iteration $i$, if we apply operation O1 then we add two cycles (each of length at least 3) and hence $l_i \geq 6$. If we apply operation O2 then we add three cycles (each of length at least 3) and hence $l_i \geq 9$. In either case, $l_i \geq 6$ for all $i \geq 1$ and so

$$m = l_0 + l_1 + \ldots + l_r \geq 3 + 6r.$$

Thus $m > 6r$ and so $r < \frac{1}{6}m$. Since we discovered above that $m = n + r$, we have that

$$\begin{aligned} m &< n + \frac{1}{6}m \\ \frac{5}{6}m &< n \\ m &< \frac{6}{5}n \end{aligned}$$

□

In fact, we can easily construct a family of cacti whose members have the following properties

1. There are no tree-edges,
2. Every cycle has length 3, and
3. Every node has either degree 2 or degree 6.

Furthermore, the limit of the ratio of the number of edges over the number of nodes of the members of the family is exactly $\frac{6}{5}$. Thus the bound described in Theorem 18 is tight.

# 7 The Linear Programming Relaxation and $\mathcal{M}$

**Theorem 19.** *Let $G \in \mathcal{M}$ have $n$ vertices. Assign nonnegative edge-costs, $c'$, to the edges of $G$ such that for any necklace of $G$ with inter-bead edges $e_1, e_2 \ldots, e_k$ we have that for each $1 \leq i \leq k$,*

$$c'_{e_i} \leq \frac{1}{2} \sum_{j=1}^{k} c'_{e_j}.$$

*Let $c$ be the metric completion on $K_n = (V, E)$ of $c'$. Then $G$ is an optimal solution to the following linear program.*

$$\begin{array}{llcll} \textit{minimize} & cx & & & \\ \textit{subject to} & x(\delta(S)) & \geq & 2 & \textit{for all } \emptyset \subset S \subset V \\ & x_e & \geq & 0 & \textit{for all } e \in E \end{array}$$

*Furthermore, if all of the edge-costs are strictly positive and for every necklace described above we have that for any edge $e_i$*

$$c'_{e_i} < \frac{1}{2} \sum_{j=1}^{k} c'_{e_j}$$

*then $G$ is the unique optimal integer solution to the linear program.*

*Proof.* Let $x$ be an optimal solution to the above linear program. If there is an edge, $e \in E$, which is not an edge of $G$ such that $x_e > 0$ then we find a path, $P$, in $G$ between the endpoints of $e$ of minimum cost. Since the cost of this path is $c_e$, we can define a new solution to the linear program

$$x'_f = \begin{cases} x_f + x_e & \text{if } f \in P \\ 0 & \text{if } f = e \\ x_f & \text{otherwise} \end{cases}$$

where $cx' = cx$. By continuing in this fashion, we may assume that the support graph of $x'$ is a subgraph of $G$.

Consider a necklace of $G$ with inter-bead edges $e_1, e_2, \ldots, e_k$. Suppose, for a contradiction, that the total (weighted) cost of the inter-bead edges relative to the edge-weights of $x'$ is strictly less than the total costs of the inter-bead edges. That is, suppose

$$c_{e_1}x'_{e_1} + c_{e_2}x'_{e_2} + \ldots + c_{e_k}x'_{e_k} < c_{e_1} + c_{e_2} + \ldots c_{e_k}.$$

Then there must be an inter-bead edge, say $e_1$, such that $x'_{e_1} < 1$. However, since $x'(\delta(S)) \geq 2$ for all $\emptyset \subset S \subset V$ it must be that $x'_{e_i} + x'_{e_1} \geq 2$ for all $2 \leq i \leq k$.

$$\begin{aligned}
c_{e_1}x'_{e_1} + c_{e_2}x'_{e_2} + \ldots + c_{e_k}x'_{e_k} &< c_{e_1} + c_{e_2} + \ldots c_{e_k} \\
c_{e_1}x'_{e_1} + c_{e_2}(2 - x'_{e_1}) + \ldots + c_{e_k}(2 - x'_{e_1}) &< c_{e_1} + c_{e_2} + \ldots c_{e_k} \\
c_{e_1}x'_{e_1} + c_{e_2}(1 - x'_{e_1}) + \ldots + c_{e_k}(1 - x'_{e_1}) &< c_{e_1} \\
c_{e_2}(1 - x'_{e_1}) + \ldots + c_{e_k}(1 - x'_{e_1}) &< c_{e_1}(1 - x'_{e_1}) \\
c_{e_2} + \ldots + c_{e_k} &< c_{e_1} \\
c_{e_1} + c_{e_2} + \ldots + c_{e_k} &< 2c_{e_1} \\
\frac{1}{2}\sum_{i=1}^{k} c_{e_i} &< c_{e_1}
\end{aligned}$$

However, this contradicts the fact that $c_{e_1} \leq \frac{1}{2}\sum_{i=1}^{k} c_{e_i}$ by our definition of $c$. Therefore,

$$c_{e_1}x'_{e_1} + c_{e_2}x'_{e_2} + \ldots + c_{e_k}x'_{e_k} \geq c_{e_1} + c_{e_2} + \ldots c_{e_k}.$$

However, every edge of $G$ is an inter-bead edge in a unique necklace and so we can add up the weighted costs of the edges of $G$ using the necklaces.

Let $E(G)$ be the set of edges of $G$ and let $N(G)$ be the set of necklaces of $G$. For any $Q \in N(G)$ let $IB(Q)$ denote the set of inter-bead edges of $Q$.

$$\begin{aligned}\sum_{e \in E(G)} c_e x'_e &= \sum_{Q \in N(G)} \sum_{e \in IB(Q)} c_e x'_e \\ &\geq \sum_{Q \in N(G)} \sum_{e \in IB(Q)} c_e \\ &= \sum_{e \in E(G)} c_e\end{aligned}$$

Thus for any optimal solution, $x$, of the linear program, $cx \geq \sum_{e \in E(G)} c_e$. But $G$ is a feasible solution to the linear program. Therefore $G$ is an optimal solution to the linear program. This proves the first part of the theorem.

Now suppose that for every edge $e_i$ in any necklace of $G$ that

$$0 < c'_{e_i} < \frac{1}{2} \sum_{j=1}^{k} c'_{e_j}.$$

Let $x$ be any optimal integer solution to the linear program. Again we reroute $x$ through $G$ to obtain an integer solution $x'$ which is still optimal. If $x$ is not the characteristic vector of $G$ then consider the last edge we reroute through $G$ via a path $P$. For any internal node, $v$ of $P$, we must have at least two units of flow on the edges incident to $v$ (since in our rerouting, the intermediate graphs are also 2-edge-connected). When we reroute one unit of flow through $v$ then there must be at least four units of flow on the edges incident to $v$. However, since $G \in \mathcal{M}$, every vertex of $G$ has degree 2 or 3. Thus there is an edge $e$, incident to $v$ such that $x'_e \geq 2$.

On the other hand, the cost of every edge is positive and so the support multigraph of $x'$ is edge-minimal and hence $x'_e = 2$. Furthermore, if $Q$ is the necklace of $G$ where $e \in IB(Q)$ then one of the inter-bead edges of $Q$ has an $x'$-value of 0 and the rest have $x'$-values of 2 (otherwise we could reduce some values and hence reduce the overall cost of an optimal solution). Let $f \in IB(Q)$ such that $x'_f = 0$. From the above work, since $cx'$ is an optimal solution it must be that the weighted cost of the necklace $Q$ is the same with respect to $x'$ as it is in $G$. That is

$$
\begin{aligned}
\sum_{g \in IB(Q)} c_g x'_g &= \sum_{g \in IB(Q)} c_g \\
\sum_{g \in IB(Q) \setminus \{f\}} 2c_g &= \sum_{g \in IB(Q)} c_g \\
\sum_{g \in IB(Q) \setminus \{f\}} c_g &= c_f \\
\sum_{g \in IB(Q)} c_g &= 2c_f \\
c_f &= \frac{1}{2} \sum_{g \in IB(Q)} c_g
\end{aligned}
$$

However,

$$c_f < \frac{1}{2} \sum_{g \in IB(Q)} c_g$$

and so we have a contradiction. Thus $x$ is the characteristic vector of $G$ and so $G$ is the unique optimal integer solution. □

**Corollary 20.** *Let $G \in \mathcal{M}$ and let $d$ be the canonical distance function of $G$. Then $G$ is an optimal solution to the linear program*

$$
\begin{array}{lllll}
\textit{minimize} & dx & & & \\
\textit{subject to} & x(\delta(S)) & \geq & 2 & \textit{for all } \emptyset \subset S \subset V \\
 & x_e & \geq & 0 & \textit{for all } e \in E
\end{array}
$$

*and furthermore, it is the unique integer optimal solution.*

# 8 Constructing Min-cut Cacti of Graphs with a Min-cut of Size 2

In our construction of the graphs of $\mathcal{M}$, we end up constructing the min-cut cacti of each of these graphs. By using these ideas, we can develop an algorithm for constructing the min-cut catci of a more general class of graphs, namely those graphs with a min-cut of size 2.

Input: A graph $G$ with a min-cut of size 2
Output: The canonical min-cut cactus, $\mathcal{H}(G)$ of $G$

Find an ear decomposition, $H_0, \ldots, H_k$ of $G$ with ears $R_1, \ldots, R_k$
Initialize $X = H_0$
for $i = 0$ to $k - 1$ do
    Let $R_{i+1}$ be a path with vertices labelled $u, l_1, \ldots, l_q, v$
    Let $a$ and $b$ be the nodes of $X$ containing the labels $u$ and $v$ respectively
    Find the transition nodes, $w_1, \ldots, w_t$, in a shortest $\{a, b\}$-path in $X$
    Identify the nodes $a, b, w_1, \ldots, w_t$ in $X$ and call the new node $z$
    Label $z$ with the union of the labels in $a, b, w_1, \ldots, w_t$
    Add a cycle of length $q + 1$ to $X$ incident only to $z$
    Label the new nodes of this new cycle with the single labels $l_1, \ldots, l_q$
    Remove any loops in $X$
Output $X$

**Lemma 21.** *At the end of each iteration of the above algorithm, $X$ is a cactus.*

*Proof.* Since $H_0$ is a cycle, clearly $X$ is initially a cactus. Suppose $X$ is a cactus at the end of iteration $i$ for some $0 \leq i \leq k-2$. For clarity, we will let $X_i$ and $X_{i+1}$ be $X$ at the end of iteration $i$ and $i+1$ respectively. Let $xy$ be any edge of $X_{i+1}$. If $xy$ is in the new cycle added in the $i+1$st iteration, then clearly $xy$ is in a unique cycle. Otherwise, $xy$ is not a new edge which was added to $X$ in iteration $i+1$. Hence $xy$ is an edge of $X_i$. Since $X_i$ is a cactus, let $C$ be the unique cycle containing $xy$. Thinking generally about cycles, if we identify some of the nodes of a cycle, we get a collection of edge-disjoint cycles and loops all incident to the new node. Since we remove loops in every iteration and $xy$ is in a cycle of $X_i$, it must also be in a cycle of $X_{i+1}$.

Suppose, for a contradiction that $xy$ is on two cycles $C_1$ and $C_2$ of $X_{i+1}$. Let $z$ be the new node of $X_{i+1}$ obtained by identifying nodes of $X_i$. If $z$ is not on either of these two cycles then they are also cycles of $X_i$ and hence $xy$ is on two distinct cycles of $X_i$ which contradicts the fact that $X_i$ is a cactus. Thus, suppose $z$ is a node of $C_1$. Since $C_1$ and $C_2$ differ, let $Q$ be a maximal subpath of $C_2$ whose internal nodes are distinct from the nodes of $C_1$. By taking the union of $C_1$ and $Q$, we are guaranteed that $X_{i+1}$ has a subgraph, $Y$, as shown in Figure 12 which contains the node $z$.

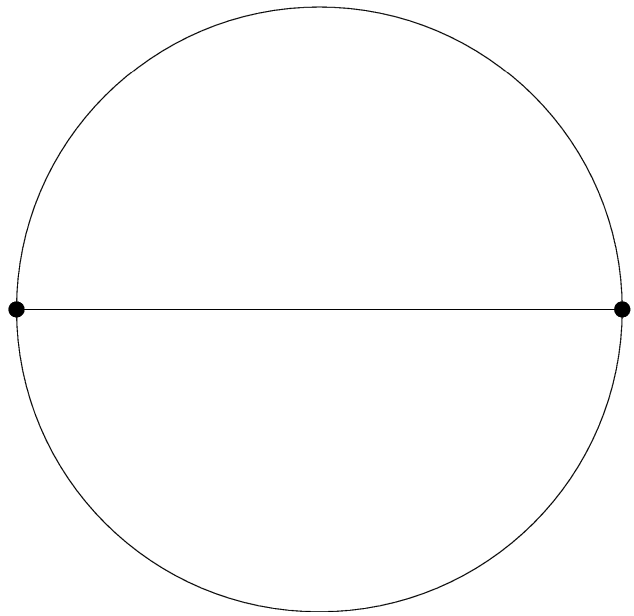

Figure 12: The subgraph $Y$

Notice that $Y$ contains two cycles which share edges and hence $Y$ cannot be a subgraph of $X_i$. Hence, $Y$ must result from the identification of nodes $z_1$, $z_2$, and $z_3$ from one of the three subgraphs of $X_i$ shown in Figure 13.

Since we are identifying the nodes $z_1$, $z_2$, and $z_3$, these are transition nodes (or also $a$ or $b$) on a shortest $\{a, b\}$-path in $X_i$. Although there are many $\{a, b\}$-paths in $X_i$, since $X_i$ is a cactus, the cycles which intersect an $\{a, b\}$-path are the same regardless of what path is chosen. Hence the shortest path corresponds to a unique chain of edge-disjoint cycles of $X_i$ and any node that lies on two cycles is a transition node of the path. Thus, for all the scenarios depicted in Figure 13, $d$ must be a transition node of $P$. However, $d$ is not identified with the other nodes to obtain $z$ and so we have a contradiction. Therefore, every edge of $X_{i+1}$ is in a unique cycle.

Hence, in order to show that $X_{i+1}$ is a cactus, we simply need to show that $X_{i+1}$ is connected. This follows trivially from the fact that $X_i$ is a cactus, and hence connected and neither of the operations of identifying nodes nor removing loops will disconnect the graph. Therefore, $X_{i+1}$ is connected and is hence a cactus. The result then follows by induction. □

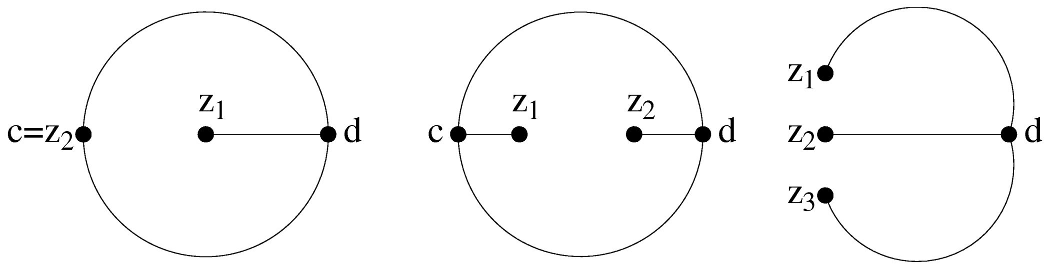


Figure 13: 3 possible subgraphs of $X_i$

**Theorem 22.** *At the end of iteration $i$ of the above algorithm, $X$ is the canonical min-cut cactus of $H_{i+1}$.*

*Proof.* A cycle is its own canonical min-cut cactus, so we have correctly initialized $X$. Suppose that $X$ is the canonical min-cut cactus of $H_i$ at the end of iteration $i-1$ (where we take the convention that iteration -1 is the initialization $X = H_0$. Let $X_{i-1}$ denote $X$ at the end of iteration $i-1$ and let $X_i$ denote $X$ at the end of iteration $i$. Notice that if $e$ is an edge in some min-cut of $H_{i+1}$ then $e$ is an inter-bead edge of some necklace of $H_{i+1}$. Thus we can apply the proof of Proposition 8 to show that the necklaces of $H_{i+1}$ are obtained from the necklaces of $H_i$ as follows.

If $(V_1, \ldots, V_l)$ is a necklace of $H_i$, $W$ is the set of internal vertices of $R_{i+1}$, $u \in V_r$, and $v \in V_s$, then $(V_1, \ldots, V_{r-1}, V_r \cup V_{r+1} \cup \ldots \cup V_{s-1} \cup V_s \cup W, V_{s+1}, \ldots, V_l)$ and $(V_{r+1}, V_{r+2}, \ldots, V_{s-1}, V_s \cup V_{s+1} \cup \ldots \cup V_l \cup V_1 \cup V_2 \cup \ldots \cup V_r \cup W)$ are both necklaces (provided they have more than one bead) are necklaces of $H_{i+1}$. If $W \neq \emptyset$ then $(V_1 \cup V_2 \cup \ldots \cup V_l, l_1, l_2, \ldots, l_q)$ is also a necklace of $H_{i+1}$. Furthermore, these are the only necklaces of $H_{i+1}$.

In our algorithm, we create exactly these necklaces in the min-cut cactus by identifying $a$, $b$, and the transition nodes of $P$. Removing loops is simply the process of disregarding necklaces that have a single bead. The min-cut cactus is automatically canonical since every node contains some label. Thus, at the end of iteration $i$, $X_i$ is the canonical min-cut cactus of $H_{i+1}$. □

Now that we know that the algorithm is correct, we would like to know its running time.

**Lemma 23.** *Let $G$ be a graph with $n$ vertices and let $\mathcal{H}(G)$ be a min-cut cactus of $G$. If $\mathcal{H}(G)$ has no empty nodes then $\mathcal{H}(G)$ has at most $n$ nodes and at most $2n-2$ edges.*

*Proof.* Let $n'$ and $m'$ be the number of nodes and edges respectively of $\mathcal{H}(G)$.

Since $\mathcal{H}(G)$ has no empty nodes, every node must be labelled by one or more vertices of $G$. Now, every vertex of $G$ appears as a label in exactly one node of $\mathcal{H}(G)$ so trivially we have that $n' \leq n$.

Since $\mathcal{H}(G)$ is a cactus, it an edge-minimal 2-edge-connected graph. Thus in any ear decomposition of $\mathcal{H}(G)$, every ear contains at least two edges. Let $L_0$ be the length of the cycle beginning the ear decomposition and let $L_1, \ldots, L_p$ be the lengths of the subsequent ears added. Thus $m' = L_0 + L_1 + \ldots + L_p$ and $n' = L_0 + (L_1 - 1) + \ldots + (L_p - 1)$. Hence $m' = n' + p$. However, the beginning cycle must contain at least 2 nodes and every ear adds at least one node so $p \leq n' - 2$. Therefore $m' \leq n' + (n' - 2) = 2n' - 2$ and so $m' \leq 2n - 2$. □

**Lemma 24.** *If $G$ is a graph with $n$ vertices and a min-cut of size 2 then every iteration of the algorithm can be completed in $O(n)$ time.*

*Proof.* The only processes in an iteration which require any significant amount of time to complete are

- finding the transition nodes,
- identifying a set of nodes, and
- removing loops from the cactus.

Identifying a set of nodes just requires us examine each of the edges in the cactus and if one the endpoints of the edge is $a$, $b$, or a transition node then we replace it with $z$. If we notice that we replaced both endpoints with $z$ then we created a loop and we can remove it right away. Thus we just need to scan the edges one by one. Since there are $O(n)$ edges in the cactus, we can complete both the identifying of nodes step and the removing of loops step in $O(n)$ time.

Hence, it just remains to show that we can find the transition nodes in $O(n)$ time.

Suppose that at the beginning of iteration $i$ where $0 \leq i \leq k-1$ we had a rooted spanning tree, $T$, of the canonical min-cut cactus of $H_i$, call it $\mathcal{H}(H_i)$. Let $r$ be the root node and let all the edges of $T$ be oriented towards $r$. Suppose further, that the arcs of $T$ are coloured so that two arcs of $T$ are the same same colour if and only if they belong to the same cycle of $\mathcal{H}(H_i)$. Now $T$ has $O(n)$ arcs since it is an oriented subgraph of $\mathcal{H}(H_i)$ and so we can construct the unique $(a, r)$-dipath in $T$ in $O(n)$ time. Similarily we will find the unique $(b, r)$-dipath in $T$. Comparing these two dipaths to find their first common node, call it $c$, can also be accomplished in $O(n)$ time. By taking the union of the $(a, c)$-subdipath and the $(b, c)$-subdipath, we have found the unique undirected $\{a, b\}$-path in $T$. Hence we have also found an $\{a, b\}$-path in $\mathcal{H}(H_i)$. As we noticed before, the transition nodes of any $\{a, b\}$-path in $\mathcal{H}(H_i)$ are the same. Thus, we can simply follow the $(a, c)$-subdipath and note whenever the colour of the arcs changes. If two consecutive arcs have different colour then we can conclude that the incident node is a transition node. By repeating this process for the $(b, c)$-subdipath, we can find the remaining transition nodes. Therefore, given this coloured rooted spanning tree of $\mathcal{H}(H_i)$, we can find the transition nodes in $O(n)$ time.

Fortunately, if such a coloured rooted spanning tree exists for $\mathcal{H}(H_i)$ then, as we find the transition nodes, we can modify $T$ to find an appropriate coloured rooted spanning tree exists for $\mathcal{H}(H_{i+1})$. Let $x$ and $y$ be consecutive transition nodes (counting $a$ and $b$ as transition nodes) along the subdipaths of $T$ used to find the transition nodes. Since we have found a $(x, y)$-dipath in $T$, when $x$ and $y$ are identified, the cycle containing $x$ and $y$ in $\mathcal{H}(H_i)$ becomes two in $\mathcal{H}(H_{i+1})$. Hence, we colour the $(x, y)$-dipath with a new colour. Furthermore, we remove the first arc of this $(x, y)$-dipath in $T$ so that when we identify $x$ and $y$ we do not get a dicycle in the resulting rooted tree. Lastly we add the new nodes labelled $l_1, \ldots, l_q$ to the tree along with the arcs $l_1 l_2, \ldots, l_{q-1} l_q, l_q z$. Then we have an appropriately coloured rooted spanning tree for $\mathcal{H}(H_{i+1})$. Thus we can appropriately modify the tree at every iteration so it just remains to show that we can start with a tree for $\mathcal{H}(H_0)$ and the result follows by induction. However, $\mathcal{H}(H_0)$ is a cycle so we can just choose a Hamiltonian dipath as our initial $T$ with all its arcs coloured the same colour since all the edges of $\mathcal{H}(H_0)$ are in the same cycle.

Therefore, the above-mentionned tree does exist for every iteration and it can be updated during our finding of the transition nodes and hence finding the transition nodes in each iteration requires $O(n)$ time. Therefore, each iteration of the algorithm takes $O(n)$ time. □

**Theorem 25.** *Let $G$ be a graph with $n$ vertices and $m$ edges. If $G$ has a min-cut of size 2 then the running time of the algorithm is $O(mn)$.*

*Proof.* The ears in any ear decomposition of $G$ along with the beginning cycle $H_0$ partition the edges of $G$. Thus any ear decomposition of $G$ contains fewer then $m$ ears. Hence the algorithm executes fewer then $m$ iterations. By Lemma 24 each iteration takes time $O(n)$ and so all the iterations take time $O(mn)$. The only other work done in the execution of the algorithm is finding an ear decomposition of $G$. We can find an ear decomposition of $G$ in $O(m)$ time. Therefore, the total running time of the algorithm is $O(mn)$. □

Theorem 25 tells us that given a graph with $n$ vertices, $m$ edges, and a min-cut of size 2 we can construct its canonical min-cut cactus in $O(mn)$ time. This is a strict improvement on the current best algorithm, due to Fleischer [2], which can construct a min-cut cactus for an arbitrary graph on $n$ vertices and $m$ edges in $O(mn\log(\frac{n^2}{m}))$ time.

**Corollary 26.** *If $G \in \mathcal{M}$ has $n$ vertices then we can find the canonical min-cut cactus of $G$ in $O(n^2)$ time.*

*Proof.* According to Theorem 18, $G$ has $O(n)$ edges. Hence by Theorem 25 we can find the canonical min-cut cactus in $O(n^2)$ time. □

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