ELG3175 Introduction to Communication Systems

Lecture 12-13

Angle Modulation

Introduction to Angle Modulation

- In angle modulation, the amplitude of the modulated signal remains fixed while the information is carried by the angle of the carrier.
- The process that transforms a message signal into an angle modulated signal is a nonlinear one.
- This makes analysis of these signals more difficult.
- However, their modulation and demodulation are rather simple to implement.

The angle of the carrier

- Let $\theta_i(t)$ represent the instantaneous angle of the carrier.
- We express an angle modulated signal by:

$$s(t) = A_c \cos(\theta_i(t))$$

where A_c is the carrier amplitude.

Instantaneous frequency

- One cycle occurs when $\theta_i(t)$ changes by 2π radians, therefore the average frequency of $s(t)$ on the interval t to $t+\Delta t$ is:

$$f_{\Delta t} = \frac{\theta_i(t+\Delta t) - \theta_i(t)}{2\pi\Delta t}$$

- Therefore the instantaneous frequency is found in the limit as Δt tends towards 0.

$$f_i(t) = \frac{1}{2\pi}\frac{d\theta_i(t)}{dt}$$

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Phase modulation

- There are two angle modulation techniques.
  - Phase modulation (PM)
  - Frequency modulation (FM)
- In PM, the phase of the carrier is a linear function of the message signal, $m(t)$. Therefore $s_{PM}(t)$ is:

$$s_{PM}(t) = A_c \cos\left(2\pi f_c t + k_p m(t) + \phi_c\right)$$

where k_p is the phase sensitivity and ϕ_c is the phase of the unmodulated carrier.

- To simplify expressions, we will assume that $\phi_c = 0$. Therefore the angle of a PM signal is given by $\theta_i(t) = 2\pi f_c t + k_p m(t)$.

FM

- For FM, the instantaneous frequency is a linear function of the message:

$$f_i(t) = f_c + k_f m(t)$$

- where k_f is the frequency sensitivity.

$$\theta_i(t) = 2\pi \int_{-\infty}^{t} f_i(\tau)d\tau = 2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau$$

$$s_{FM}(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\right]$$

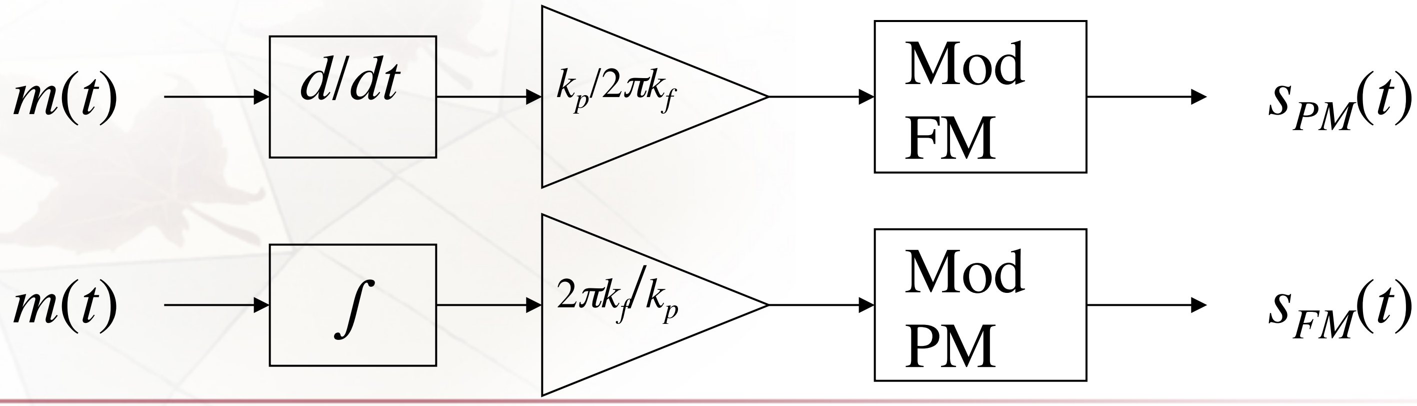
Instantaneous frequency of a PM signal / Instantaneous phase of an FM signal

- From $s_{PM}(t)$, we find

$$f_i(t)_{PM} = f_c + \frac{k_p}{2\pi}\frac{dm(t)}{dt}$$

From $s_{FM}(t)$, we find

$$\phi_i(t)_{FM} = 2\pi k_f \int_0^t m(\tau)d\tau$$



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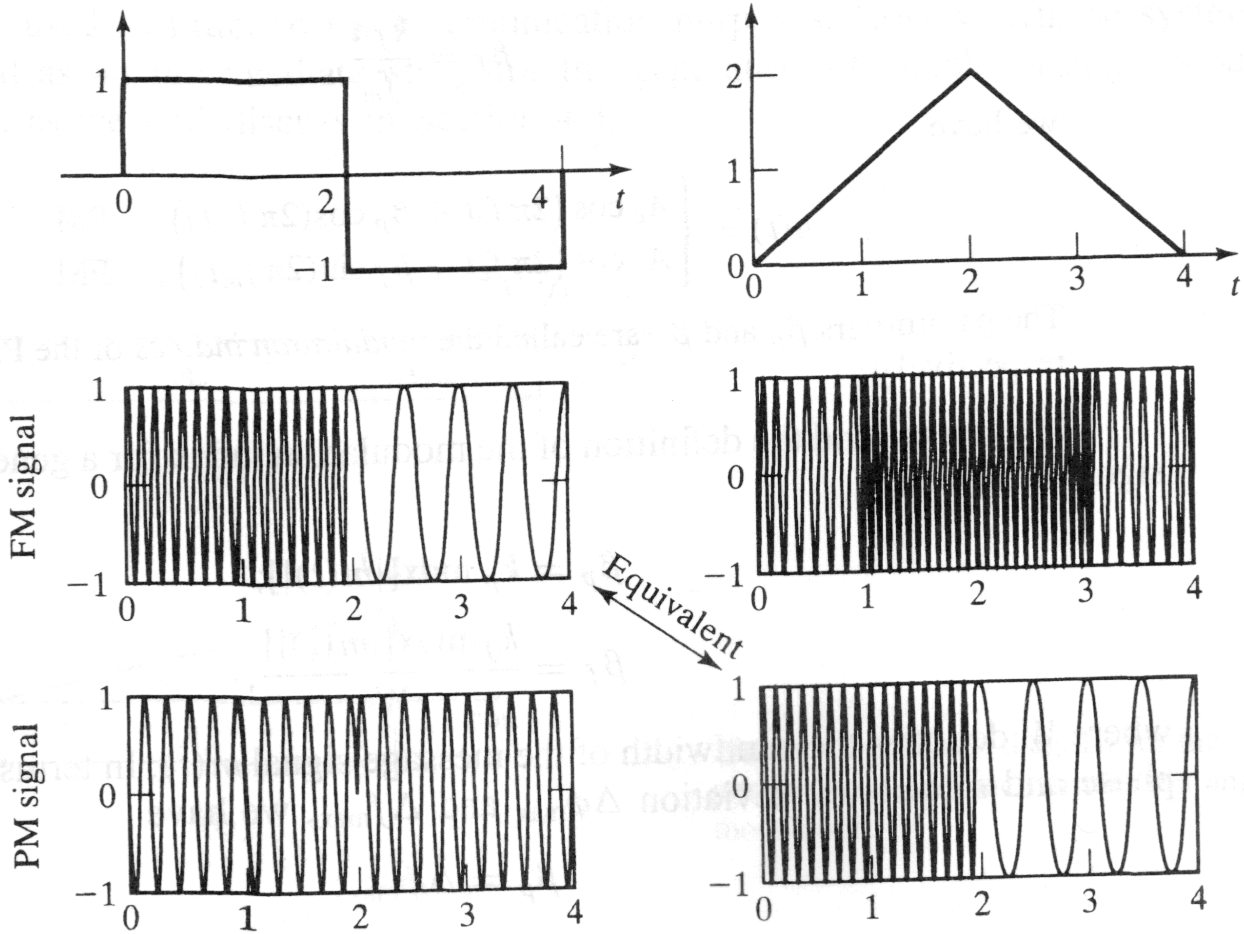


Figure 4.2 Frequency and phase modulation of square and sawtooth waves.

Example

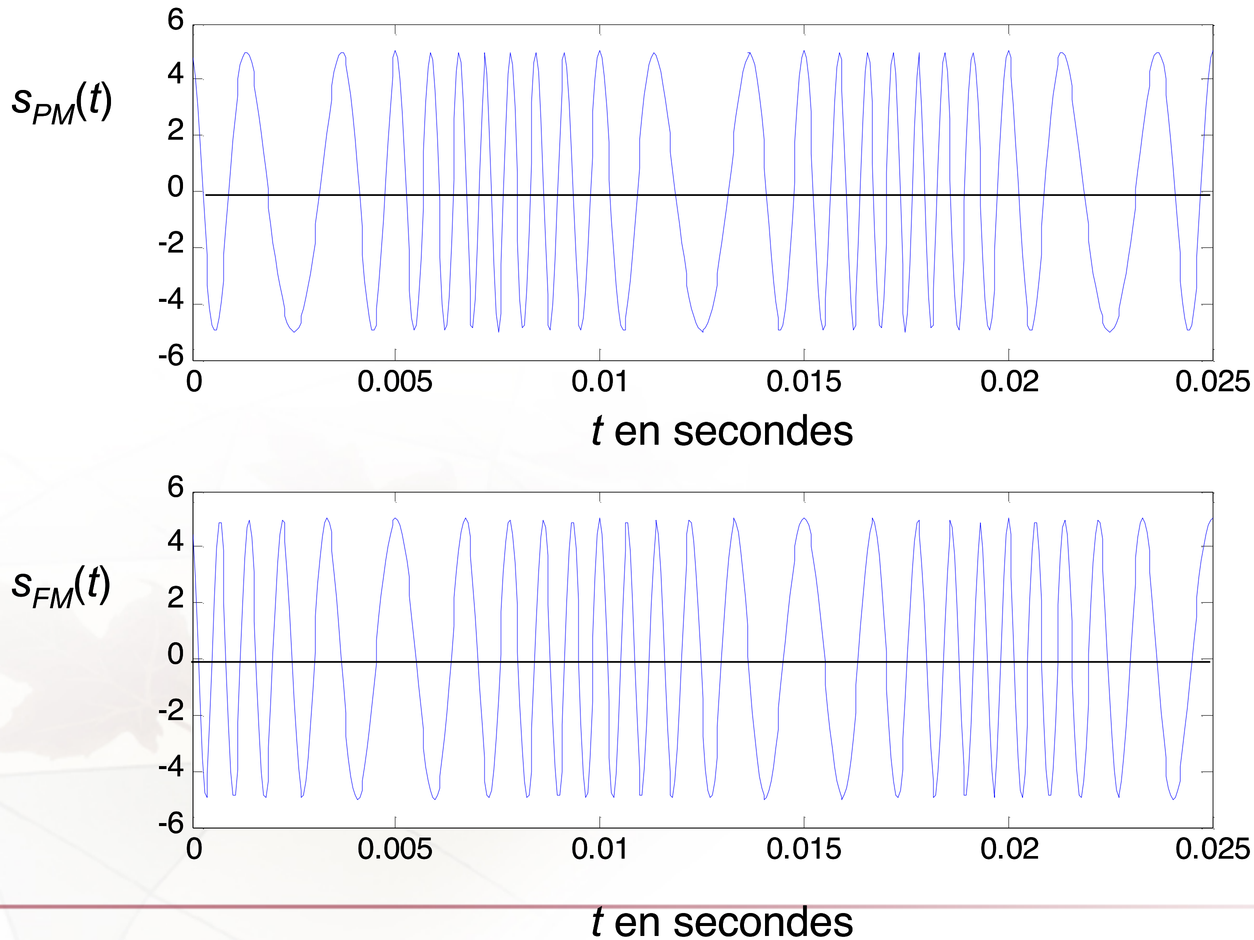
- Find $s_{FM}(t)$ and $s_{PM}(t)$ if $m(t) = A\cos(2\pi f_m t)$.
  - SOLUTION

$$s_{PM}(t) = A_c \cos\left[2\pi f_c t + Ak_p \cos\left(2\pi f_m t\right)\right]$$

$$\int_{-\infty}^{t} A\cos\left(2\pi f_m \tau\right) d\tau = \frac{A}{2\pi f_m}\sin\left(2\pi f_m t\right)$$

$$s_{FM}(t) = A_c \cos\left[2\pi f_c t + \frac{Ak_f}{f_m}\sin\left(2\pi f_m t\right)\right]$$

- The PM and FM of the example are shown here for $A_c = 5$, $A = 1$, $f_c = 1$ kHz, $f_m = 100$ Hz, $k_p = 2\pi$ rads/V and $k_f = 500$ Hz/V.



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Characteristics of Angle Modulated Signals

	PM Signal	FM Signal
Instantaneous phase $\phi_i(t)$	$k_p m(t)$	$2\pi k_f \int_0^t m(\tau)d\tau$
Instantaneous frequency	$f_c + \frac{k_p}{2\pi}\frac{dm(t)}{dt}$	$f_c + k_f m(t)$
Maximum phase deviation $\Delta\phi_{max}$	$k_p \lvert m(t)\rvert_{\max}$	$2\pi k_f \lvert x(t)\rvert_{\max}$ où $x(t) = \int_{-\infty}^{t} m(\tau)d\tau$
Maximum frequency deviation Δf_{max}	$\frac{k_p}{2\pi}\lvert x(t)\rvert_{\max}$ où $x(t) = \frac{dm(t)}{dt}$	$k_f \lvert m(t)\rvert_{\max}$
Power	$\frac{A_c^2}{2}$	$\frac{A_c^2}{2}$

Modulation index

- Assume that $m(t) = A_m \cos(2\pi f_m t)$. The resulting FM signals is:

$$s_{FM}(t) \quad = \quad A_c \cos\left(2\pi f_c t + \frac{A_m k_f}{f_m} \sin(2\pi f_m t)\right)$$

- For the FM signal

$$\beta_F = \frac{k_f A_m}{f_m} = \frac{\Delta f_{\max}}{f_m}$$

FM Modulation index

- For any $m(t)$ which has bandwidth B_m, we define the modulation index as :

$$\beta_F = \frac{k_f \, |m(t)|_{\max}}{B_m} = \frac{\Delta f_{\max}}{B_m}$$

Example

- The signal $m(t) = 5\text{sinc}^2(10t)$.

Find the modulation index for FM modulation with $k_f = 20$ Hz/V.

  - SOLUTION

- $B_m = 10$Hz, therefore $\beta_F = 20\times5/10 = 10$.

Narrowband FM

- Consider an FM signal :

$$s_{FM}(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\right]$$

$$\text{where} \quad \left|2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\right| << 1$$

- We say that $s_{FM}(t)$ is a narrowband FM signal.
- For example, consider when $m(t) = A_m \cos(2\pi f_m t)$.

$$s_{FM}(t) = A_c \cos\left(2\pi f_c t + \frac{A_m k_f}{f_m}\sin(2\pi f_m t)\right)$$

$$s_{FM}(t) = A_c \cos\left(2\pi f_c t + \beta_F \sin(2\pi f_m t)\right)$$

Narrowband FM

- When β_F << 1, the FM signal is NBFM.
- cos(A+B) = cos(A)cos(B)-sin(A)sin(B). Therefore

$$
\begin{aligned}
s_{FM}(t) &= A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right] \\
&= A_c \cos(2\pi f_c t)\cos\left(2\pi k_f \int_0^t m(\tau)d\tau\right) - A_c \sin(2\pi f_c t)\sin\left(2\pi k_f \int_0^t m(\tau)d\tau\right) \\
&\approx A_c \cos(2\pi f_c t) - A_c\left(2\pi k_f \int_0^t m(\tau)d\tau\right)\sin(2\pi f_c t)
\end{aligned}
$$

$$(\text{if } A << 1, \cos(A) \approx 1 \text{ and } \sin(A) \approx A.)$$

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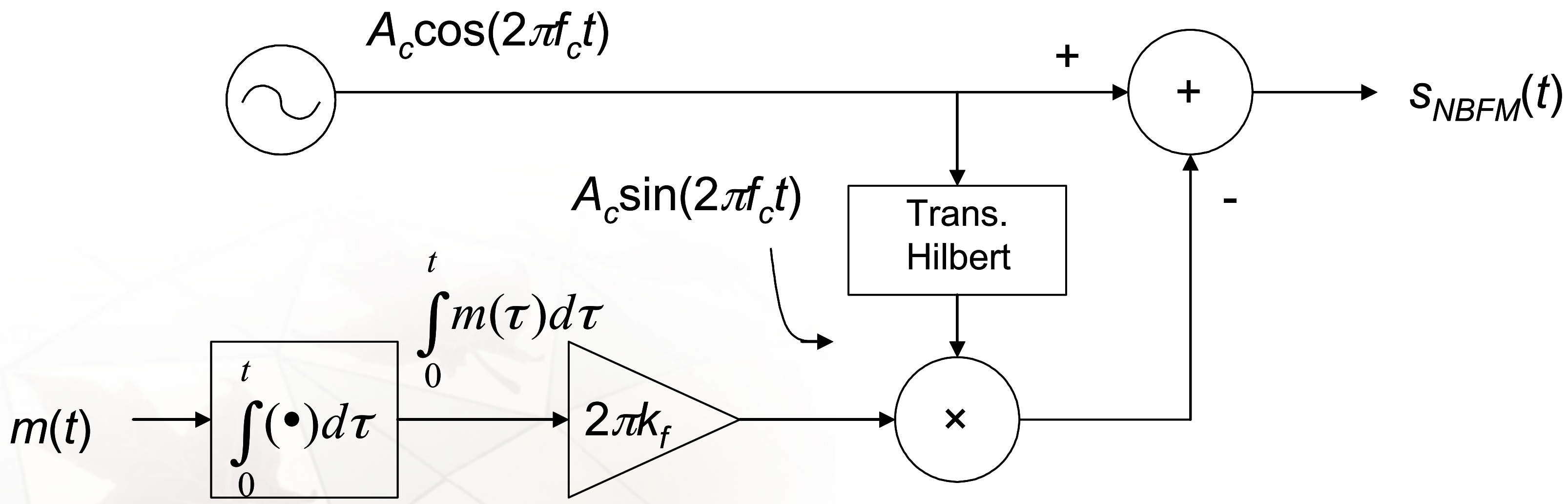
NBFM Modulator

Bandwidth of NBFM approx. = $2B_m$



Wideband FM - WBFM

- For an FM signal to be NBFM, $\beta_F << 1$.
- Any signal that is not narrowband is therefore wideband.
- However, typically $\beta_F > 1$ for an FM signal to be considered wideband.
- The bandwidth of WBFM signals is larger than NBFM since Δf_{max} is increased.

The Fourier series of the WBFM signal when $m(t) = A_m \cos 2\pi f_m t$.

- We can express the complex envelope of the WBFM signal using Bessel functions of first kind and order n as

$$\tilde{s}_{FM}(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta_F) e^{j2\pi n f_m t}$$

- And the WBFM signal itself becomes:

$$\begin{aligned} s_{FM}(t) &= \mathrm{Re}\{\tilde{s}_{FM}(t) e^{j2\pi f_c t}\} \\ &= \mathrm{Re}\left\{ \sum_{n=-\infty}^{\infty} A_c J_n(\beta_F) e^{j(2\pi f_c t + 2\pi n f_m t)} \right\} \\ &= \sum_{n=-\infty}^{\infty} A_c J_n(\beta_F) \cos(2\pi(f_c + n f_m)t) \end{aligned}$$

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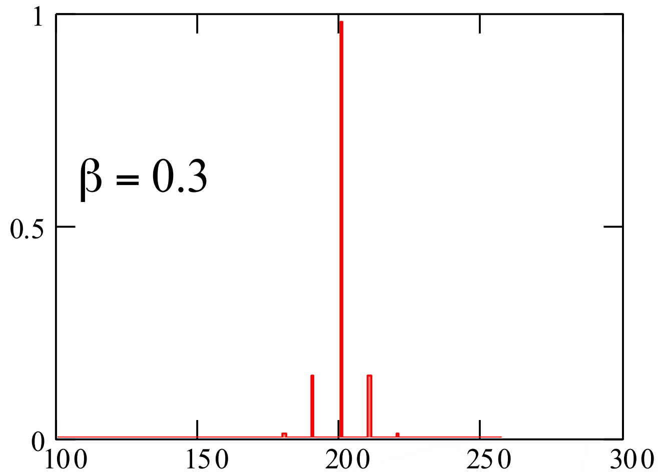
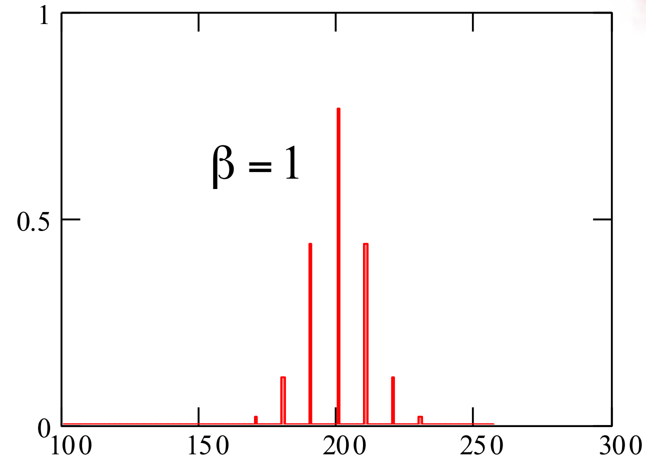
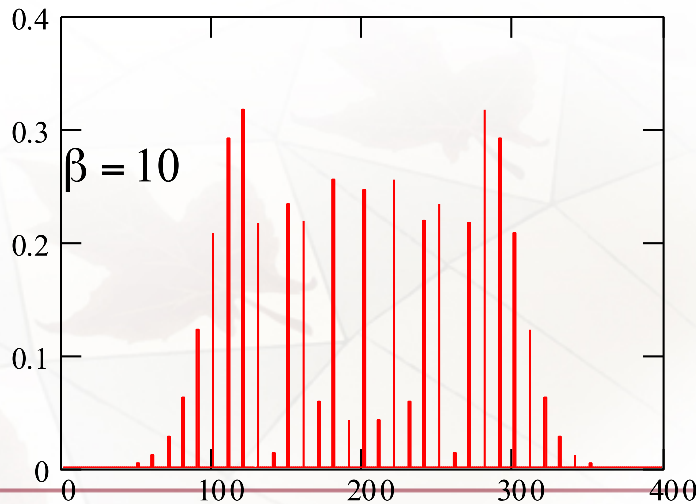
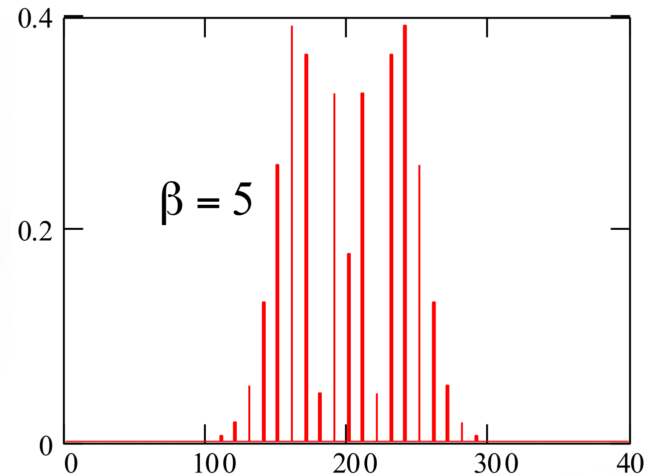
Spectrum: Examples









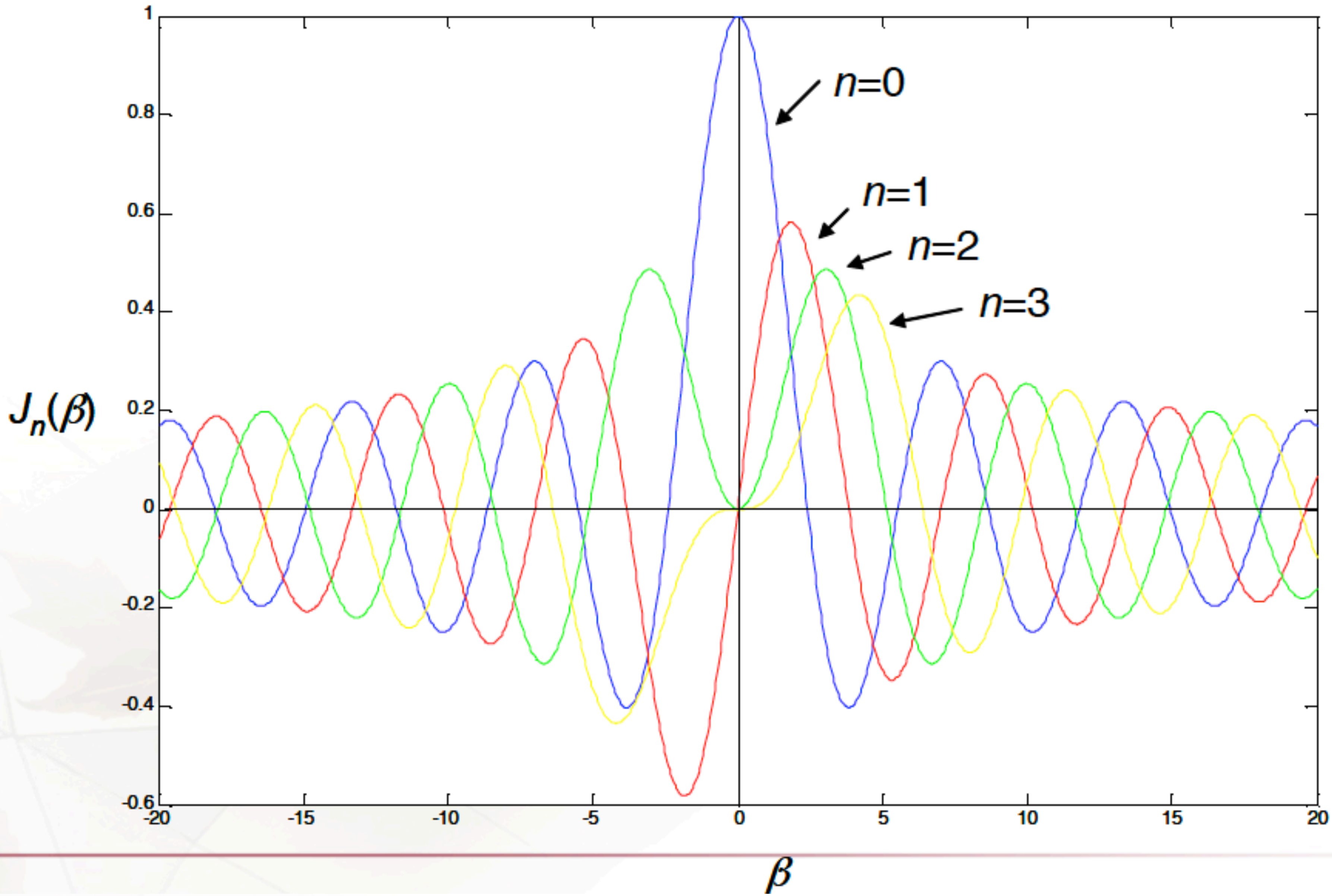
Spectrum of the WBFM signal when $m(t) = A_m \cos 2\pi f_m t$.

- The spectrum of this signal is:

$$S_{FM}(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta_F)\left[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)\right]$$

- This expression shows that the FM signal's spectrum is made up of an infinite number of impulses at frequencies $f = f_c + nf_m$.
- Therefore, theoretically, this WBFM signal has infinite bandwidth.
- However, the properties of the Bessel function show that most of these impulses contribute little to the overall power of the signal and are negligible.
  - We define the practical bandwidth as the range of frequencies which contains at least 99% of the total power of the WBFM signal.

The function $J_n(\beta)$



Properties of $J_n(\beta)$

1) If n is an integer :

$J_n(\beta) = J_{-n}(\beta)$ for even n

and

$J_n(\beta) = -J_{-n}(\beta)$ for odd n

2) when $\beta << 1$

$J_0(\beta) \approx 1$

$J_1(\beta) \approx \beta/2$

and

$J_n(\beta) \approx 0,\ n > 1$

4) $Im\{J_n(\beta)\}=0$

3) $$\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$$

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Power of the FM signal

- The power of an FM signal is:

$$P_{FM} = \frac{A_c^2}{2}$$

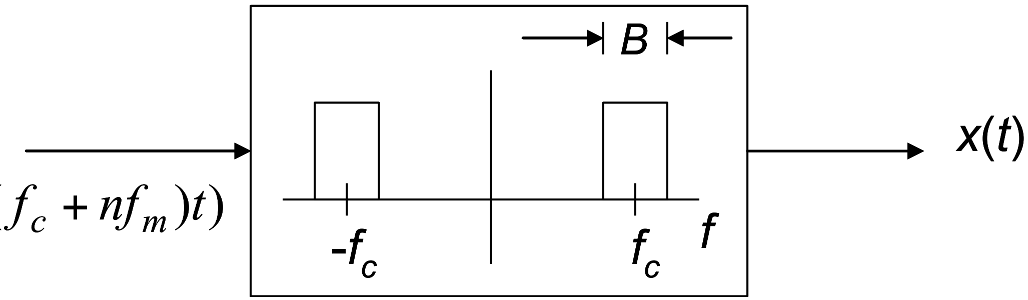
$$s_{FM}(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta_F) \cos(2\pi(f_c + nf_m)t)$$

- The power of the above expression is:

$$P = \frac{A_c^2}{2} \sum_{n=-\infty}^{\infty} J_n^2(\beta_F)$$

Filtering a WBFM signal to limit its bandwidth.

$$s_{FM}(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta_F)\cos(2\pi(f_c + nf_m)t)$$



We want to choose B so that the power of $x(t)$
Is at least 0.99× the power of $s_{FM}(t)$.

$$x(t) = \sum_{n=-X}^{X} A_c J_n(\beta_F)\cos(2\pi(f_c + nf_m)t)$$

where X is the largest integer that satisfies :

$$f_c + Xf_m \le f_c + \frac{B}{2} \quad \text{and} \quad f_c - Xf_m \ge f_c - \frac{B}{2}$$

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- The power of $x(t)$ is:

$$P_x = \frac{A_c^2}{2} \sum_{n=-X}^{X} J_n^2(\beta_F)$$

- Therefore we must choose X so that:

$$\sum_{n=-X}^{X} J_n^2(\beta_F) \geq 0.99$$

- We know that $J_n^2(\beta_F) = J_{-n}^2(\beta_F)$. Therefore

$$J_0^2(\beta_F) + 2\sum_{n=1}^{X} J_n^2(\beta_F) \geq 0.99$$

Values of $J_n(\beta)$

n	β=0.1	β=0.2	β=0.5	β=1	β=2	β=3	β=5	β=10
0	0.997	0.99	0.938	0.765	0.224	-0.2601	-0.178	-0.246
1	0.05	0.1	0.242	0.44	0.577	0.3391	-0.323	0.043
2	0.001	0.005	0.031	0.115	0.353	0.4861	0.047	0.255
3	$2\times10^{-5}\approx0$	1.6×10^{-4}	0.0026	0.02	0.129	0.3091	0.365	0.058
4				0.002	0.034	0.1320	0.391	-0.220
5					0.007	0.0430	0.261	-0.234
6					0.001	0.0114	0.131	-0.014
7						0.0025	0.053	0.217
8							0.018	0.318
9							0.006	0.292
10							0.001	0.207
11								0.123
12								0.063
13								0.029

Example

- The signal $m(t) = A_m\cos(2\pi f_m t)$ is to be transmitted using FM techniques. Find the practical bandwidth if

  (a) A_m = 5V, f_m = 20 Hz and k_f = 4 Hz/V

  (b) A_m = 10V, f_m = 400 Hz and k_f = 200 Hz/V.
- SOLUTION

  (a) IN this example, β_F = (5)(4)/(20) = 1. We need to find X so that S = $J_0^2(\beta_F) + 2\sum_{n=1}^{X} J_n^2(\beta_F) \geq 0.99$.
- From the table, if X = 1, S = $(0.765^2+2\times0.44^2)$ = 0.9648. If X = 2, S = $0.9648+2\times0.115^2$ = 0.9912. Therefore X = 2 and $B = 4f_m$.

  (b) Here, β_F = (10)(200)/(400) = 5. We can show that X = 6 yields S = 0.994. Therefore $B = 12f_m$.

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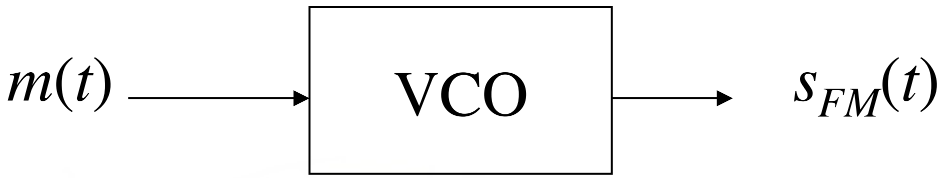
Carson's Rule

- For $m(t) = A_m\cos(2\pi f_m t)$, When β is an integer, we always find that $X = \beta+1$.
- Therefore we can estimate that the practical bandwidth of an FM signal is $B = 2(\beta_F+1)f_m$.
- For any random $m(t)$ with maximum value A_m and bandwidth B_m, the true bandwidth is difficult to find.
- According to Carson, the worst case is when the spectrum of $m(t)$ is concentrated around $f = B_m$ (such as a sinusoid).
- Based on experiments by Carson, the bandwidth of a WBFM signal, B_{FM}, can be estimated by

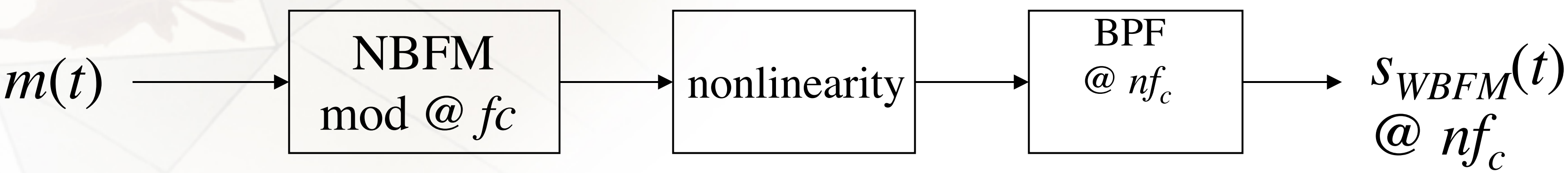
$$B_{FM} = 2(\beta_F + 1)B_m \quad (***)$$

Generation of WBFM Signals

- Direct method
  - Voltage Controlled Oscillator (VCO)



- Indirect method
  - Armstrong's method



Armstrong's method

- Nonlinearity
  - $v_o = a_1 v_i + a_2 v_i^2 + a_3 v_i^3 + \ldots$
  - $v_i(t) = s_{NBFM}(t)$.
  - Let $s_{NBFM}(t) = A_c \cos(2\pi f_c t + 2\pi k_f \int m(t)dt) = A_c \cos(\theta_i(t))$.
  - $v_o(t) = a_1 s_{NBFM}(t) + a_2 s^2{}_{NBFM}(t) + a_3 s^3{}_{NBFM}(t) \ldots$
  - $v_o(t) = a_1 A_c \cos(\theta_i(t)) + a_2 A_c^2 \cos^2(\theta_i(t)) + a_3 A_c^3 \cos^3(\theta_i(t)) \ldots$
  - $v_o(t) = a_1 A_c \cos(\theta_i(t)) + a_2 A_c^2/2 + (a_2 A_c^2/2)\cos(2\theta_i(t)) + (3a_3 A_c^3/4)\cos(\theta_i(t)) + (a_3 A_c^3/4)\cos(3\theta_i(t)) \ldots$
  - $n\theta_i(t) = 2\pi(nf_c)t + 2\pi(nk_f)\int m(t)dt$ (carrier frequency = nf_c and $k_f' = nk_f$ therefore $\beta_F' = n\beta_F$).
- BPF is used to pass the spectral component centred @ f = nf_c.

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Demodulation of FM signals

- Differentiator plus envelope detection
- Frequency discriminator.
- Frequency counter.

Differentiator and envelope detector

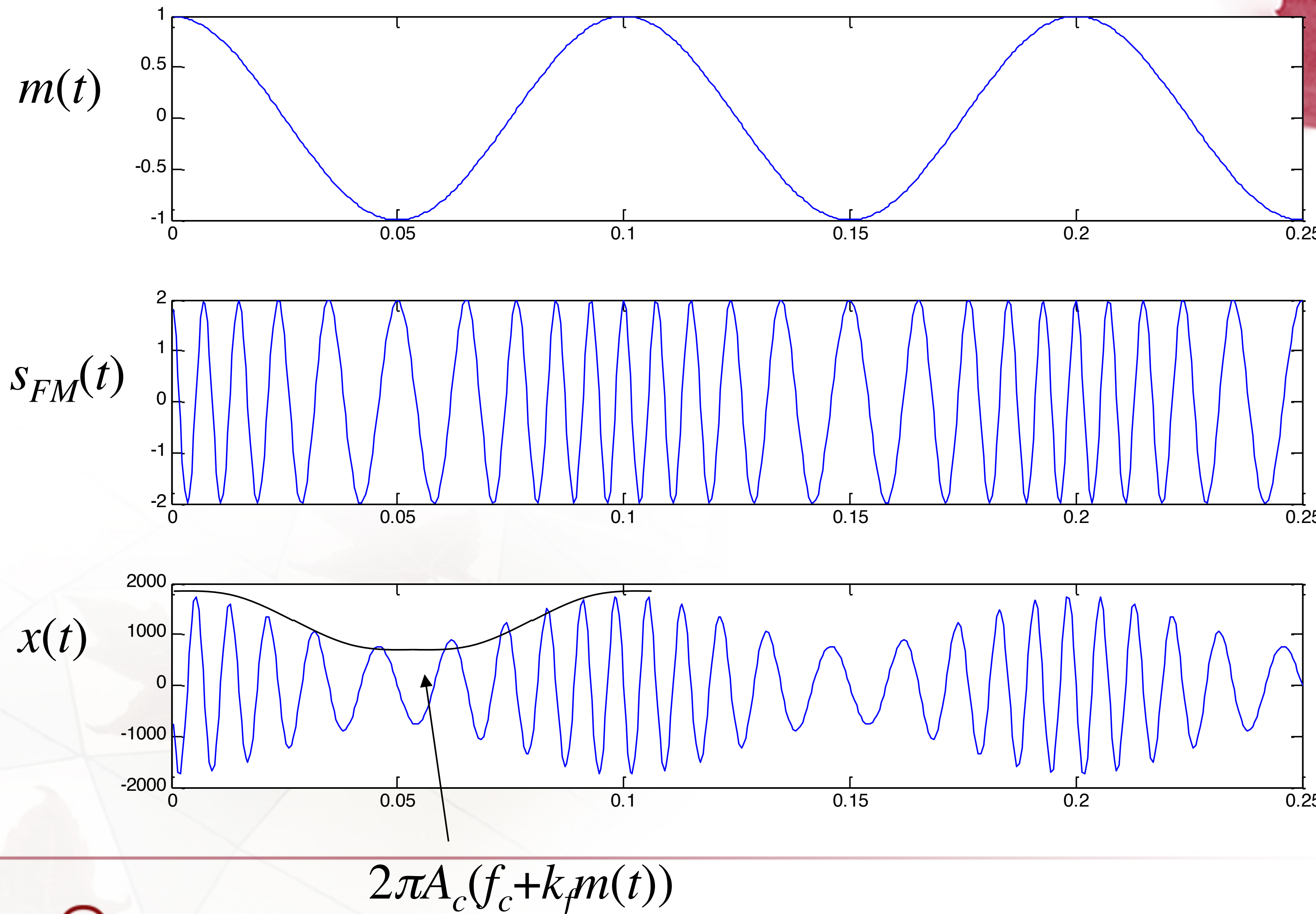
Differentiator and envelope detector

$$
\begin{aligned}
x(t) &= \frac{ds_{FM}(t)}{dt} \\
&= \frac{d}{dt}\left(A_c \cos\left(\theta_i(t)\right)\right) \\
&= -\frac{d\theta_i(t)}{dt} A_c \sin\left(\theta_i(t)\right) \\
&= 2\pi A_c f_i(t) \sin\left(2\pi f_c t + 2\pi k_f \int m(t)dt + \pi\right) \\
&= 2\pi A_c \left(f_c + k_f m(t)\right) \sin\left(2\pi f_c t + 2\pi k_f \int m(t)dt + \pi\right)
\end{aligned}
$$

$f_c >> |k_f m(t)|$ then $2\pi A_c(f_c + k_f m(t)) > 0$.

Example

- $m(t)$ = cos2π10t, f_c = 100, A_c = 2, k_f = 40 Hz/V.
- $s_{FM}(t)$ = 2cos(2π200t+4sin2π10t)
- $x(t)$ = 4π(100+40cos2π10t)sin(2π100t+4sin2π10t+π)



$m(t)$
1
0.5
0
-0.5
-1
0
0.05
0.1
0.15
0.2
0.25
$s_{FM}(t)$
2
1
0
-1
-2
$x(t)$
2000
1000
0
-1000
-2000
$2\pi A_c(f_c+k_f m(t))$

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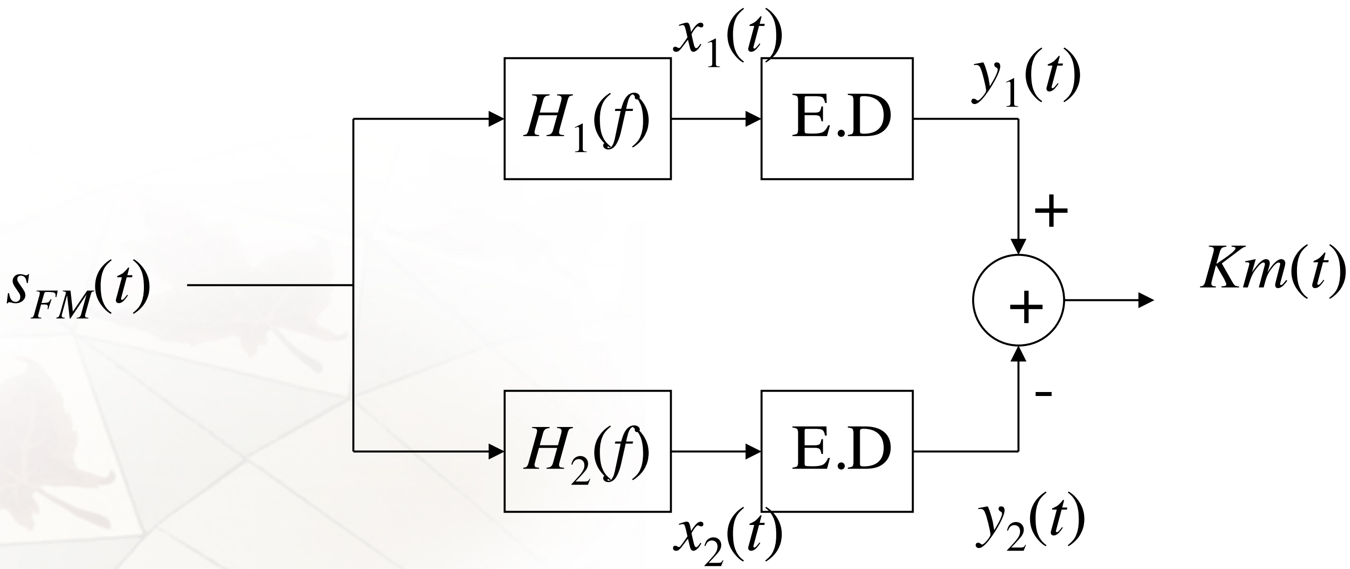
Differentiator and envelope detector

- Output of envelope detector
  - $y(t) = 2\pi A_c(f_c+k_f m(t)) = 2\pi A_c f_c + 2\pi A_c k_f m(t)$
  - Assuming that m(t) has no DC component ($M(f) = 0$ for $f = 0$), then
- Output of DC block
  - $z(t) = 2\pi A_c k_f m(t) = Km(t)$.

Frequency discriminator

- Similar to differentiator
- Input to envelope detector has lower amplitude.



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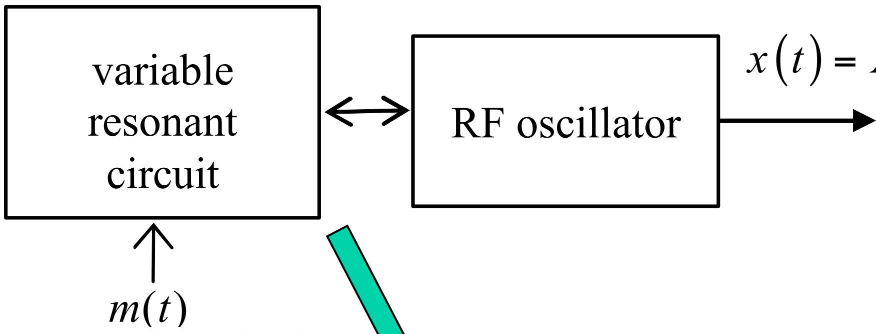
FM Modulator

General principle:

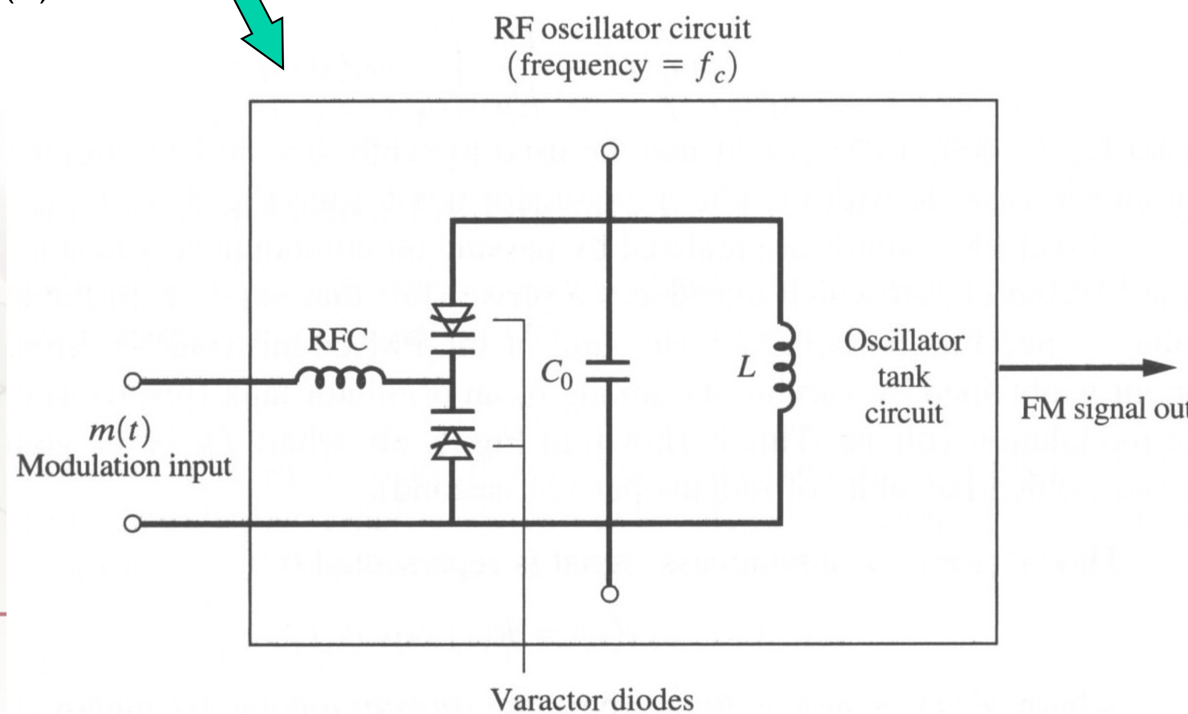


$$x(t) = A_c \cos\left[\omega_c t + \int_0^t \Omega(\tau)\,d\tau + \varphi_0\right]$$

Practical implementation:

Difficulty: frequency stability.

Suitable for narrowband FM only.



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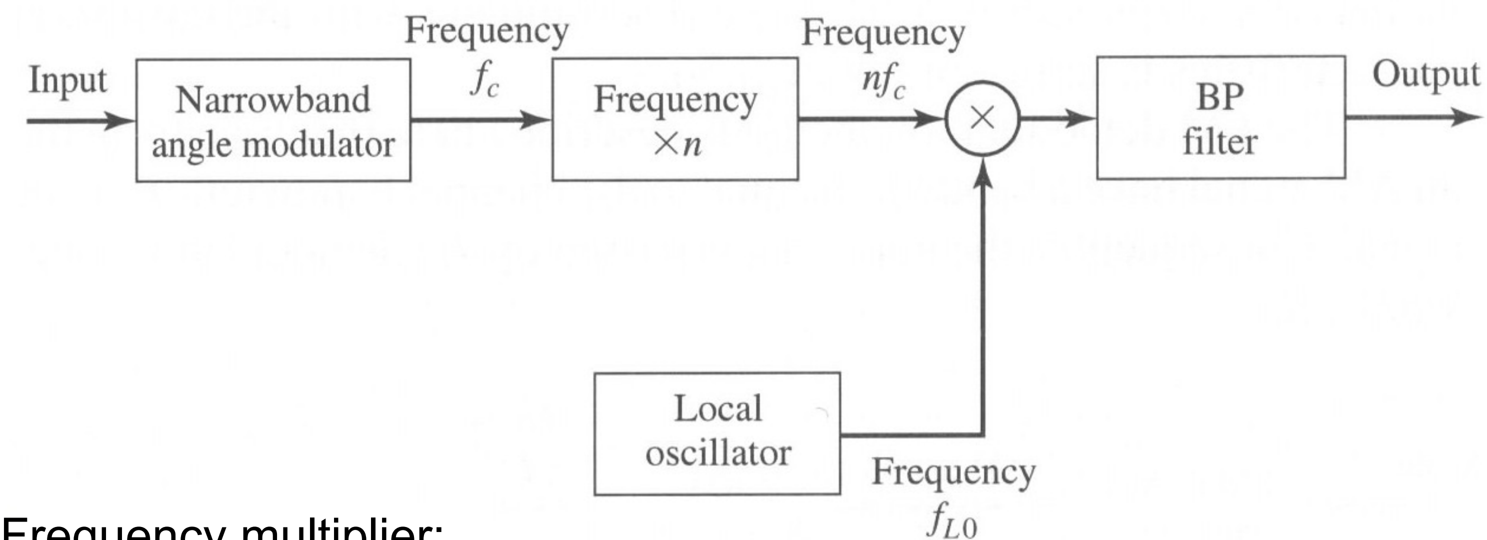
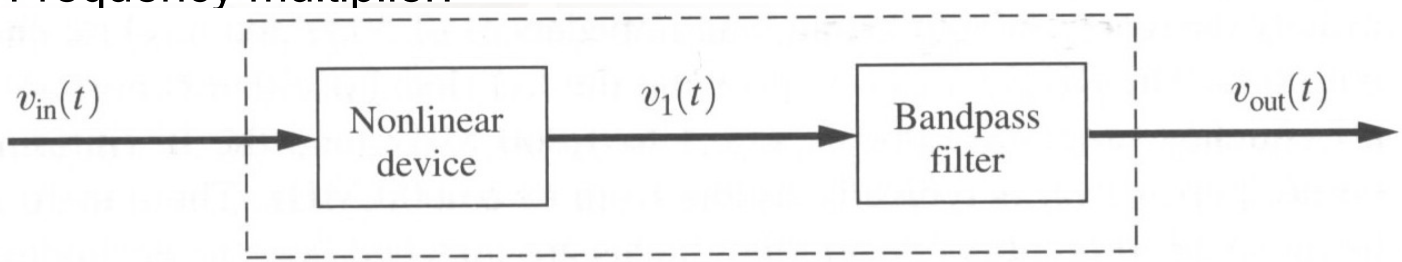
Indirect Wideband Angle Modulator

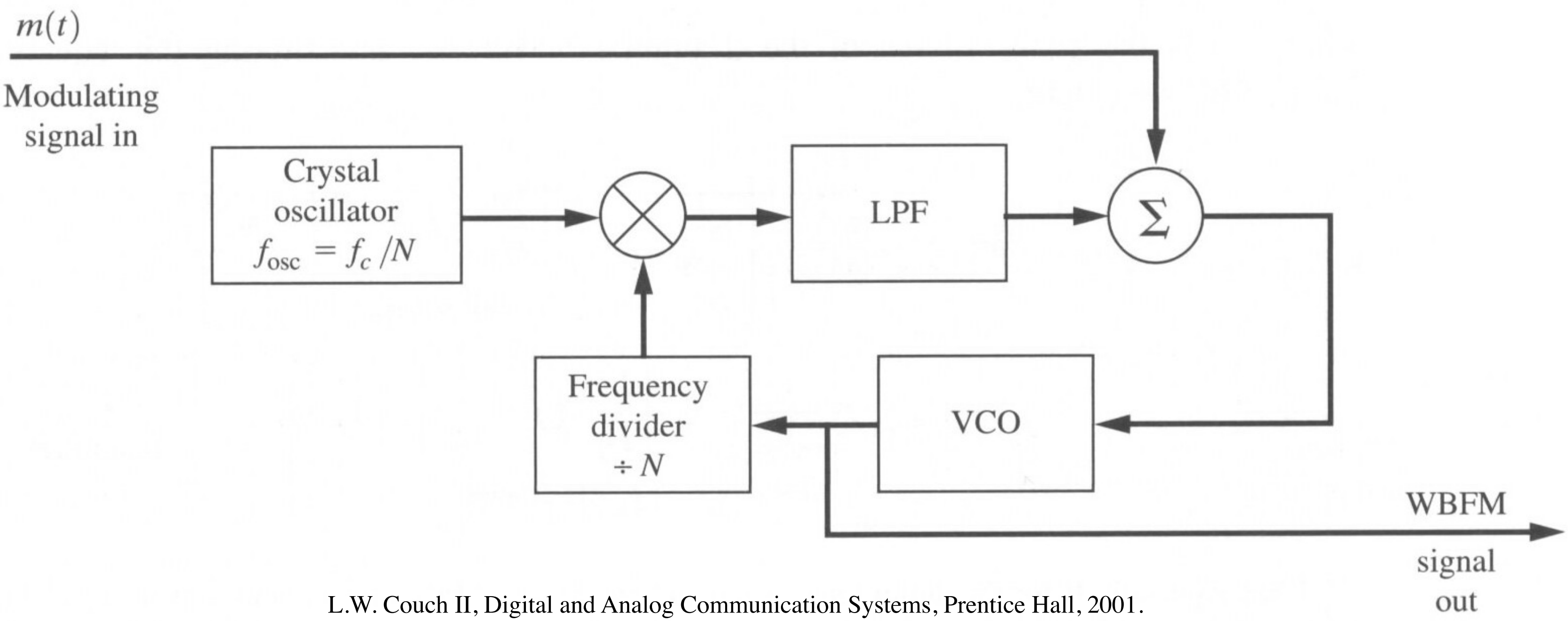


Frequency multiplier:



$$\cos^2 \psi(t) = \frac{1}{2}\left[1 + \cos\left(2\psi(t)\right)\right] \xrightarrow{\text{BPF}} \frac{1}{2}\cos\left(2\psi(t)\right)$$

Direct Wideband Angle Modulator



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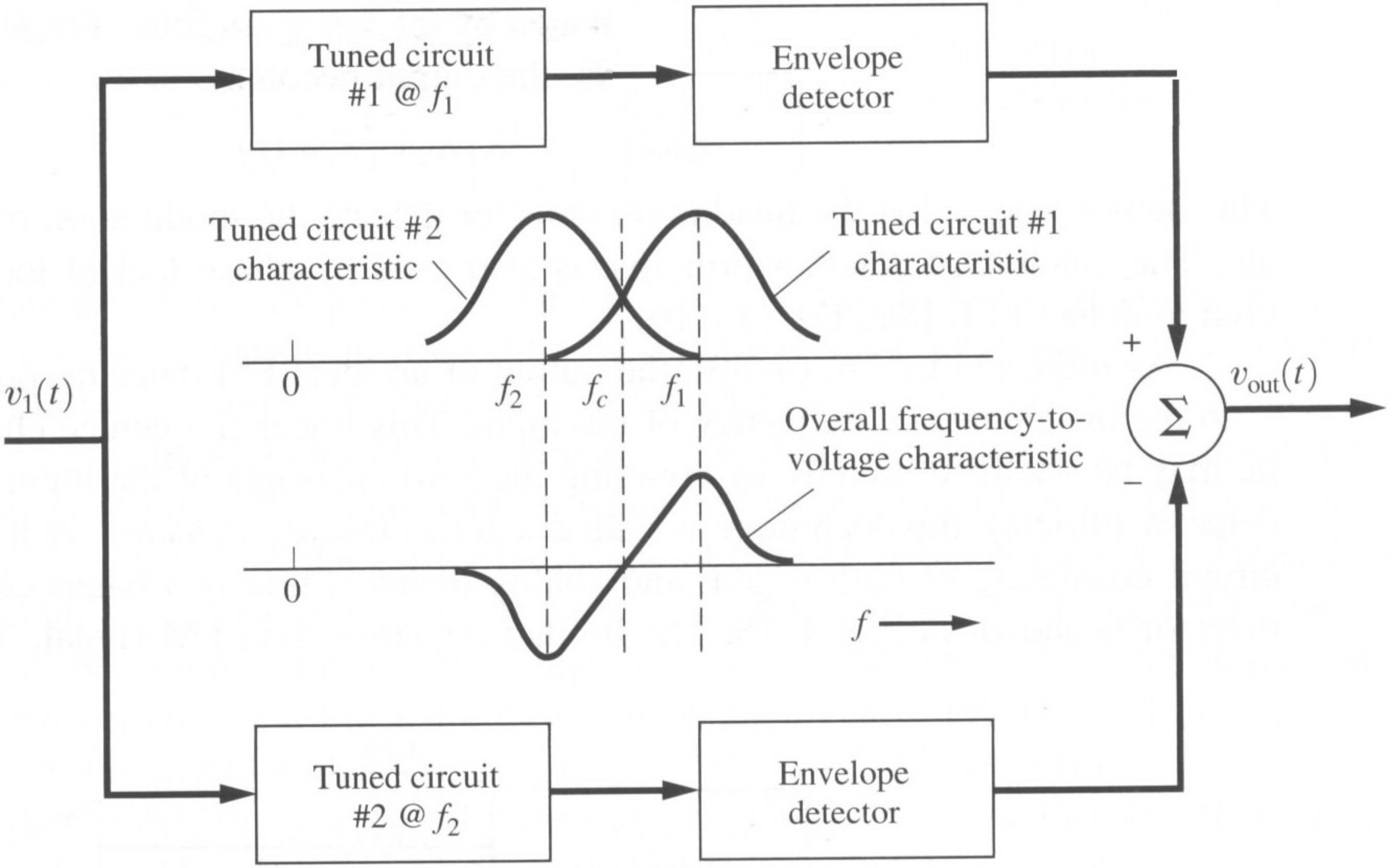
<u>how it operates? </u> consider it without feedback first

why is feedback required?

why is frequency divider required?

Balanced Discriminator: Block Diagram



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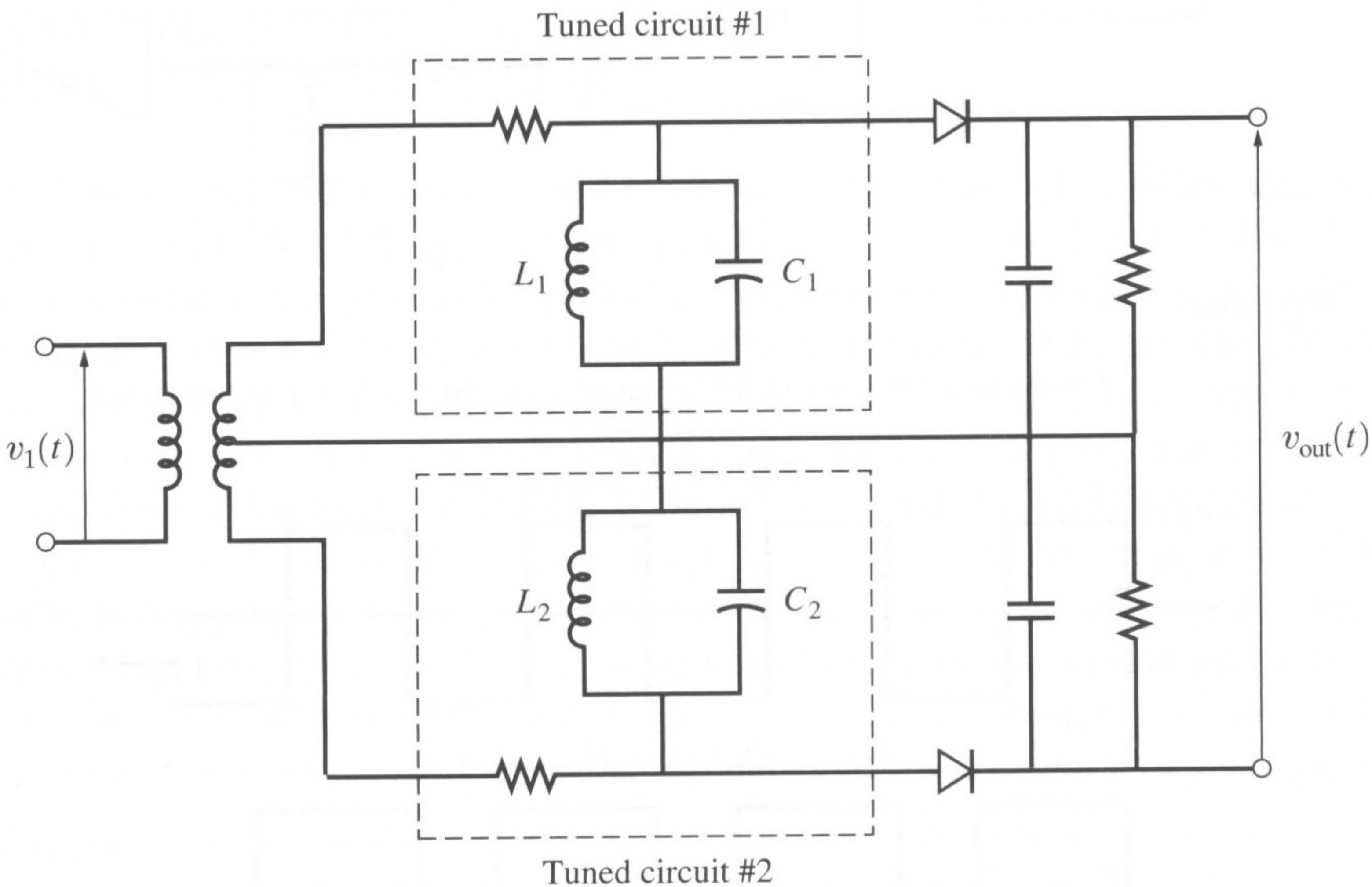
Balanced Discriminator: Circuit Diagram



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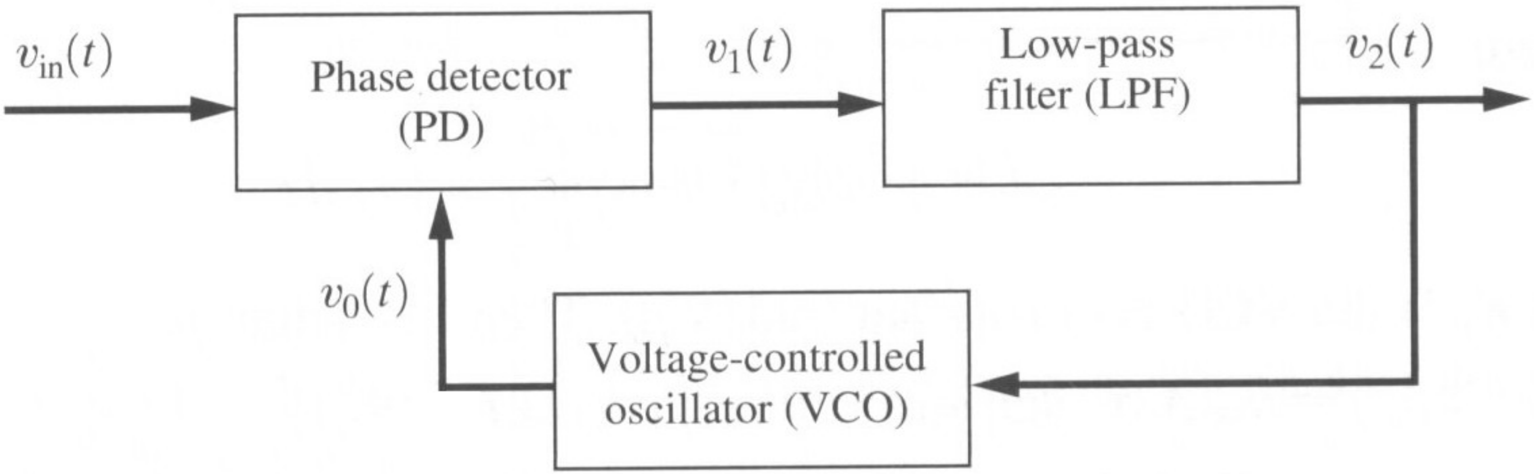
Phased Locked Loop (PLL) Detector



$$\boxed{v_{in}(t) = A_{in}\sin[\omega_c t + \varphi_{in}(t)]}$$

$$\boxed{v_0(t) = A_0\cos[\omega_c t + \varphi_0(t)]}$$

$$\boxed{\omega_{VCO}(t) = \frac{d}{dt}\left(\omega_c t + \varphi_0(t)\right) = \omega_c + \alpha v_2(t)}$$

$$\boxed{v_1(t) = \frac{A_1 A_2}{2}\sin[\varphi_{in}(t) - \varphi_0(t)] + (2\omega_c)\text{term}}$$

Informally,

$$\varphi_{in}(t) \approx \varphi_0(t)$$

$$v_2(t) \approx \frac{1}{\alpha}\frac{d}{dt}\varphi_{in}(t)$$

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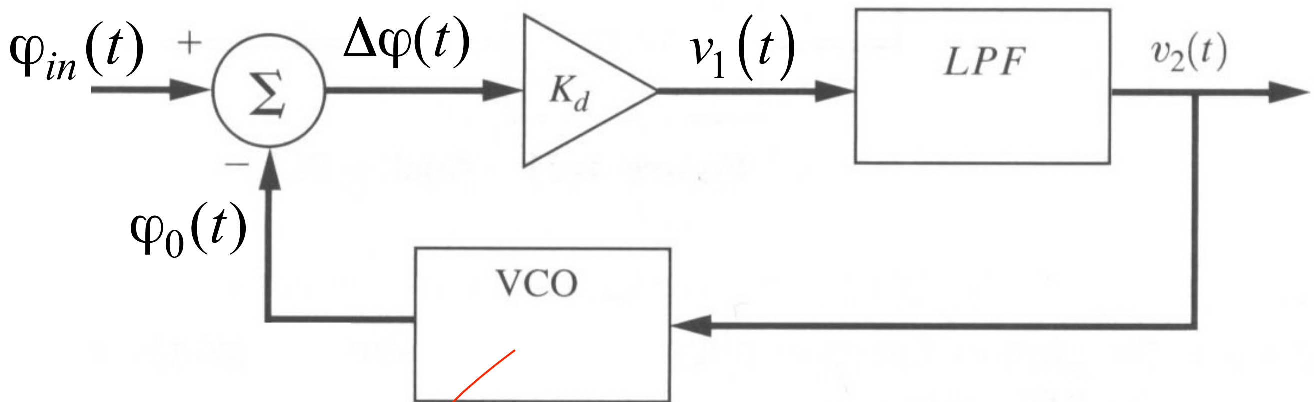
PLL Detector: Linear Model



$$v_1(t) = \frac{A_1 A_2}{2}\sin[\varphi_{in}(t) - \varphi_0(t)] + +(2\omega_c)\text{term}$$

$$v_2(t) = \frac{A_1 A_2}{2}\sin[\varphi_{in}(t) - \varphi_0(t)] \approx \approx \frac{A_1 A_2}{2}(\varphi_{in}(t) - \varphi_0(t)) = K_d \Delta\varphi(t)$$

$$\frac{d}{dt}\varphi_0(t) = \alpha v_2(t)$$

$$v_2(t) = \frac{1}{\alpha}\frac{d}{dt}[\varphi_{in}(t) - \Delta\varphi(t)] \approx \frac{1}{\alpha}\frac{d}{dt}\varphi_{in}(t)$$

Comparison of AM and FM/PM

- AM is simple (envelope detector) but no noise/ interference immunity (low quality).
- AM bandwidth is twice or the same as the modulating signal (no bandwidth expansion).
- Power efficiency is low for conventional AM.
- DSB-SC & SSB – good power efficiency, but complex circuitry.
- FM/PM – spectrum expansion & noise immunity. Good quality.
- More complex circuitry. However, ICs allow for cost-effective implementation.

Important Properties of Angle-Modulated Signals: Summary

- FM/PM signal is a nonlinear function of the message.
- The signal's bandwidth increases with the modulation index.
- The carrier spectral level varies with the modulation index, being 0 in some cases.
- Narrowband FM/PM: the signal's bandwidth is twice that of the message (the same as for AM).
- The amplitude of the FM/PM signal is constant (hence, the power does not depend on the message).