Lectures of May 26th, 2006
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Example 1 *Let X be a continuous random variable with PDF $f_X(x)$. Let the random variable $Y = X^2$. Find PDF of Y, in terms of $f_X(x)$.*

Answer:

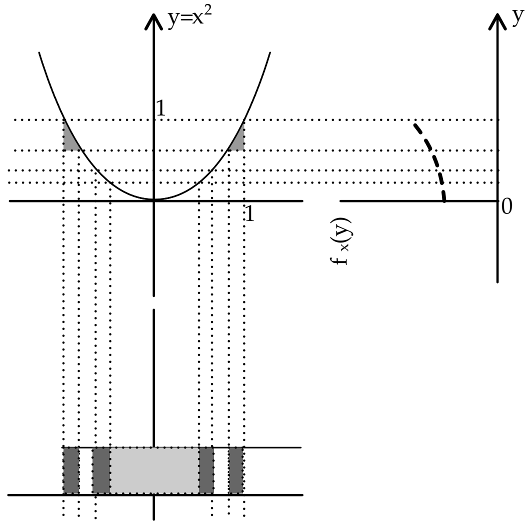

Figure 1: Intuitive Approach

$$F_Y(y)\text{=P}[Y \leq y]\text{=P}[x^2 \leq y]$$

$$F_Y(y) = \begin{cases} P[-\sqrt{y} \leq x \leq \sqrt{y}], & \text{if } y \geq 0, \\ 0, & \text{if } y < 0. \end{cases}$$

CDF: if $y \geq 0$, $F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$

PDF: $f_Y(y) = F'_Y(y)$

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 0, \\ f_X(\sqrt{y})\frac{1}{2\sqrt{y}} - f_X(-\sqrt{y})\frac{-1}{2\sqrt{y}}, & \text{if } y \geq 0. \end{cases}$$

$$F_Y(y) = \begin{cases} 0, & \text{if } y < 0, \\ \frac{1}{2\sqrt{y}}[f_X(\sqrt{y}) + f_X(-\sqrt{y})], & \text{if } y \geq 0. \end{cases}$$

Now, consider the general case: Given f_X of a continuous RV X, determine the PDF f_Y of $Y{:=}g(x)$, where g is a smooth function.

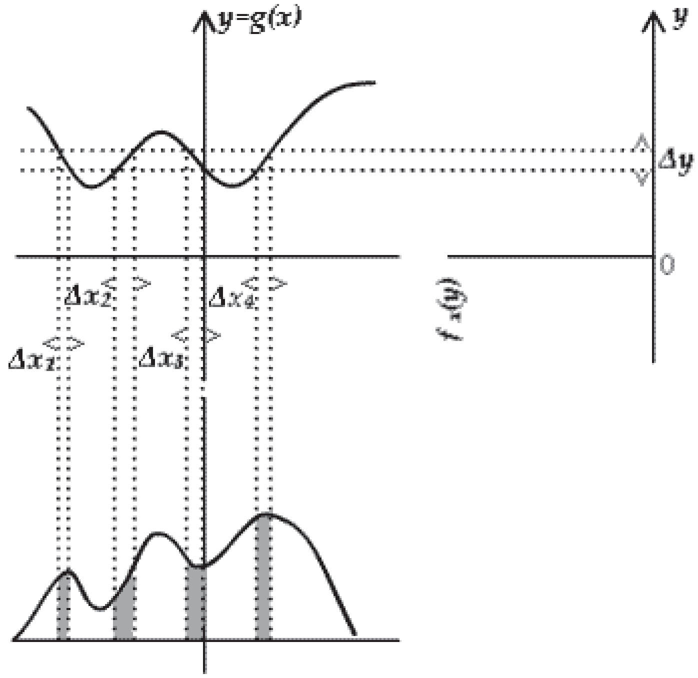

Figure 2: $y = g(x)$

Suppose x_1, x_2, ..., x_k are all solutions to equations $y = g(x)$, for a given value y.

$f_Y(y)|\Delta y| \approx f_X(x_1)|\Delta x_1| + f_X(x_2)|\Delta x_2| + ... + f_X(x_k)|\Delta x_k|$

$f_Y(y) \approx \sum_{i=1}^{k} f_X(x_i)\frac{|\Delta x_i|}{|\Delta y|}$

In the limit, when $\Delta y \to 0$, we have

$f_Y(y) = \sum_{i=1}^{k} [f_X(x_i)\frac{|dx|}{|dy|}]_{x=x_i}$

Example 2 *Let X be uniformly distributed over (0,2π). Let $Y = cos(X)$. find the PDF of Y.*

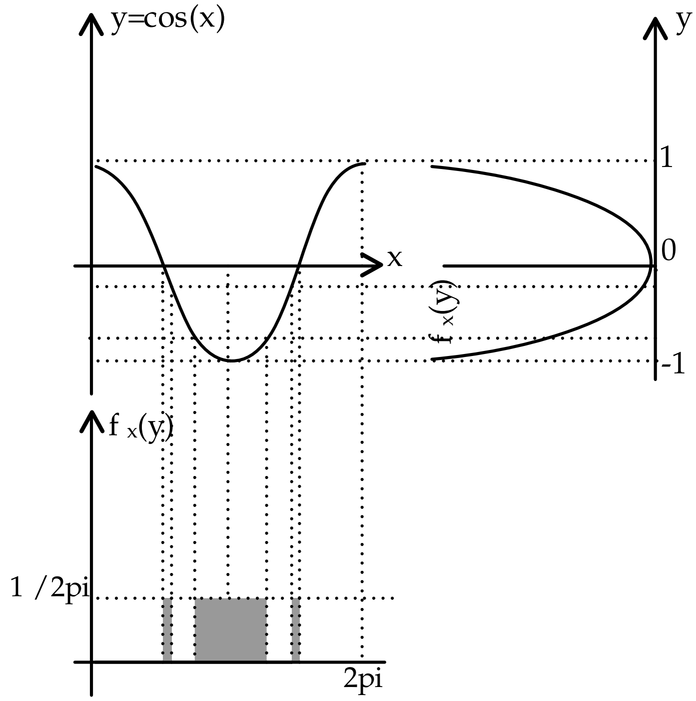

Figure 3: $y = cos(x)$

Answer:

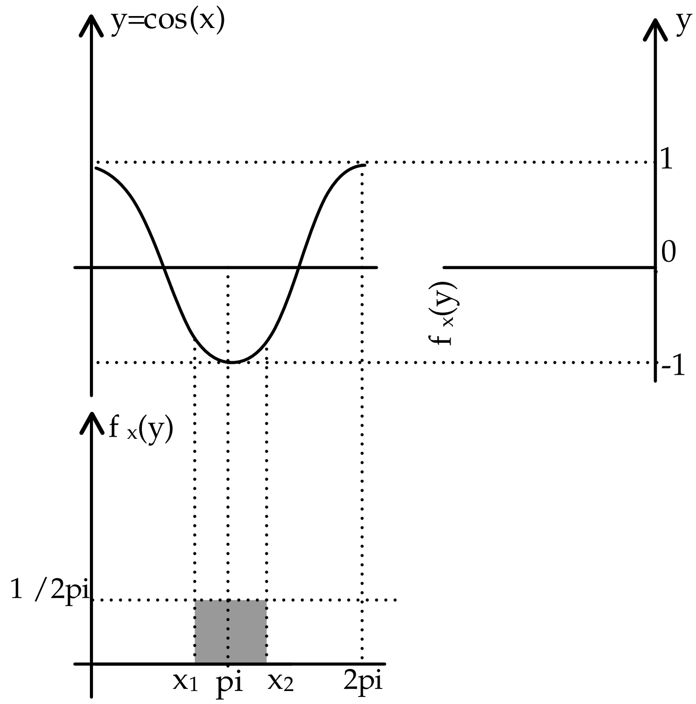

Figure 4: $y = cos(x)$

For every $y \in (-1,1)$ denotes by x_1 and x_2 the solution of equation $y = cosx$ and $x_1 = cos^{-1}(y)$ (where we are treating $cos^{-1}(.)$ as function with range [0,π]).
$\frac{dy}{dx} = -sin(x)$, then $|\frac{dx}{dy}| = \frac{1}{|sin(x)|}$

$$f_Y(y) = \left[f_X(x)\frac{1}{|sin(x)|}\right]_{x=x_1} + \left[f_X(x)\frac{1}{|sin(x)|}\right]_{x=x_2}$$

$$= \frac{1}{2\pi}\frac{1}{|sin(cos^{-1}(y))|} + \frac{1}{2\pi}\frac{1}{|\sin(2\pi - cos^{-1}(y))|}$$

$$= \frac{1}{2\pi}\frac{1}{\sqrt{1-y^2}} + \frac{1}{2\pi}\frac{1}{\sqrt{1-y^2}}$$

$$= \frac{1}{\pi}\frac{1}{\sqrt{1-y^2}}$$

1 Expected value of $Y = g(X)$

Suppose that X is a random variable and Y is defined as $Y = g(X)$ for some g.

$E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_{-\infty}^{\infty} g(x) f_X(x) dx$

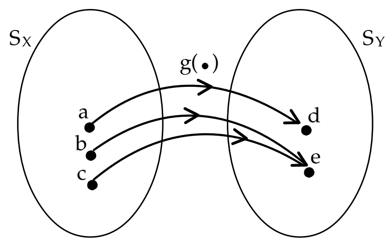

Figure 5: Y:=$g(x)$

According to the figure above,
$E[y] = P[d] \cdot d + P[e] \cdot e$
Proof:
$P[a] \cdot g[a] + P[b] \cdot g[b] + P[c] \cdot g[c]$
$= P[a] \cdot d + P[b] \cdot e + P[c] \cdot e$
$= P[d] \cdot d + e[P[b] + P[c]]$
$= P[d] \cdot d + P[e] \cdot e$ $\blacksquare$

2 Two inequalities

1. **Markov Inequality**

Suppose X is a non-negative random variable, then for any $a \geq 0$,

$P[X \geq a] \leq \frac{E[X]}{a}$
(Markov inequality provides a bound on the right tail of the distribution of a non-negative random variable, based on mean only)

2. **Chebyshev inequality**

For any random variable, $P[|X - E[X]| \geq a] \leq \frac{VAR[X]}{a^2}, \forall a \geq 0$

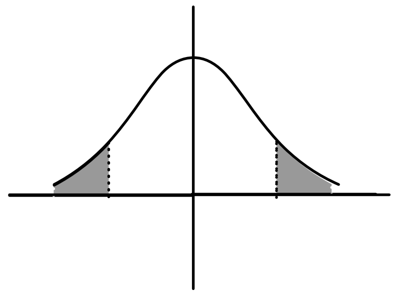

Figure 6: $Y := g(x)$

(Chebyshev Inequality provides a bound on two tails of a distribution, in terms of mean and variance.)