Pipelining and Parallel Processing in IIR Digital Filters

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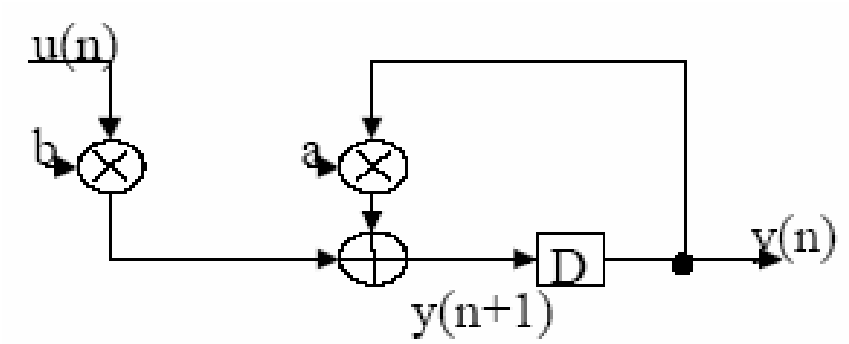
Outline

- Introduction
- Pipelining in 1st-Order IIR Digital Filters
- Pipelining in Higher-Order IIR Digital Filters
- Parallel Processing for IIR Filters
- Combined Pipelining and Parallel Processing for IIR Filters

Introduction

consider a 1st-order LTI IIR Filter

$$y(n+1) = ay(n) + u(n)$$



iteration period: *Tm + Ta*

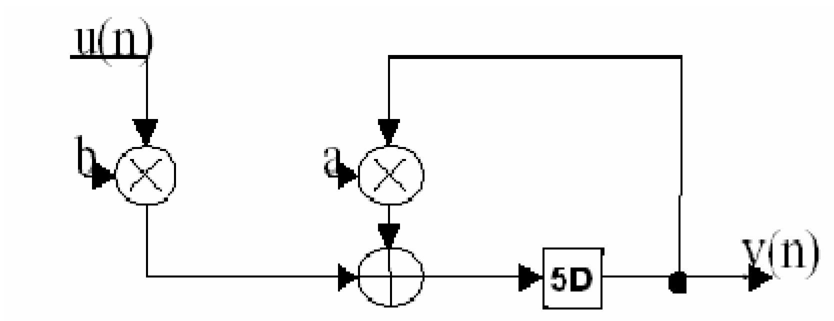
Tm: multiplication time

Ta: addition time

Interleaved Time Series

- a naïve try:

  inefficient interleaving and slow sample rate

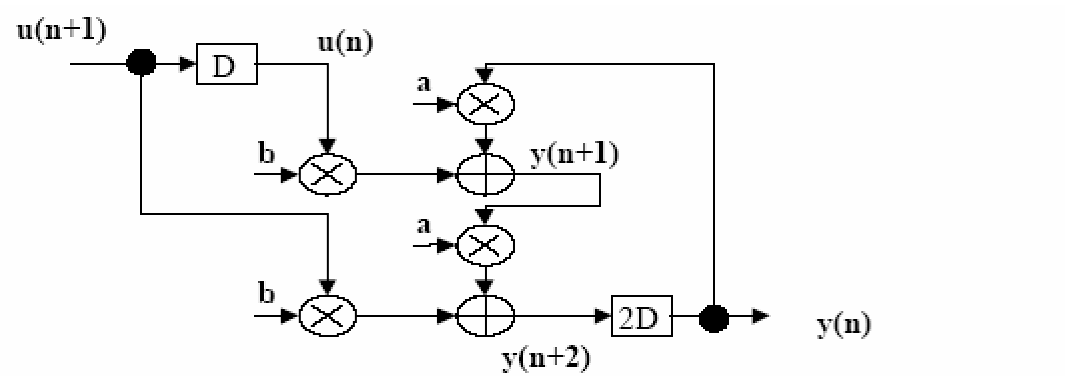


Time (n)	0	1	2	3	4	5	6	7	8	9	10
State x(n)	$y^{(1)}(0)$	$y^{(2)}(0)$	$y^{(3)}(0)$	$y^{(4)}(0)$	$y^{(5)}(0)$	$y^{(1)}(1)$	$y^{(2)}(1)$	$y^{(3)}(1)$	$y^{(4)}(1)$	$y^{(5)}(1)$	$y^{(1)}(2)$

Look-Ahead Computation

- more iterated recursion

$$y(n+2) = a\left[ay(n) + bu(n)\right] + bu(n+1)$$



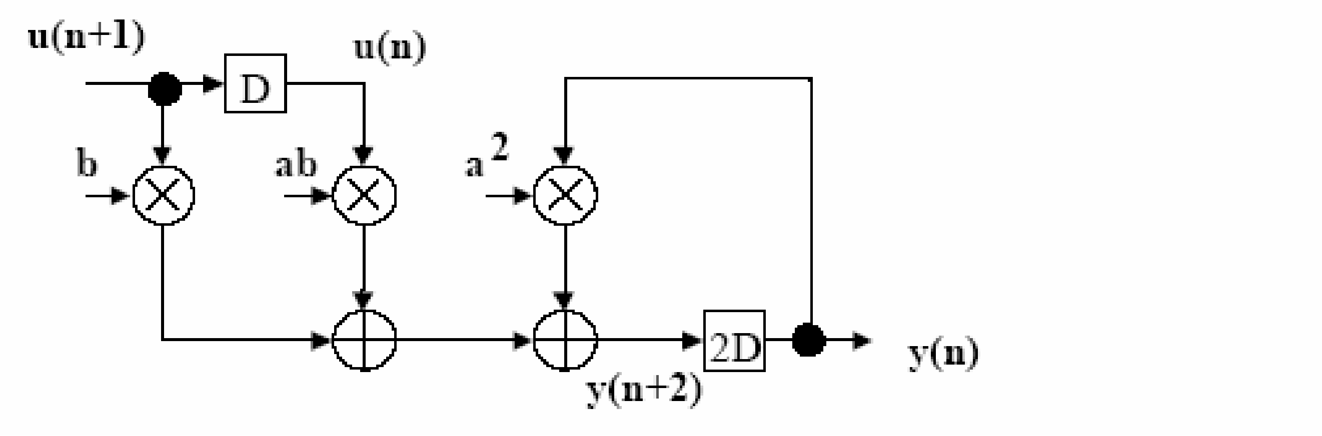
iteration bound: *2 (Tm + Ta)/2*

the same as the previous version

Look-Ahead Computation

- another equivalent recursion

$$y(n+2) = a^2 \cdot y(n) + ab \cdot u(n) + b \cdot u(n+1)$$

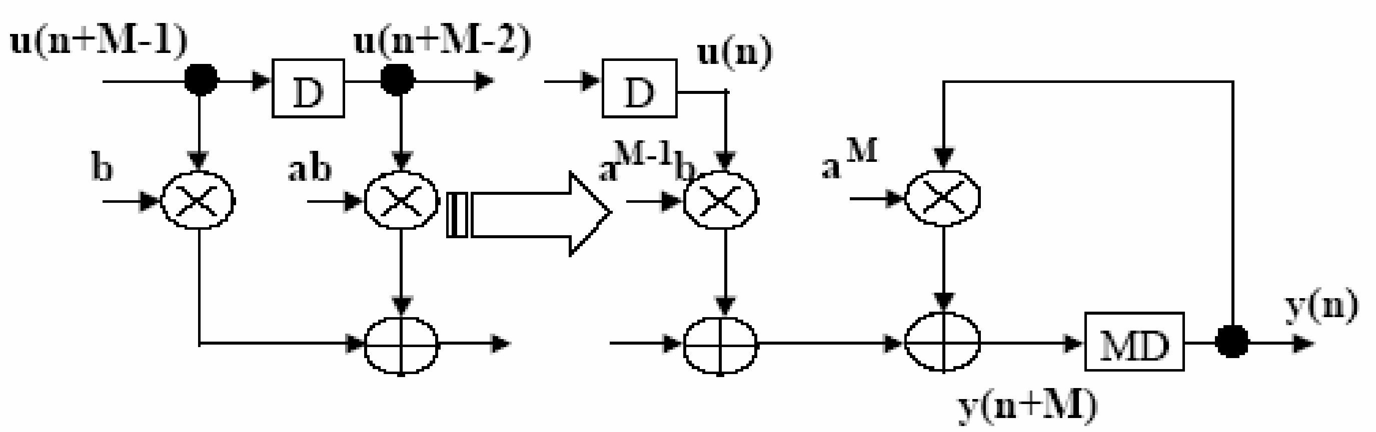


iteration bound: *(Tm + Ta)/2*

Look-Ahead Computation

- general case:
  applying M-1 steps of look-ahead

$$y(n+M) = a^M \cdot y(n) + \sum_{i=0}^{M-1} a^i \cdot b \cdot u(n+M-1-i)$$

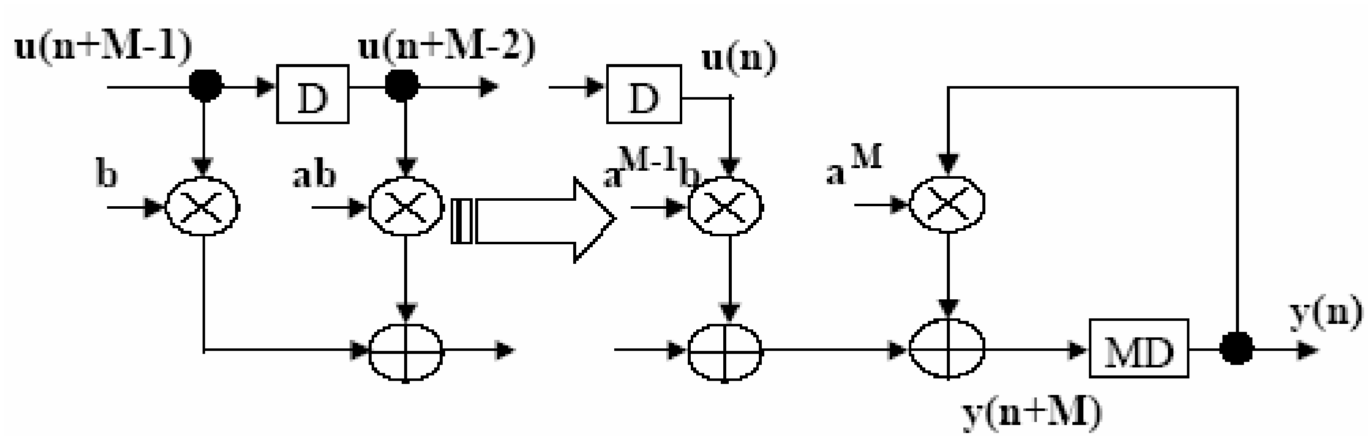


iteration bound: *(Tm + Ta)/M*
linear increase in complexity

Interleaved Time Series

- pipelined interleaving

M = 5



Time (n)	0	1	2	3	4	5	6	7	8	9
State x(n)	y(-4)	y(-3)	y(-2)	y(-1)	y(0)	y(1)	y(2)	y(3)	y(4)	y(5)

Pipelining in 1st-Order IIR Digital Filters

- **revisit the 1st-order IIR filter**

$$y(n+1) = ay(n) + u(n)$$

$$H(z) = \frac{1}{1 - a \cdot z^{-1}}$$

- **Example: 3-stage pipelining**

$$H(z) = \frac{1 + a \cdot z^{-1} + a^2 \cdot z^{-2}}{1 - a^3 \cdot z^{-3}}$$

adding poles and zeros at $z = ae^{\pm(j2\pi/3)}$

Look-Ahead Pipelining with Power-of-2 Decomposition

consider a 1st-order LTI IIR Filter

$$H(z) = \left(b \cdot z^{-1}\right) / \left(1 - a \cdot z^{-1}\right)$$

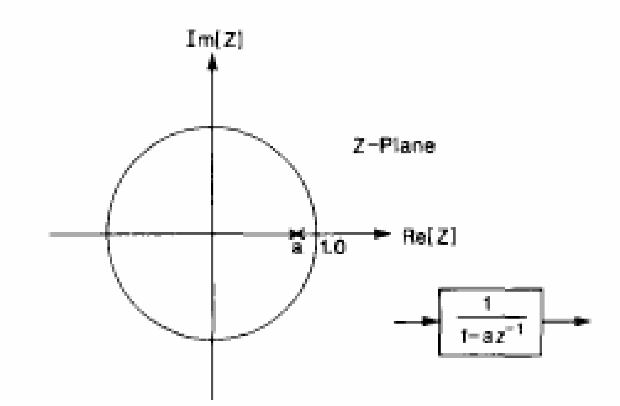
applying the decomposition technique

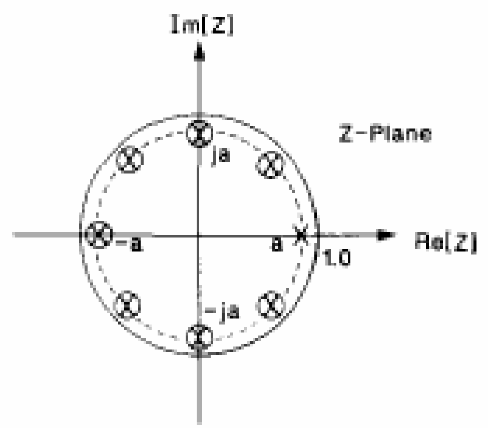
$$H(z) = \frac{b \cdot z^{-1} \prod_{i=0}^{\log_2 M - 1} \left(1 + a^{2^i} \cdot z^{-2^i}\right)}{1 - a^M \cdot z^{-M}}$$

$\log_2 M$ sets of transformation

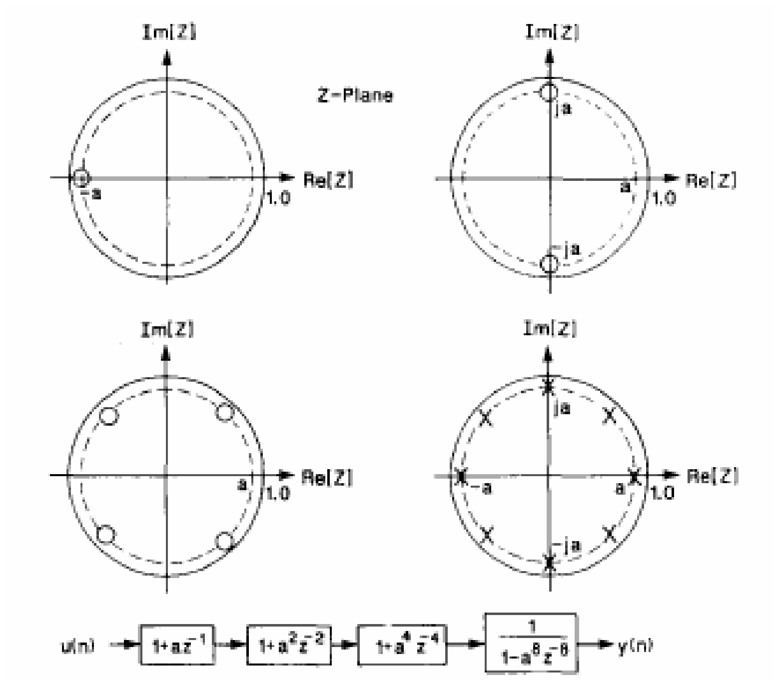
logarithmic increase in hardware complexity

original IIR





pipelined IIR (M = 8)



pipelined IIR with Decomposition (M = 8)

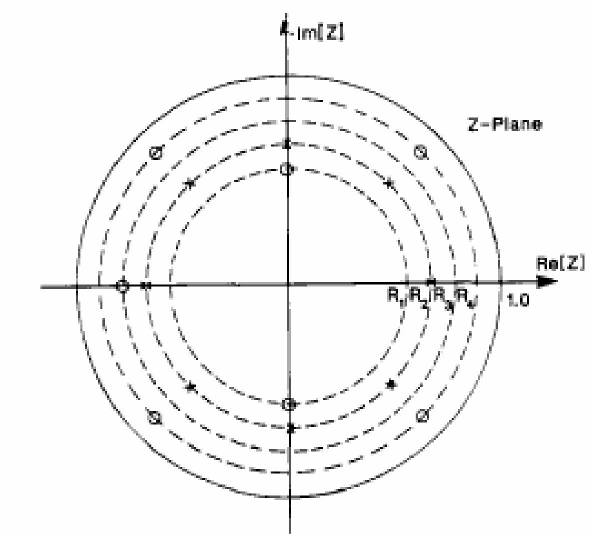
Finite Precision Problems

- pole position sensitivity to filter coefficients

$$p = (a^M + \Delta)^{1/M} \approx a\left(1 + \frac{\Delta}{Ma^M}\right)$$

more sensitive for small value of a

- inexact pole/zero cancellation



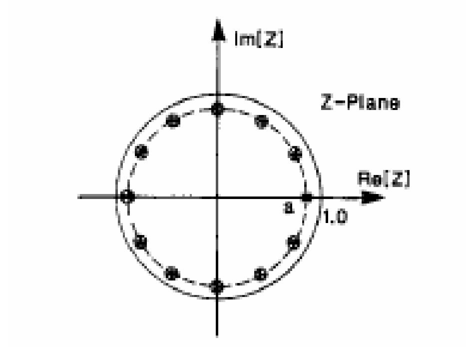
Look-Ahead Pipelining with General Decomposition

- **the 1st-order IIR filter again**

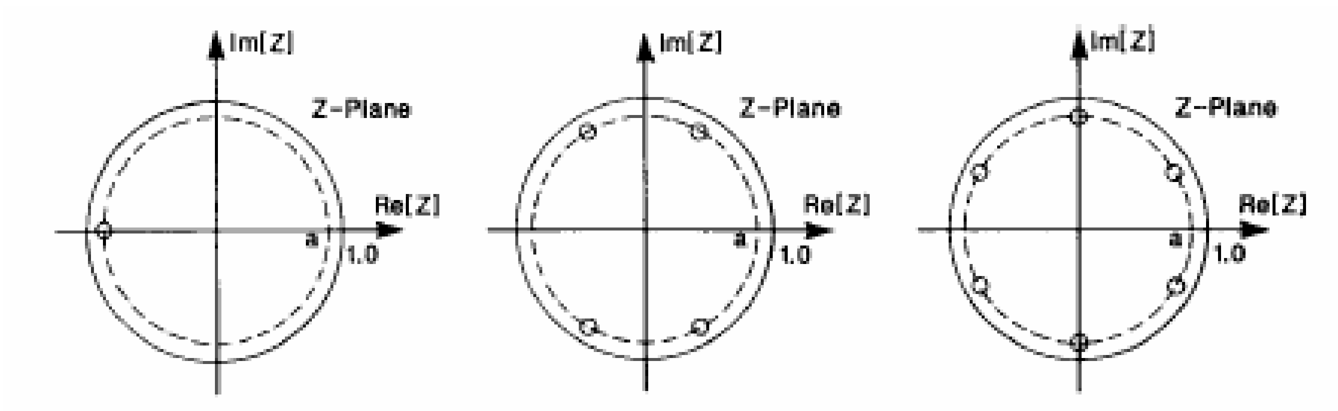
$$H(z) = \frac{1}{1 - a \cdot z^{-1}}$$

- **12-stage pipelined: 2x3x2 decomposition**

$$H(z) = \frac{\sum_{i=0}^{11} a^i \cdot z^{-i}}{1 - a^{12} \cdot z^{-12}} = \frac{\left(1 + az^{-1}\right)\left(1 + a^2 z^{-2} + a^4 z^{-4}\right)\left(1 + a^6 z^{-6}\right)}{1 - a^{12} \cdot z^{-12}}$$



pipelined IIR (M = 12)



pipelined IIR with Decomposition (M = 12)

$$H(z) = \frac{\sum_{i=0}^{11} a^i \cdot z^{-i}}{1 - a^{12} \cdot z^{-12}} = \frac{\left(1 + az^{-1}\right)\left(1 + a^2 z^{-2} + a^4 z^{-4}\right)\left(1 + a^6 z^{-6}\right)}{1 - a^{12} \cdot z^{-12}}$$

Look-Ahead Pipelining with General Decomposition

- 2x2x3 decomposition

$$H(z) = \frac{\left(1 + az^{-1}\right)\left(1 + a^2 z^{-2}\right)\left(1 + a^4 z^{-4} + a^8 z^{-8}\right)}{1 - a^{12} \cdot z^{-12}}$$

- 3x2x2 decomposition

$$H(z) = \frac{\left(1 + az^{-1} + a^2 z^{-2}\right)\left(1 + a^3 z^{-3}\right)\left(1 + a^6 z^{-6}\right)}{1 - a^{12} \cdot z^{-12}}$$

Pipelining in Higher-order IIR Digital Filters

- **Clustered Look-Ahead Pipelining**

consider a 2nd-order IIR filter:

$$H(z) = \frac{1}{1 - \frac{5}{4}z^{-1} + \frac{3}{8}z^{-2}}$$

with poles at 1/2 and 3/4

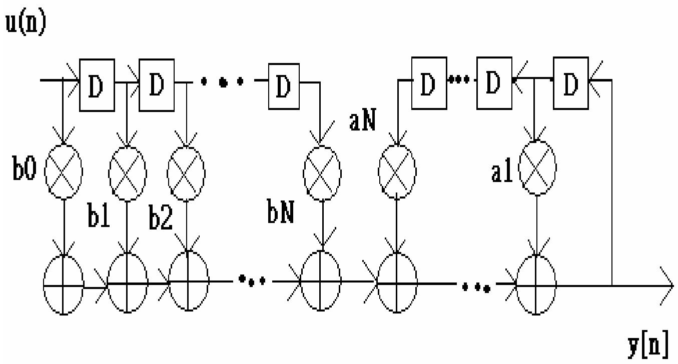
$$H(z) = \frac{1}{1 - \frac{5}{4}z^{-1} + \frac{3}{8}z^{-2}} \cdot \frac{1 + \frac{5}{4}z^{-1}}{1 + \frac{5}{4}z^{-1}}$$

$$= \frac{1 + \frac{5}{4}z^{-1}}{1 - \frac{19}{16}z^{-2} + \frac{15}{32}z^{-3}}$$

using 2-stage pipelining

Pipelining in Higher-order IIR Digital Filters

- **Clustered Look-Ahead Pipelining**



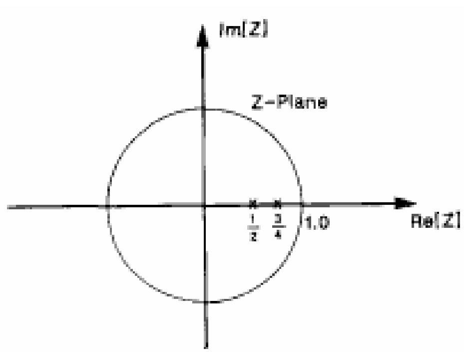
if using higher value of M (ex. M = 3):

multiplying $\left(1+5/4\,z^{-1}+19/16\,z^{-2}\right)$

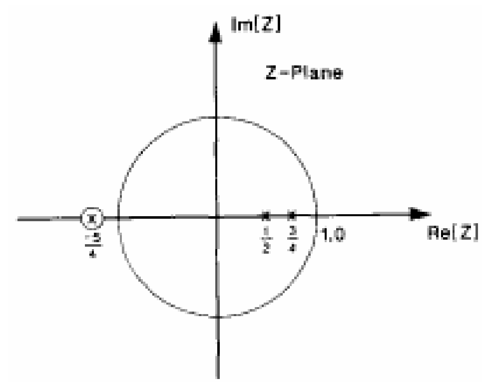
$$H(z)=\frac{1+\frac{5}{4}z^{-1}+\frac{19}{16}z^{-2}}{1-\frac{65}{64}z^{-3}+\frac{57}{128}z^{-4}}$$

linear increase hardware complexity

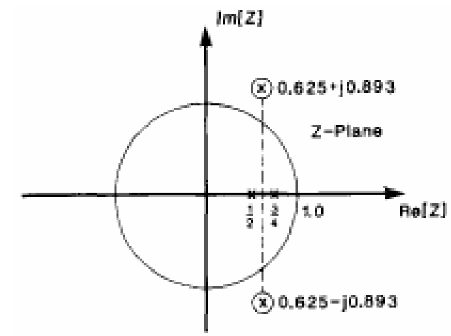
Instability Problems



original
IIR



pipelined
IIR
(M = 2)



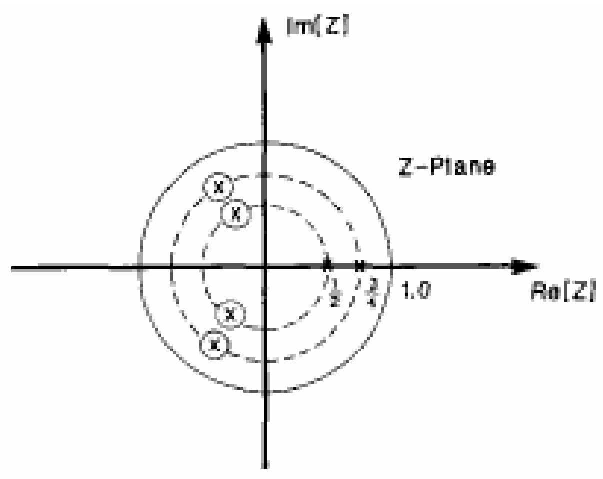
pipelined
IIR
(M = 3)

numerical method to find M for stability

Pipelining in Higher-order IIR Digital Filters

- **Scattered Look-Ahead Pipelining**

revisit the 2nd-order IIR filter: $H(z) = \dfrac{1}{1 - \frac{5}{4}z^{-1} + \frac{3}{8}z^{-2}}$



guaranteed stability if the original filter is stable

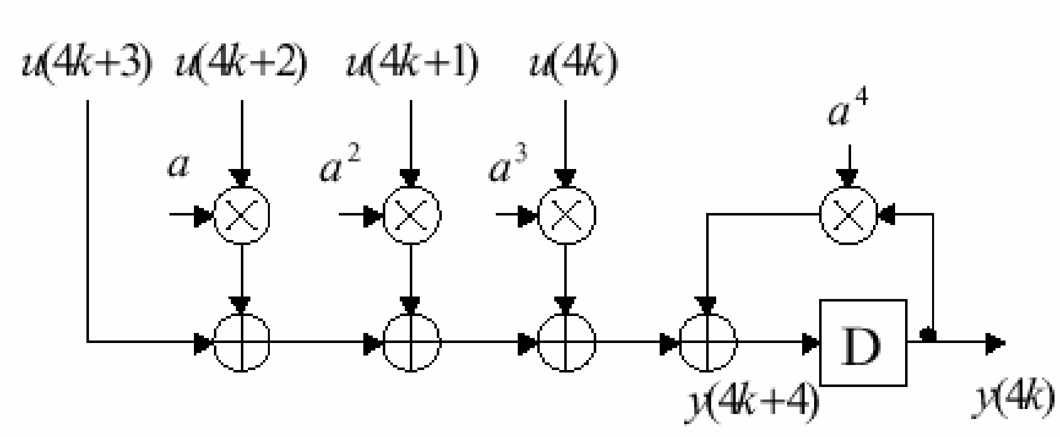
using decomposition to obtain area efficiency

Parallel Processing in IIR Filters

- consider a 1st – order IIR filter

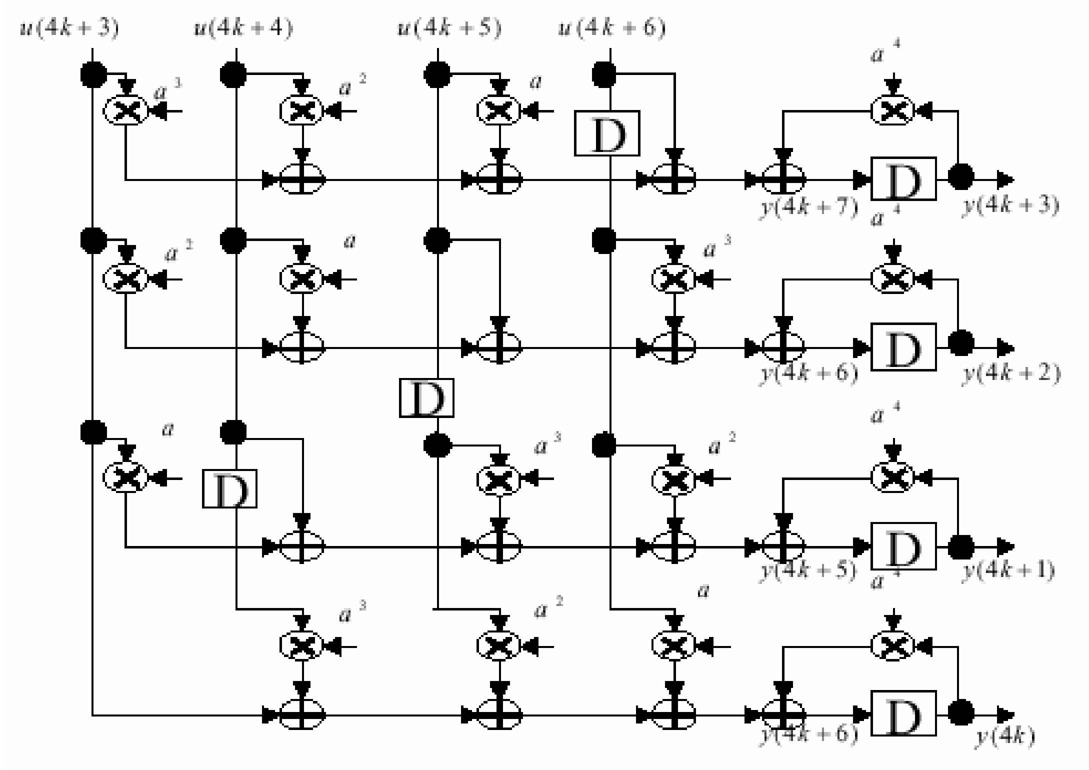
$$H(z) = \frac{z^{-1}}{1 - az^{-1}} \qquad y(n+1) = ay(n) + u(n)$$

$$y(4k+4) = a^4 y(4k) + a^3 u(4k) + a^2 u(4k+1) + au(4k+2) + u(4k+3)$$



L = 4

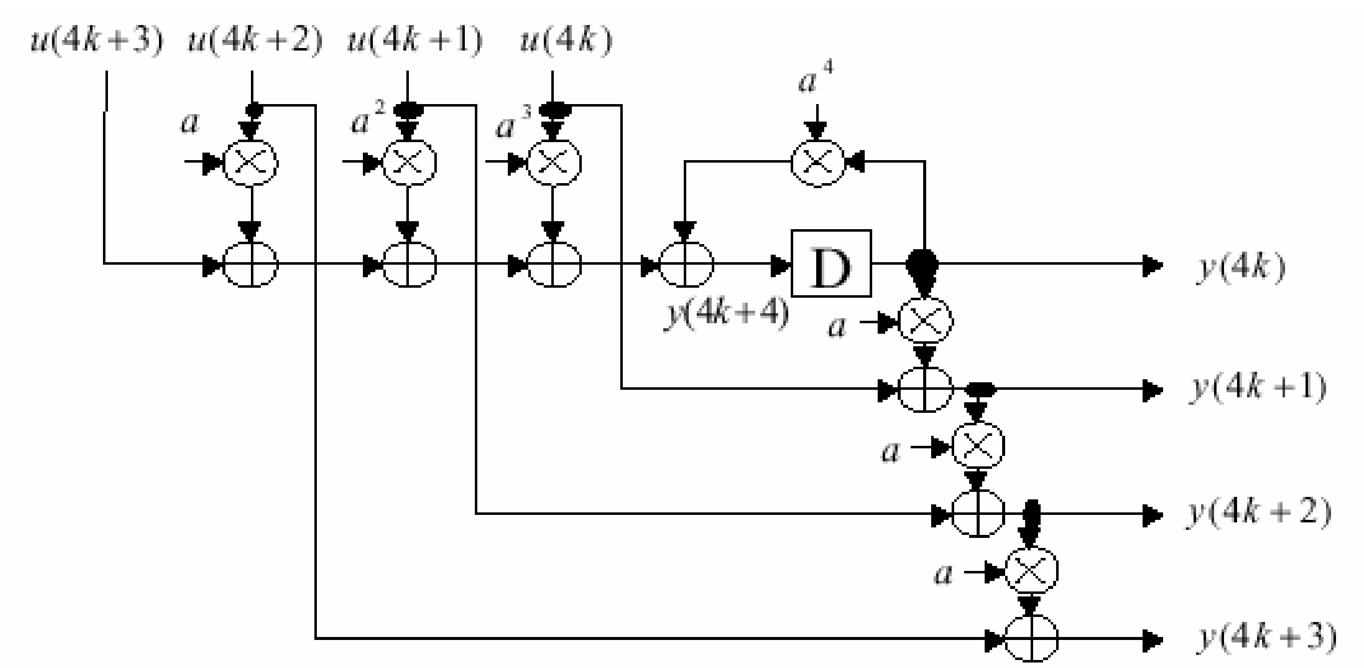
A Straightforward Structure



L = 4

Hardware complexity : L^2 Multiply-add operation

Incremental Block Processing



L = 4

Hardware complexity : $2L-1$ multiply-add operation

Round-off Noise Robustness

- pole movement

$$z = a \qquad \text{v.s.} \qquad z = a^4$$

$$\text{Round-off noise} \quad \propto \frac{1}{1-a^2}$$

for one pole IIR filter

Parallel Processing in IIR Filters

- consider a 2nd – order IIR filter

$$H(z) = \frac{\left(1 + z^{-1}\right)^2}{1 - \frac{5}{4}z^{-1} + \frac{3}{8}z^{-2}}$$

$$y(n) = \frac{5}{4}y(n-1) - \frac{3}{8}y(n-2) + f(n)$$

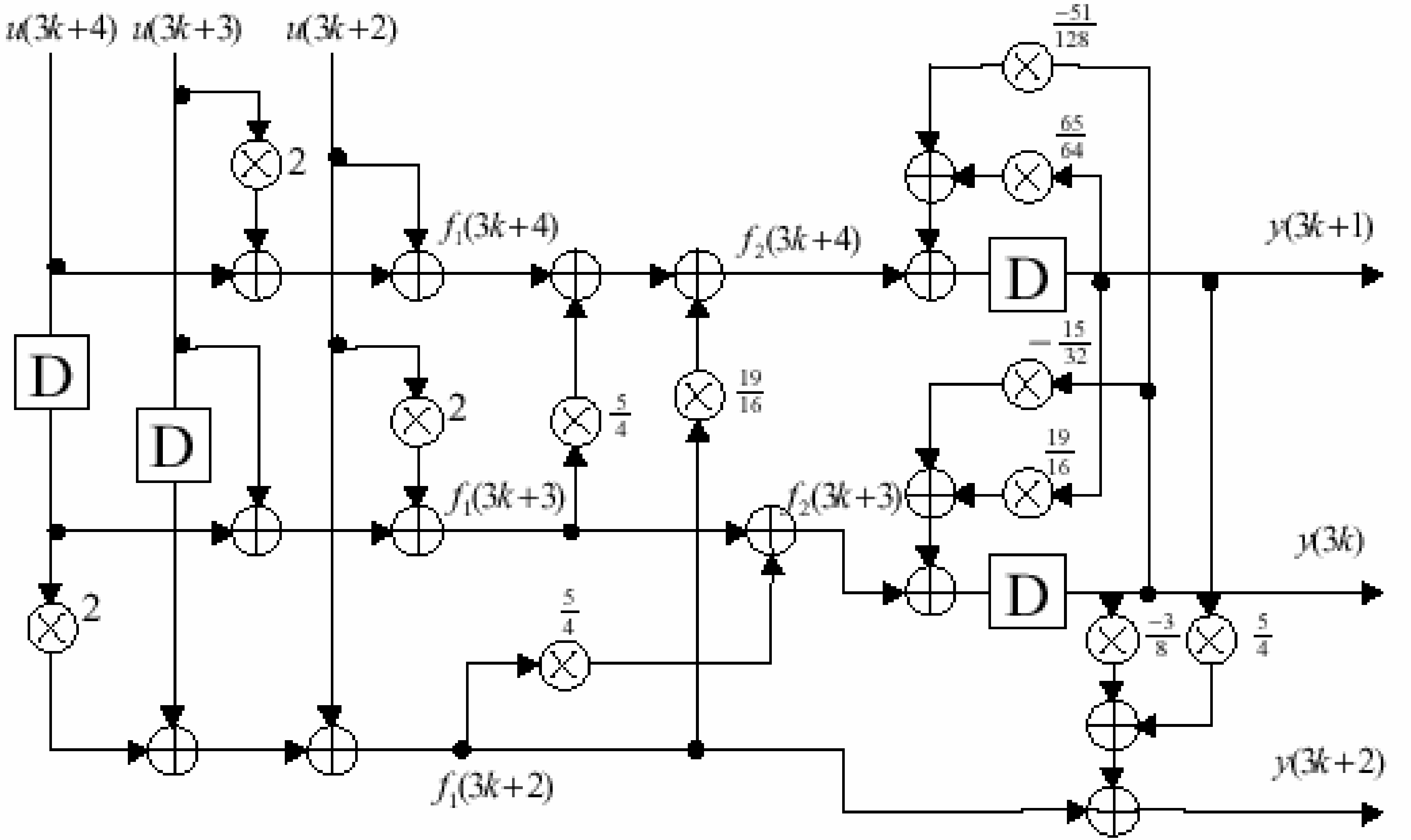
$$f(n) = u(n) + 2u(n-1) + u(n-2)$$

$$\begin{cases} y(3k+3) = \frac{19}{16} y(3k+1) - \frac{15}{32} y(3k) + \frac{5}{4} f(3k+2) \\ \qquad\qquad + f(3k+2) \\ y(3k+4) = \frac{65}{64} y(3k+1) - \frac{57}{128} y(3k) + \frac{19}{16} f(3k+2) \end{cases}$$

$$+ \frac{5}{4} f(3k+3) + f(3k+4)$$

$$y(3k+2) = \frac{5}{4} y(3k+1) - \frac{8}{3} y(3k) + f(3k+2)$$

$$\begin{bmatrix} y(3k+3) \\ y(3k+4) \end{bmatrix} = \begin{bmatrix} \frac{-15}{32} & \frac{19}{16} \\ \frac{-57}{128} & \frac{65}{64} \end{bmatrix} \cdot \begin{bmatrix} y(3k) \\ y(3k+1) \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$



u(3k+4)
u(3k+3)
u(3k+2)
2
f1(3k+4)
f2(3k+4)
f1(3k+3)
f2(3k+3)
f1(3k+2)
D
5/4
19/16
−51/128
65/64
−15/32
−3/8
y(3k+1)
y(3k)
y(3k+2)

Round-off Noise Robustness

- pole movement

$$z = \frac{1}{2}, \frac{3}{4} \qquad \text{V.S.} \qquad z = \left(\frac{1}{2}\right)^3, \left(\frac{3}{4}\right)^3$$

Combined Pipelining and Parallel Processing For IIR Filters

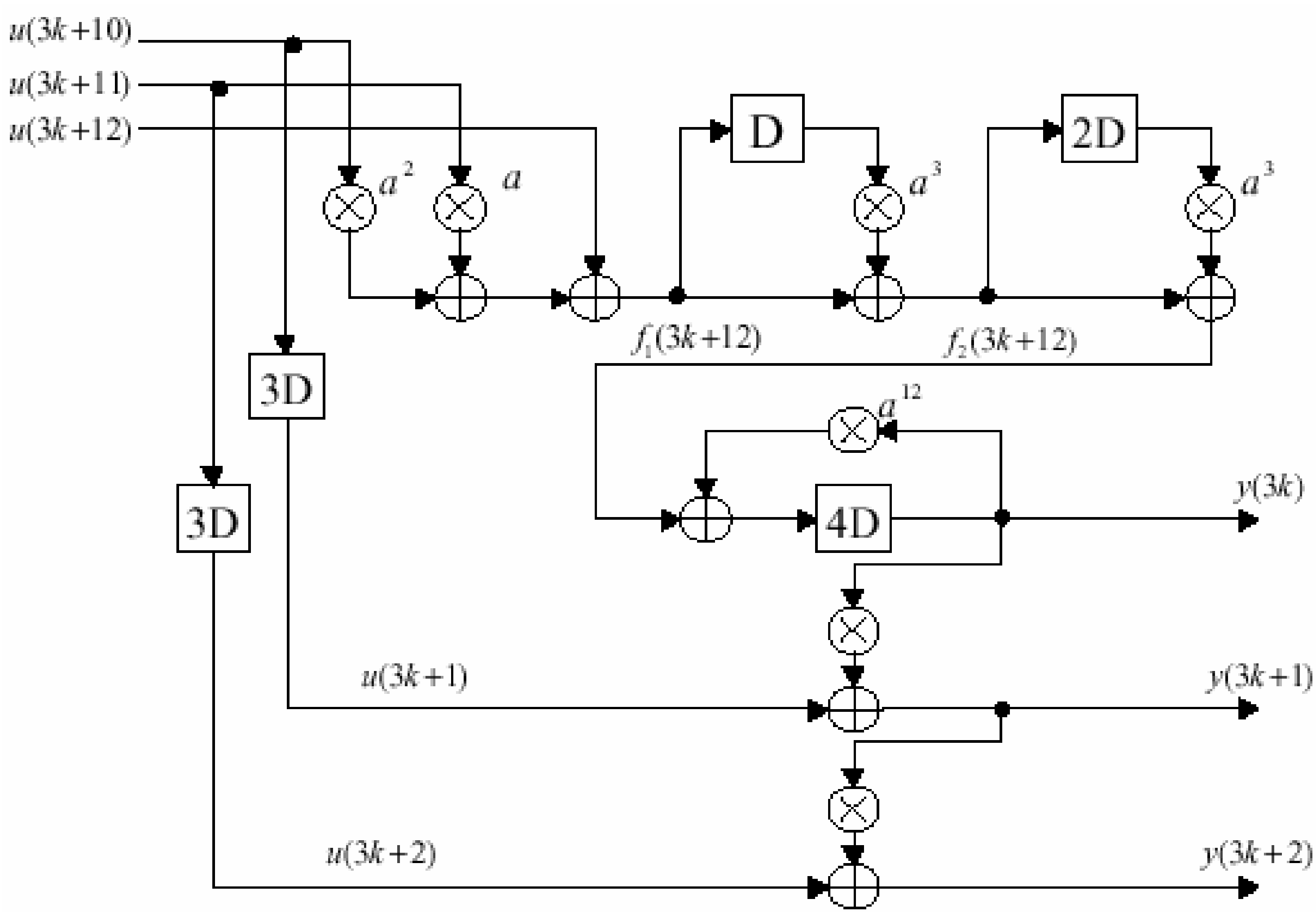
- revisit the 1st – order IIR filter $y(n+1) = ay(n) + u(n)$

with L = 4 and M = 3

$$
\begin{aligned}
y(3k+12) &= a^{12}y(3k) \\
&+ a^{11}u(3k+1) + a^{10}u(3k+2) + a^{9}u(3k+3) \\
&+ a^{8}u(3k+4) + a^{7}u(3k+5) + a^{6}u(3k+6) \\
&+ a^{5}u(3k+7) + a^{4}u(3k+8) + a^{3}u(3k+9) \\
&+ a^{2}u(3k+10) + au(3k+11) + u(3k+12) \\
&= a^{12}y(3k) + a^{6}f_2(3k+6) + a^{3}f_1(3k+9) + f_1(3k+12)
\end{aligned}
$$

where

$$
\begin{cases}
f_1(3k+12) = a^2u(3k+10) + au(3k+11) + u(3k+12) \\
f_2(3k+12) = a^3 f_1(3k+9) + f_1(3k+12)
\end{cases}
$$



u(3k+10)
u(3k+11)
u(3k+12)
a^2
a
D
a^3
2D
a^3
f_1(3k+12)
f_2(3k+12)
3D
a^12
3D
4D
y(3k)
u(3k+1)
y(3k+1)
u(3k+2)
y(3k+2)

$$\because M = 4 \,\&\, N = 1$$

$\Rightarrow$ **4 poles :** $a^3, -a^3, ja^3, -ja^3$

$$\because L = 3$$

$\Rightarrow$ **pole distance :** $|a|^3$

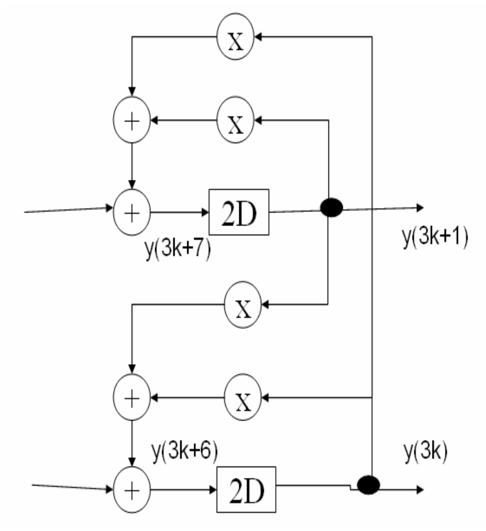
The multiplication complexity :

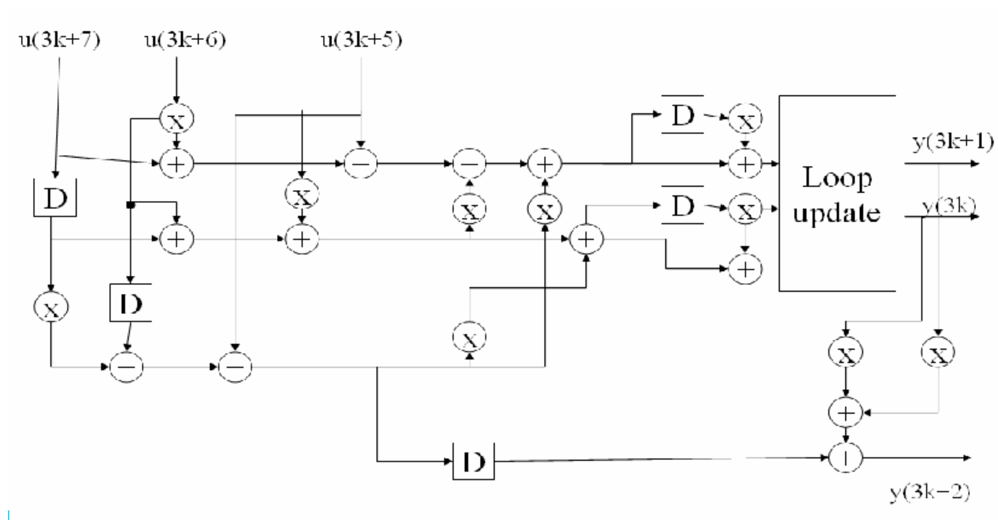
$$(L-1) + \log_2 M + 1 + (L-1) = 2L - 1 + \log_2 M$$

Combined Pipelining and Parallel Processing For IIR Filters

- revisit the 2nd – order IIR filter

$$H(z) = \frac{\left(1 + z^{-1}\right)^2}{1 - \frac{5}{4} z^{-1} + \frac{3}{8} z^{-2}}$$





loop update

L = 3 and M = 2

$$Y(3k+6) = A \bullet \left(Y(3k+3) + F_2\right) = A \bullet \left(A \bullet \left(Y(3k) + F_1\right)\right) = A^2 Y(3k) + F$$

where $$Y(3k) = \begin{pmatrix} y(3k) & y(3k+1) \end{pmatrix}^T$$

The eigenvalues of A^2 **:** $\left(\frac{1}{2}\right)^6, \left(\frac{3}{4}\right)^6$

$\Rightarrow$ **Take the square root :** $\left(\frac{1}{2}\right)^3, -\left(\frac{1}{2}\right)^3, \left(\frac{3}{4}\right)^3, -\left(\frac{3}{4}\right)^3$

Case Analysis

- Example: 4-th order Chebyshev low-pass filter with M =4

$$H(z) = \frac{A(1+z^{-1})^4}{(1+Bz^{-1}+Cz^{-2})(1+Dz^{-1}+Ez^{-2})}$$

simple model:

$$T = \frac{V \bullet Ccharge}{k \bullet (V - Vth)^2}$$

$$P = Ctotal \bullet V^2 \bullet f$$

result (if using V=5 and Vth=1): V' = 2.38

power ratio = 58.91%

Case Analysis

- Example: 2nd order IIR filter with L =3

$$H(z) = \frac{\left(1 + z^{-1}\right)^2}{1 - \frac{5}{4} z^{-1} + \frac{3}{8} z^{-2}}$$

result (if using V=5 and Vth=1): V' = 2.3365

power ratio = 29.116%