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1 Jacobian of a Transformation

Let transformation $(V, W) = g(x, y)$ be expressed as

$$\begin{cases} \mathrm{V} = \mathrm{g}_1(x, y) \\ \mathrm{W} = \mathrm{g}_2(x, y) \end{cases}$$

The Jacobian $J(x, y)$ of a transformation g is defined as

$$J(x,y) = \begin{bmatrix} \delta V/\delta x & \delta V/\delta y \\ \delta W/\delta x & \delta W/\delta y \end{bmatrix}$$

1) Jacobian $J(x, y)$ is a functioin of (x, y)

2) When the transformation is linear

$$\begin{bmatrix} \mathrm{V} \\ \mathrm{W} \end{bmatrix} = \mathrm{M} \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \end{bmatrix}$$

for some 2x2 matrix M. $J(x, y)$ is independent of (x, y). i.e. a constant.

3) If $V = g(x)$, then $J(x) = dv/dx$ or $(dg(x)/dx)$

4) If $J(x, y)$ is the jacobian of g, and $J(V, W)$ is the jacobian of g^{-1}, then

$|\mathrm{J(x,y)}| = |1/\mathrm{J(V,W)}|_{(V,W)=g(x,y)}$

It can be shown $f_{VW}(v, w) = \frac{f_{XY}(x,y)}{|J(x,y)|}|_{(x,y)=g^{-1}(v,w)}$

**Recall when $Y = g(X)$, then

$f_Y(y) = \frac{f_X(x)}{|dy/dx|}|_{x=g^{-1}(y)}$

2 Correspondence

Ex. *L*et X and Y be independent zero-mean unit variance gaussian RV's.

Let

$$\begin{cases} \text{V} = \sqrt{X^2 + Y^2} \\ \text{W} = \angle(x, y) \end{cases}$$

Find $f_{VW}(x, y)$

Solution: The inverse transform is

$$\begin{cases} x = v\cos w \\ y = v\sin w \end{cases}$$

$w \in [0, 2\pi]$, $v \in [0, +\infty]$

$$J(v, w) = \det \begin{bmatrix} \frac{\delta x}{\delta w} & \frac{\delta x}{\delta w} \\ \frac{\delta y}{\delta v} & \frac{\delta y}{\delta w} \end{bmatrix} = det \begin{bmatrix} \cos w, & -v \sin w \\ \sin w, & v \cos w \end{bmatrix}$$

$= v \cos^2 w + v \sin^2 w = \mathbf{v}$

$f_{VW}(v, w) = f_{XY}(v \cos w, v \sin w) \cdot |J(v, w)|$

$= f_x(v \cos w) \cdot f_y(v \sin w) \cdot v = (\frac{1}{\sqrt{2\pi}} e^{[\frac{-v^2 \cos^2 w}{2}]})(\frac{1}{\sqrt{2\pi}} e^{[\frac{-v^2 \sin^2 w}{2}]}) \cdot v$

$= \frac{v}{2\pi} e^{\frac{-v^2}{2}}$

*This is called **Rayleigh** distribuion.

The fact that $f_{VW}(v, w)$ is independent of w suggests
$f_W(w) = \frac{1}{2\pi}$, $w \in [0, 2\pi)$

3 Expected value of function of several Random Variables

Let $X_1, X_2 \ldots X_n$ be a set of random variables with joint distribution

$Z := g(X_1, X_2 \ldots, X_n)$

$E[Z] = \int \int \int\ g(x_1, x_2 \ldots x_n) f_{X1,X2} \ldots,_{Xn}(x_1, x_2 \ldots x_n) dx_1 dx_2 \ldots dx_n$

If Z=X+Y

E[Z]=E[X]+E[Y] ?

verify:

E[Z]= $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)f_{XY}(x,y)dxdy$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf_{XY}(x,y)dxdy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf_{XY}(x,y)dxdy$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf_{XY}(x,y)dxdy$

$= \int_{-\infty}^{\infty} x\int_{-\infty}^{\infty} f_{XY}(x,y)dy$

$= \int_{-\infty}^{\infty} xf_X(x)dx$

= E[X]

similarly $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf_{XY}(x,y)dxdy$

=E[Y]

We are interested in determining the distribution of $(X_1 \dots X_n)$ and function g.

If X and Y are independent, then

E[g$_1(X)g_2(Y)] = E[g_1(X)]E[g_2(Y)]$

4 Correlation

For any random variables X and Y

The Correlation of X and Y is defined as:

$E(XY)$

IF X,Y are independent then

E(XY)=E(X)E(Y)

When E(XY)=0 (or the correlation between X and Y are said to be Orthogonal)

5 Covariance

The Cov(X,Y) of random variable's X and Y is defined as

$\rightarrow Cov(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x - E(x)(y - E(y))f_{XY}(x,y)dxdy$

=E([x-E(x)][y-E(y)])

$\rightarrow Cov[X + \alpha, Y + \beta] = Cov(X,Y)\forall\alpha,\beta$

When Cov(X,Y)=0, X and Y are uncorrelated

If X,Y are independent ,then it's uncorrelated

Cov(X,Y)=E(XY)-E(X)E(Y)

If Y is a deterministic random varialbe Y=a with probability 1

Then Cov(X,Y)=Var(X)

Correlation Coefficient:

$\rho_{XY} = Cov(X,Y)/\sqrt{VAR(X)}\sqrt{VAR(Y)}$

Cov(X,Y)=0, then $\rho_{XY} = 0, -1 \leq \rho_{XY} \leq\ 1$