Integrality Gap and Approximation Algorithm for the Multi-survivable Network Design Problem

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Abstract

Given a complete graph with non-negative edge costs and non-negative integer vertex connectivity requirements, we consider the problem of finding a minimum cost subgraph that has at least r_{ij} edge-disjoint paths between every pair of distinct vertices i, j, where r_{ij} is the minimum of the vertex connectivity requirements r_i and r_j. This is known as the survivable network problem. When multi-edges are allowed in the solution, we call this the multi-survivable network design problem. Let r_m be the maximum vertex connectivity requirement and r_l be the smallest nonzero vertex connectivity requirement. We show that the integrality gap for the linear programming relaxation of the multi-survivable network design problem is at most $\frac{3}{2}\frac{r_m}{r_l}$ when r_m is even, and at most $\frac{3}{2}\frac{r_m}{r_l} + \frac{1}{2r_l}$ when r_m is odd. We also present an approximation algorithm with this bound as a performance guarantee. We show that these results are improvements for the best known bounds for certain special cases of this problem.

1   Introduction

The minimum cost k-edge connected subgraph problem, $k \geq 2$ is an important problem in network design. Given a network of centres and specific non-negative costs for connecting any two centres with a link, this problem consists of finding a cheapest way to construct a network so that it can survive the loss of $k - 1$ links, i.e., so that the network remains connected even if $k - 1$ of the links are lost. The problem has many practical, cost-saving applications, such as the design of reliable communication and transportation networks. Here, "cost" can be interpreted as the construction cost, and "reliable" as the continuation of services in the event of a catastrophe causing the loss of a "link" (i.e. a communications cable or a transportation road). For such applications, finding a solution that is either optimal or very close to optimal means substantial financial savings for the company constructing the network.

In some applications, it is useful to allow more than one link (i.e., multi-links) to be built between a pair of centres in order to build a reliable network at lower cost. For example, to connect two components of a network, it may be more practical and more cost efficient to use multiple duplicate links between two centres of the network components (one centre from each component) rather than a single link between two centres and other single links between different pairs of centres of the two network components. Instances of this occur in the laying of underwater wires between islands and a mainland, where a link failure is considered to be the failure of a wire. Also, in case of a wire failure, sensors in nuclear power plants are each connected by three wires to the main system.

A more general problem in network design is the survivable network design problem. This problem consists of finding a cheapest way to construct a network so that it can survive the loss of certain numbers of links in certain areas of the network. This is used, for example, in designing fibre optic telecommunications networks [17, 14].

1.1   Multi-SNDP, SNDP, Metric kEC, and Multi-kEC Problems

Given a graph $G = (V, E)$, and any vertex set $S \subset V$, let $\delta(S)$ denote the set of edges $\{uv \in E : u \in S, v \notin S\}$.

The *survivable network design problem* (SNDP) (also called the *generalized Steiner network problem*) is as follows: Let $G = (V, E)$ be a complete graph with non-negative edge costs $c \in \mathbb{R}^E$, and vertex connectivity requirements $r_i \in \mathbb{N}_{\geq 0}$, $i \in V$. For each pair of distinct vertices $i, j \in V$ let $r_{ij} = \min\{r_i, r_j\}$. The problem is to find a minimum-cost subgraph that has, for each pair of distinct vertices $i, j \in V$, at least r_{ij} edge-disjoint paths between i and j. In such a solution, the loss of any k edges still allows all vertices with connectivity requirement greater than k to be connected.

If multiple copies of the edges are allowed in the solution of SNDP, the problem will be referred to as the *multi-survivable network design problem* (multi-SNDP). A *multi-subgraph* is a subgraph with multiple copies of at least one of its edges. The equivalent cut requirement version of multi-SNDP is: Given a cut requirement function $f : 2^V \to \mathbb{Z}$ defined on the subsets of V by $f(S) = \max_{i \in S, \ j \notin S} r_{ij}$ for all $S \subset V$, find a minimum-cost multi-subgraph having at least $f(S)$ edges in each cut $\delta(S)$, $S \subset V$. Note that if G is not complete, we can make it into a complete graph by adding in all the "missing" edges and assigning each of them an appropriately large cost.

When the edge costs $c \in \mathbb{R}^E$ of a graph are non-negative and satisfy the triangle inequality $c_{uw} \leq c_{uv} + c_{vw}$ for all distinct vertices $u, v, w \in V$, the costs are called *metric*. A SNDP problem with metric edge costs will be referred to as *metric SNDP*.

A graph is said to be *k-edge connected*, $k \geq 1$, $k \in \mathbb{Z}$, if any $k - 1$ edges can be removed without disconnecting the graph, or, equivalently, if there are at least k edge-disjoint paths between every pair of vertices, or, equivalently, if every cut $\delta(S), S \subset V$, has at least k edges.

Given a complete graph, $G = (V, E)$, with non-negative edge costs $c \in \mathbb{R}^E$, the *k-edge connected subgraph problem* (kEC), $k \geq 1$, is the special case of the SNDP where $r_i = k$ for all $i \in V$. In particular, it is the problem of finding a minimum cost k-edge connected spanning subgraph of G, $k \geq 1$. Note that 1EC is the well known minimum cost spanning tree problem. A kEC problem with metric edge costs will be referred to as *metric kEC*.

If multiple copies of the edges are allowed in the solution of kEC, the problem will be referred to as the *multi-k-edge connected subgraph problem* (multi-kEC). Problems kEC and multi-kEC are special cases of SNDP and multi-SNDP, respectively. In this paper, we examine multi-SNDP, and apply our results to SNDP, metric SNDP, multi-kEC, kEC, and metric kEC.

The SNDP is an NP-hard problem, since the Steiner tree problem, which has been shown to be NP-hard [19], is a special case. Thus, multi-SNDP is an NP-hard problem. Both kEC and multi-kEC ($k \geq 2$) are also known to be NP-hard problems [15]. (The minimum cost spanning tree problem is not NP-hard.) As it is considered highly unlikely that efficient algorithms for exactly solving NP-hard problems exist, we look at designing efficient algorithms that return approximate, or near-optimal, solutions. If an algorithm runs in polynomial time and returns a solution whose value is always within a constant factor α of the optimal value, then the algorithm is known as an *α-approximation algorithm*, and the factor α is called the *approximation* or *performance guarantee* of the algorithm.

A related concept to approximation guarantees is that of integrality gaps. Given a minimization problem P, its integer linear program Q, and its linear programming (LP) relaxation Q_{LP}, the *integrality gap* for the linear programming relaxation is the largest ratio $opt(Q)/opt(Q_{LP})$ over all possible edge costs. If it can be proved that this largest ratio has value k, the linear programming relaxation is said to have a *k integrality gap*. Furthermore, if the proof of this is constructive and of polynomial time, then it forms a k-approximation algorithm for the problem P.

The best general constant factor approximation guarantee that has existed for multi-SNDP, SNDP, kEC and multi-kEC has, to this point, been 2. This guarantee is due to Kamal Jain [18], who, in 2001, presented a 2-approximation algorithm for finding, in a general graph that may not be simple and has non-negative edge costs, a minimum cost subgraph having at least a specified number of edges in each cut [18]. Initially, multiple copies of edges are not allowed in the solution, however, as is noted near the end of his paper, the 2-approximation algorithm holds as well for the case when multiple copies of edges are allowed in the solution. In addition to providing an approximation guarantee, Jain's algorithm provides a constructive proof that the integrality gap of the linear programming relaxation of the general problem he considers is at most 2. In particular, Jain's result implies that the integrality gap of the LP relaxation of kEC and multi-kEC is ≤ 2.

Not widely known is that Goemans and Bertsimas [16] in 1993 presented an approximation algorithm (the improved tree heuristic) for multi-SNDP that, given distinct vertex connectivity requirements $\rho_o = 0 < \rho_1 < \rho_2 < \ldots < \rho_q$, has

3

approximation guarantee $\sum_{j=1}^{q} \frac{f(\rho_j - \rho_{j-1})}{\rho_j}$, where $f(c) = \frac{3}{2}c$ when c is even, and $f(c) = \frac{3c+1}{2}$ when c is odd [16]. The proof used to obtain this is rather complex and involved, and, although not indicated, implicitly yields an upper bound for integrality gap as well. As a special case, this approximation algorithm has an approximation guarantee (and integrality gap) for multi-kEC of $\frac{3}{2}$ when k is even, and $\frac{3}{2} + \frac{1}{2k}$ when k is odd.

We present an upper bound for the integrality gap for the linear programming relaxation of multi-SNDP; that is, we show that the integrality gap for the linear programming relaxation of the multi-survivable network design problem is at most $\frac{3}{2}\frac{r_m}{r_l}$ when r_m is even, and at most $\frac{3}{2}\frac{r_m}{r_l} + \frac{1}{2r_l}$ when r_m is odd, where r_m is the maximum vertex connectivity requirement and r_l is the smallest nonzero vertex connectivity requirement. We present as well an approximation algorithm for multi-SNDP which has a performance guarantee the same as this upper bound. These are both improvements for the best known bounds for certain special cases of the problem. We show the application of these results (integrality gap and approximation algorithm) to SNDP, metric SNDP, multi-kEC, kEC, metric kEC, and Steiner multi-kEC. Additionally, we discuss the relationship between our approximation algorithm and the improved tree heuristic of Goemans and Bertsimas [16]. We show that our multi-kEC approximation algorithm, with a given modification, applies as well for metric 2EC; and that problems multi-2EC and metric 2EC are closely related.

Our results for integrality gap and the approximation algorithms for multi-SNDP and multi-kEC depend upon new upper bounds for the optimal values of the minimum cost T-join and the minimum cost spanning tree in terms of the optimal value of the LP relaxation of multi-SNDP, which are also presented.

1.2   Related Problems and Overview

Given a general graph $G = (V, E)$ with non-negative edge costs $c \in \mathbb{R}^E$ and vertex connectivity requirements $r_i \in \{0, k\}$ for all $i \in V$, $k \geq 1$, the *Steiner k-edge connected subgraph problem* (Steiner kEC) is that of finding a minimum k-edge connected subgraph of G that spans the set of terminal vertices $S = \{i \in V \mid r_i = k\} \subseteq V$; i.e., such that there are at least $r_{ij} = \min(r_i, r_j)$ edge-disjoint paths between all pairs of vertices $i, j \in V$. This problem is NP-hard [23]. Problem kEC is a special case of Steiner kEC, which is a special case of SNDP. If G is a complete graph and multiple copies of the edges are allowed in the solution of Steiner kEC, the problem will be referred to as the *Steiner k-edge connected multi-subgraph problem* (Steiner multi-kEC). Problem multi-kEC is a special case of Steiner multi-kEC, which is a special case of multi-SNDP.

Problem kEC is a special case of the *graph augmentation problem*: Given a complete graph $G = (V, E)$, a subgraph (V, F), $F \subset E$, and a non-negative cost function $c \in \mathbb{R}^E$, find a set of edges $E' \subseteq E \backslash F$ of minimum cost such that the "augmented graph" $(V, F \cup E')$ satisfies a given property [9]. Among other things, this problem can be used for improving the reliability (or survivability) of already existing networks. For kEC we have $F = \emptyset$, and the property to

be satisfied is that the augmented graph $(V, F \cup E') = (V, E')$ must be k-edge connected. As is mentioned earlier, kEC is a special case of SNDP and multi-kEC is a special case of multi-SNDP.

For a summary of the current known results on the different versions of kEC, see [21].

The remainder of this paper is organized as follows. In Section 2, we give some required background information on minimum cost perfect matchings, T-joins, and spanning trees. In Section 3, we present the main results of the paper, along with new upper bounds for the optimal values of the minimum cost T-join and the minimum cost spanning tree in terms of the optimal value of the LP relaxation of multi-SNDP. We also present a family of examples where our integrality gap for the LP relaxation of multi-SNDP, and our approximation algorithm for multi-SNDP, are improvements for the best known bounds of the problem. In Section 4, we show the applications of our results to SNDP and metric SNDP. In Section 5, we show the applications of our results to multi-kEC, kEC, metric kEC, and Steiner multi-kEC, and discuss the relationship of our approximation algorithm for multi-kEC to Goemans and Bertsimas' improved tree heuristic. In Section 6, we show how problems multi-2EC and metric 2EC are closely related. We also show that the integrality gap for the LP relaxation of metric 2EC has an upper bound of $\frac{3}{2}$, and how a $\frac{3}{2}$-approximation algorithm for metric 2EC can be obtained. This algorithm (but not the proof) is similar to the $\frac{3}{2}$-approximation algorithm for metric 2EC and metric 2VC given by Frederickson and Ja'Ja' in [10]. We show how our proof can provide a simpler, more compact proof of their result.

2 Minimum Cost Perfect Matchings, T-joins, and Spanning Trees

Before presenting our main results, the following background information is required.

A *minimum cost perfect matching* in a connected graph $G = (V, E)$ with non-negative edge costs, is a minimum cost subset of edges of E such that each vertex in V is incident to exactly one edge in the subset. Using an efficient implementation of Edmonds' blossom algorithm, a minimum cost perfect matching in G can be found in $O(n(m + n \log n))$ time [7], where n is the number of vertices and m is the number of edges in G. If G is a complete graph, this becomes $O(n^3)$.

Given a connected graph $G = (V, E)$ with non-negative edge costs $c \in \mathbb{R}^E$, and given $T \subseteq V$, a *minimum cost T-join* is a minimum cost set of edges $\tilde{E} \subseteq E$ such that $|\delta(v) \cap \tilde{E}|$ is odd if and only if $v \in T$. Note that a connected graph has a T-join if and only if $|T|$ is even.

Let x_e represent the number of times each edge $e \in E$ is included in the solution. For any edge set $E' \subseteq E$ and $x \in \mathbb{R}^E$, let $x(E')$ denote the sum $\sum(x_e : e \in E')$. The following theorem states that we can find the cost of an

optimal T-join by solving a linear program that has no integer constraints.

Theorem 1 (Cook et. al. [6], Theorem 5.28) *If $G = (V, E)$, $T \subseteq V$ with $|T|$ even, and $c \in \mathbb{R}^E$ with $c \geq 0$, then the minimum cost of a T-join of G is equal to the optimal value of*

$$minimize \quad cx \tag{1}$$
$$subject \ to \quad x(\delta(S)) \quad \geq \quad 1, \quad for \ all \ S \subseteq V \ s.t.$$
$$|S \cap T| \ odd, \tag{2}$$
$$x_e \quad \geq \quad 0, \quad for \ all \ e \in E. \tag{3}$$

A solution to the minimum cost T-join problem can be found in polynomial time in the following way: Find a shortest (i.e. minimum cost) path, P_{uv}, in G between each pair of vertices $u, v \in T$; and let f_{uv} be the cost of this path. Next, form a complete graph $\hat{G} = (T, \hat{E})$ using the vertices in T and the edge costs f_{uv}, and find a minimum cost perfect matching $\hat{M} \subseteq \hat{E}$ in it. The symmetric difference of the edge-sets of the corresponding paths P_{uv} for $uv \in \hat{M}$, is a minimum cost T-join of G [8, 6]. The shortest path between two vertices in a connected graph with non-negative edge costs can be determined in $O(n \log n + m)$ time [22]. For a complete graph, this becomes $O(n^2)$ time. Thus, finding a shortest path between each pair of vertices in a complete graph (i.e., finding all pairs shortest paths) takes, in the worst case, $O(n^3)$ time. Therefore, finding a minimum cost T-join can be done, in the worst case, in $O(n^3 + n^3) = O(n^3)$ time.

A *minimum cost spanning tree* of a complete graph $G = (V, E)$ is a tree of G that includes all the vertices of G and has minimum cost. Such a spanning tree can be found in $O(n \log n + m)$ time [11]. This becomes $O(n^2)$ for a complete graph.

Before presenting a corollary that is foundational to our results, we need the following definitions and proposition.

Given a graph $G = (V, E)$, a partition $P = (V_1, V_2, \ldots, V_{k_P})$ of the vertex set V is *feasible* if each part V_i induces a connected subgraph $G(V_i)$ for all $i = 1, 2, \ldots, k_P$. The graph G_P is the *shrunk graph*, obtained by identifying all the vertices in V_i into a vertex $\tilde{v}_i$ and deleting all the edges of G that have both endpoints in the same part, V_i, of the partition P. G_P has k_P vertices, and its edge set, E_P, contains all the edges of G that have endpoints in different parts, V_i, of the partition P. Given a connected graph $G = (V, E)$, consider the collection of spanning trees of G. The incidence vector, x, corresponding to a spanning tree, T, has $x_e = 1$ if T contains e, and $x_e = 0$ otherwise, for all $e \in E$.

Proposition 1 (Chopra [4]) *Given a connected graph $G = (V, E)$, the dominant of the spanning tree polytope is defined by*

$$\left\{ x \in \mathbb{R}_+^m \ : \ \sum_{e \in E_P} x_e \geq k_P - 1, \ for \ all \ feasible \ partitions \ P \right\}. \tag{4}$$

6

Notice that for a complete graph, all partitions of G are feasible.

Chopra [4] shows that, given a partition P of G, the partition inequality $\sum_{e \in E_P} x_e \geq k_P - 1$ defines a facet of (4) if and only if the shrunk graph G_P is two-vertex connected. In particular, this means that for a complete graph all the partition inequalities of (4) are facet-inducing and thus necessary in the description.

The following corollary is foundational to our results, and follows directly from Proposition 1. It states hat we can find the cost of a minimum cost spanning tree by solving a linear program that has no integrality constraints.

Corollary 1 *If $G = (V, E)$ is a connected graph, and $c \in \mathbb{R}^E$ with $c \geq 0$, then the cost of a minimum cost spanning tree of G is equal to the optimal value of*

$$minimize \quad cx \tag{5}$$

$$subject\ to \quad \sum_{e \in E_P} x_e \quad \geq \quad k_P - 1, \quad for\ all\ feasible\ partitions\ P, \tag{6}$$

$$x_e \quad \geq \quad 0, \quad for\ all\ e \in E. \tag{7}$$

3 Integrality Gap and Approximation Algorithm

Problem multi-SNDP can be formulated as an integer linear program (ILP) as follows:

$$minimize \quad cx \tag{8}$$

$$subject\ to \quad x(\delta(S)) \quad \geq \quad \max_{ij \in \delta(S)} r_{ij}, \quad for\ all\ \emptyset \subset S \subset V, \tag{9}$$

$$x_e \quad \geq \quad 0, \quad for\ all\ e \in E, \tag{10}$$

$$x_e \quad integer.$$

The linear programming (LP) relaxation of multi-SNDP is:

$$minimize \quad cx \tag{11}$$

$$subject\ to \quad x(\delta(S)) \quad \geq \quad \max_{ij \in \delta(S)} r_{ij}, \quad for\ all\ \emptyset \subset S \subset V, \tag{12}$$

$$x_e \quad \geq \quad 0, \quad for\ all\ e \in E. \tag{13}$$

The following notation and definitions will be used throughout the rest of this paper: Given any multi-SNDP problem, i.e., given a complete graph $G = (V, E)$ with non-negative edge costs $c \in \mathbb{R}^E$ and vertex connectivity requirements $r_i \in \mathbb{N}_{\geq 0}$, $i \in V$, let $r_m = \max\{r_{ij} \mid i, j \in V\}$, $r_l = \min\{r_{ij} \mid r_{ij} > 0, \ i, j \in V\}$, and $V^+ = \{i \in V \mid r_i > 0\}$. Let G^+ be the complete graph with vertex set V^+, respective vertex connectivity requirements from G, and edge costs c_{ij}^+, $i, j \in V^+$, the cost of a minimum-cost i-j path in G. In other words, if $G^\triangle$ is the metric completion of G, then $G^+ = G^\triangle - \{i \in V \mid r_i = 0\}$. Furthermore, let

opt(multi-SNDP) be the cost of an optimal multi-SNDP solution (i.e., the optimal value of the integer linear program (8)) on G, and let *opt(multi-SNDP$_{LP}$)* be the cost of an optimal solution to the LP relaxation of multi-SNDP (i.e., the optimal value of the linear program (11)) on G.

Given a complete graph $G = (V, E)$ with non-negative edge costs c, the *metric completion*, $G^\triangle = (V, E)$, of G is obtained by setting the edge cost $c_{ij}^\triangle$ to be that of a minimum-cost path connecting vertices i and j in G, for every edge $ij \in E$. Clearly, the edge costs $c^\triangle$ are metric. Note that, given a complete graph G' with metric edge costs, its metric completion, $G^\triangle$, does not necessarily have the same edge costs (although both have metric edge costs).

Theorem 2 *(Theorem 3 Goemans and Bertsimas [16]) Let $G = (V, E)$ be a complete graph with non-negative edge costs $c \in \mathbb{R}^E$ and a non-negative connectivity requirement r_{ij} for every (unordered) pair of vertices $i, j \in V$. Let $G^\triangle = (V, E)$ be the metric completion of G, with metric costs $c^\triangle \in \mathbb{R}^E$. Then the optimal values of multi-SNDP on G and on $G^\triangle$ are equal (and the optimal values of their linear programming relaxations are equal).*

When Theorem 2 is applied to multi-kEC, $k \geq 1$, the following corollary is obtained.

Corollary 2 *Given a complete graph $G = (V, E)$ with non-negative edge costs $c \in \mathbb{R}^E$ and the metric completion $G^\triangle$ of G,*

$$opt(\text{multi-}kEC \text{ on } G) = opt(\text{multi-}kEC \text{ on } G^\triangle).$$

Lemma 1 *Given a complete graph $G = (V, E)$ with non-negative edge costs $c \in \mathbb{R}^E$,*

$$opt(multi\text{-}SNDP) \leq opt(SNDP).$$

Also,

$$opt(multi\text{-}SNDP_{LP}) \leq opt(SNDP_{LP}).$$

Proof: The problem multi-SNDP is a relaxation of SNDP (relaxing the constraints that each edge can be chosen at most once in the solution). Similarly, the linear programming relaxation of multi-SNDP is a relaxation of the linear programming relaxation of SNDP. ∎

The following proposition can be obtained directly by combining Lemma 1 with Theorem 2.

Proposition 2 *Given a complete graph $G = (V, E)$ with non-negative edge costs $c \in \mathbb{R}^E$, let $G^\triangle = (V, E)$ be the metric completion of G, with metric costs $c^\triangle \in \mathbb{R}^E$. Then*

$$opt(multi\text{-}SNDP \text{ on } G) \leq opt(SNDP \text{ on } G^\triangle)$$

and

$$opt(multi\text{-}SNDP_{LP} \text{ on } G) \leq opt(SNDP_{LP} \text{ on } G^\triangle).$$

The following theorem follows directly from the parsimonious property of Goemans and Bertsimas [16].

Theorem 3 *Given a complete graph $G = (V, E)$ with metric edge costs $c \in \mathbb{R}^E$ and vertex connectivity requirements $r_i \in \mathbb{N}_{\geq 0}$, $i \in V$. The optimal value of the LP relaxation of multi-SNDP, LP (11), is equal to the optimal value of*

$$minimize \ \ cx \tag{14}$$

$$subject \ to \quad x(\delta(S)) \quad \geq \quad \max_{ij \in \delta(S)} r_{ij}, \quad for \ all \ \emptyset \subset S \subset V, \tag{15}$$

$$x(\delta(i)) \quad = \quad 0, \quad for \ all \ i \in V \backslash V^+, \tag{16}$$

$$x_e \quad \geq \quad 0, \quad for \ all \ e \in E. \tag{17}$$

Proposition 3 *Given a complete graph $G = (V, E)$ with non-negative edge costs $c \in \mathbb{R}^E$,*

$$opt(multi\text{-}SNDP_{LP} \ on \ G^+) = opt(multi\text{-}SNDP_{LP} \ on \ G).$$

Proof: Let $G^\triangle, c^\triangle$ be the metric completion of G. Let x^* be an optimal solution to LP(14) on $G^\triangle$. Let y be x^* restricted to G^+, with edge costs $c^{\triangle, V^+}$ as defined at the beginning of the section. Since x^* satisfies the constraints (16), i.e., since $x_e^* = 0$ on all the edges incident to vertices i with $r_i = 0$, we have that $c^{\triangle, V^+} y = c^\triangle x^*$. By Theorem 3, $c^\triangle x^* = opt(multi\text{-}SNDP_{LP} \ on \ G^\triangle)$.

Clearly, y satisfies the LP relaxation of multi-SNDP (LP(11)) on G^+. Suppose that y is not an optimal solution of LP(11) on G^+. Let W be an optimal solution of LP(11) on G^+. Let $cost(*)$ be the cost of $*$. Then $cost(w) < cost(y)$. Form a solution w' on $G^\triangle$ by setting $w' = 0$ on all the edges that are incident to a vertex $i \in V \backslash V^+$, and $w' = w$ otherwise. Note that w' is a solution to LP(14). Then $cost(w') = cost(w) < cost(y) = cost(x^*)$. This is a contradiction, since x^* is an optimal solution to LP(14). Thus, y is an optimal solution of LP(11) on G^+, i.e., $cost(y) = opt(multi\text{-}SNDP_{LP} \ on \ G^+)$.

Hence, we have that $opt(multi\text{-}SNDP_{LP} \ on \ G^+) = c^{\triangle, V^+} y = c^\triangle x^* = opt(multi\text{-}SNDP_{LP} \ on \ G^\triangle)$. From Theorem 2, $opt(multi\text{-}SNDP_{LP} \ on \ G^\triangle) = opt(multi\text{-}SNDP_{LP} \ on \ G)$. The result follows. ∎

Let $opt(T\text{-join})$ be the cost of a minimum cost T-join on G.

Lemma 2 *Given a complete graph $G = (V, E)$ with non-negative edge costs $c \in \mathbb{R}^E$ and vertex connectivity requirements $r_i \in \mathbb{N}_{\geq 0}$, $i \in V$. Let $T \subseteq V^+$, $|T|$ even. The following holds on G:*

$$opt(T\text{-join}) \ \leq \ \frac{1}{r_l} \ opt(multi\text{-}SNDP_{LP}) \ \leq \ \frac{1}{r_l} \ opt(multi\text{-}SNDP).$$

Proof: Let x^* be an optimal solution to the multi-SNDP linear programming relaxation LP(11). In particular, x^* satisfies the constraints (12). Notice that $\max_{ij \in \delta(U)} r_{ij} \geq 0$, and hence $x^*(\delta(U)) \geq 0$, for all $\emptyset \subset U \subset V$. We show that $\frac{1}{r_l} x^*$ satisfies the constraints (2), i.e., that $\frac{1}{r_l} x^*$ satisfies $x(\delta(U)) \geq 1$ for all

$\emptyset \subset U \subset V$ such that $|U \cap T|$ is odd. We do this by examining the following two cases.

Case 1: For some $\emptyset \subset U \subset V$, either $r_i = 0$ for all $i \in U$, or $r_j = 0$ for all $j \in V \backslash U'$, or both. Thus, we have that either $\max_{i \in U} r_i = 0$, or $\max_{i \in V \backslash U} r_i = 0$, or both. Notice that, in a complete graph,

$$\max_{ij \in \delta(U)} r_{ij} = \min\{\max_{i \in U} r_i, \max_{j \in V \backslash U} r_j\}.$$

Then $\max_{ij \in \delta(U)} r_{ij} = 0$ and thus $x^*(\delta(U)) \geq 0$. Without loss of generality, assume that $r_i = 0$ for all $i \in U$. Since there does not exist a vertex $i \in T$ such that $r_i = 0$, we have that $T \subseteq V \backslash U$. Thus, we have that $|V \backslash U \cap T| = |T|$ is even, and $|U \cap T| = 0$ is even. Thus, for all $\emptyset \subset U \subset V$ for which either $r_i = 0$ for all $i \in U$, or $r_j = 0$ for all $j \in V \backslash U$, or both, then $\frac{1}{r_l} x^*$ satisfies $x(\delta(U)) \geq 0$ and $|U \cap T|$ is even.

Case 2: For some $\emptyset \subset U \subset V$, there exists $i \in U$, $j \in V \backslash U$ such that $r_i > 0$ and $r_j > 0$. Thus, $r_{ij} > 0$ and x^* satisfies $x(\delta(U)) \geq r_{ij}$. Since

$$r_l = \min\{r_i \mid r_i > 0, \ i \in V\} = \min\{r_{ij} \mid r_{ij} > 0, \ i, j \in V, \ i \neq j\},$$

therefore $r_{ij} \geq r_l$. Therefore x^* satisfies $x(\delta(U)) \geq r_l$, and hence $\frac{1}{r_l} x^*$ satisfies $x(\delta(U)) \geq 1$.

Thus, $\frac{1}{r_l} x^*$ satisfies the constraints (2) for all $\emptyset \subset U \subset V$.

In addition, since x^* satisfies the constraints (13), $\frac{1}{r_l} x^*$ also satisfies the inequalities $x_e \geq 0$ for all $e \in E$. Thus, $\frac{1}{r_l} x^*$ satisfies the constraints (2) and (3). This means that $\frac{1}{r_l} x^*$ is a feasible solution of LP(1). Using Theorem 1, we therefore have that, for $T \subseteq V$, $|T|$ even,

$$opt(\textit{T-join}) \ \leq \ \frac{1}{r_l} \ cx^* \ = \ \frac{1}{r_l} \ opt(\textit{multi-SNDP}_{LP}) \ \leq \ \frac{1}{r_l} \ opt(\textit{multi-SNDP}).$$

∎

Given a complete graph $G = (V, E)$ with non-negative edge costs $c \in \mathbb{R}^E$, let $opt(MST)$ be the cost of a minimum cost spanning tree (i.e., the optimal value of LP(5)) on G.

Lemma 3 *Given a complete graph $G = (V, E)$ with non-negative edge costs $c \in \mathbb{R}^E$ and vertex connectivity requirements $r_i \in \mathbb{N}_{\geq 0}$, $i \in V$, the following holds:*

$$opt(\textit{MST on } G^+) \ \leq \ \frac{2}{r_l} \ opt(\text{multi-SNDP}_{LP} \textit{ on } G) \ \leq \ \frac{2}{r_l} \ opt(\textit{multi-SNDP on } G).$$

Proof: Let $G^\triangle, c^\triangle$ be the metric completion of G. Let x^* be an optimal solution to LP(14) on $G^\triangle$. Let y be x^* restricted to (V^+, E_{V^+}), the subgraph of $G^\triangle$ with vertex set V^+, edge set E_{V^+}, and edge costs $c^{\triangle, V^+}$, where E_{V^+} is the set of edges of $G^\triangle$ with both ends in V^+ and $c^{\triangle, V^+}$ are the costs $c^\triangle$ restricted to

(V^+, E_{V^+}). Since x^* satisfies the constraints (16), i.e., since $x_e^* = 0$ on all the edges incident to vertices i with $r_i = 0$, we have that

$$c^{\triangle, V^+} y = c^\triangle x^* \tag{18}$$

and

$$y(\delta(S)) \quad \geq \quad \max_{ij \in \delta(S)} r_{ij}, \quad \text{for all } \emptyset \subset S \subset V^+, \tag{19}$$

$$y_e \quad \geq \quad 0, \quad \text{for all } e \in E_{V^+}. \tag{20}$$

Since y satisfies the constraints (19) and since, for all $\emptyset \subset S \subset V^+$, $\max_{ij \in \delta(S)} r_{ij} \geq r_l$, therefore $\frac{2}{r_l} y$ satisfies

$$y(\delta(S)) \quad \geq \quad \frac{2}{r_l}(\max_{ij \in \delta(S)} r_{ij}) \quad \geq \quad 2, \qquad \text{for all } \emptyset \subset S \subset V^+. \tag{21}$$

Let $P = (S_1, S_2, \ldots, S_{k_P})$ be a partition of (V^+, E_{V^+}). Note that P is always feasible since (V^+, E_{V^+}) is a complete graph. Using (21) above, we have, for $\frac{2}{r_l} y$,

$$\sum_{e \in E_P} y_e \quad = \quad \frac{1}{2} \left[y(\delta(S_1)) + y(\delta(S_2)) + \ldots + y(\delta(S_{k_P})) \right]$$

$$\geq \quad \frac{1}{2} (k_P \cdot 2)$$

$$= \quad k_P$$

$$\geq \quad k_P - 1.$$

Thus, $\frac{2}{r_l} y$ satisfies $\sum_{e \in E_P} y_e \geq k_P - 1$, for all feasible partitions $P = (S_1, S_2, \ldots, S_{k_P})$ of (V^+, E_{V^+}).

Note that y also satisfies the constraints (20). Thus, $\frac{2}{r_l} y$ satisfies the constraints (6) and (7) for the graph (V^+, E_{V^+}). This means that $\frac{2}{r_l} y$ is a feasible solution of LP(5) on (V^+, E_{V^+}). Using Corollary 1 and equation (18),

$$opt(MST \text{ on } (V^+, E_{V^+})) \quad \leq \quad \frac{2}{r_l} c^{\triangle, V^+} y$$

$$= \quad \frac{2}{r_l} c^\triangle x^*$$

Applying Theorems 3 and 2, we therefore have that

$$opt(MST \text{ on } (V^+, E_{V^+})) \quad = \quad \frac{2}{r_l} opt(multi\text{-}SNDP_{LP} \text{ on } G^\triangle)$$

$$= \quad \frac{2}{r_l} opt(multi\text{-}SNDP_{LP} \text{ on } G)$$

$$\leq \quad \frac{2}{r_l} opt(multi\text{-}SNDP \text{ on } G).$$

11

Noticing that (V^+, E_{V^+}) is precisely G^+ completes the proof. ∎

Lemmas 2 and 3 will be used to create a solution for multi-SNDP within a given bound of the optimal. Before doing this, we need to define two special graphs. These two graphs will be used for the remainder of this section. Let $G = (V, E)$ be a complete graph with non-negative edge costs $c \in \mathbb{R}^E$ and vertex connectivity requirements $r_i \in \mathbb{N}_{\geq 0}$, $i \in V$. Vertices with $r_i = 0$ are called *Steiner vertices*. Let $M = (V, F)$ be a minimum cost spanning tree of G^+. Let $V^{odd} \subset V$ be the set of vertices of G^+ having odd degree in M. Let $J \subseteq E$ be a minimum cost T-join on G^+, where $T = V^{odd}$. Let E' be the edge set consisting of $\lceil \frac{r_m}{2} \rceil$ copies of the edges in F, and J' be the edge set consisting of $\lfloor \frac{r_m}{2} \rfloor$ copies of the edges in J. Let $K^+ = (V, E' \cup J')$. Let K be the multi-subgraph of G that is formed from K^+ by adding the Steiner vertices of G to K^+ and by replacing the edges of K^+ by the minimum-cost paths in G that gave rise to their edge costs. It will be seen at the end of this section that K is the heuristic solution to multi-SNDP that is returned by our approximation algorithm.

The following proposition is needed:

Proposition 4 *Let $G = (V, E)$ be a complete graph with non-negative edge costs $c \in \mathbb{R}^E$ and vertex connectivity requirements $r_i \in \mathbb{N}_{\geq 0}$, $i \in V$. The graph K^+ is a r_m-edge connected spanning multi-subgraph of G^+, the graph K is a r_m-edge connected multi-subgraph of G, and the total cost of K is equal to the total cost of K^+.*

Proof: We show first that K^+ is a r_m-edge connected spanning multi-subgraph of G^+. Since a connected graph has a T-join if and only if $|T|$ is even, and it is known that every graph has an even number of odd degree vertices, therefore, G^+ has a V^{odd}-join.

Clearly, all the vertices of the subgraph $H = (V, F \cup J)$ of G^+ have even degree (note that H may contain multiple copies of the edges). This means that H is an Eulerian graph, i.e., it contains an Euler tour. Hence, H is two-edge connected, i.e., it has two edge disjoint paths between every pair of vertices.

Thus, when r_m is even, K^+ partitions into $\frac{r_m}{2}$ Eulerian subgraphs, i.e., $\frac{r_m}{2}$ 2-edge connected subgraphs. Thus, K^+ has $\frac{r_m}{2} \cdot 2$ edge-disjoint paths (counting parallel edges as distinct edges), and is thus r_m-edge connected. When r_m is odd, K^+ partitions into $\frac{r_m-1}{2}$ Eulerian subgraphs and one minimum spanning tree, i.e., $\frac{r_m-1}{2}$ 2-edge connected subgraphs and one connected subgraph. Thus, K^+ has $\frac{r_m-1}{2} \cdot 2 + 1 = r_m$ edge-disjoint paths (counting parallel edges as distinct edges), and is thus r_m-edge connected. Hence, K^+ is a r_m-edge connected spanning multi-subgraph of G^+.

Form a multi-subgraph, K, of G from K^+ by adding the Steiner vertices of G to K^+ and by replacing the edges of K^+ by the minimum-cost paths in G that gave rise to their edge costs. This is done with no decrease in edge-connectivity. By construction, the total cost of K is equal to the total cost of K^+. ∎

Lemmas 2 and 3 lead to the following theorem:

Theorem 4 *Given a complete graph $G = (V, E)$ with non-negative edge costs $c \in \mathbb{R}^E$ and vertex connectivity requirements $r_i \in \mathbb{N}_{\geq 0}$, $i \in V$. The integrality gap for the linear programming relaxation of multi-SNDP is at most $\frac{3}{2} \frac{r_m}{r_l}$ when r_m is even, and at most $\frac{3r_m + 1}{2r_l} = \frac{3}{2} \frac{r_m}{r_l} + \frac{1}{2r_l}$ when r_m is odd.*

Proof: Let $cost(K)$ be the cost of K, and $cost(K^+)$ be the cost of K^+. Let $opt(M \ on \ G^+)$ be the cost of a minimum cost spanning tree of G^+, and let $opt(J \ on \ G^+)$ be the cost of an optimal V^{odd}-join on G^+. Using Lemma 2 and Proposition 3,

$$
\begin{aligned}
opt(J \ on \ G^+) &\leq \frac{1}{r_l} \ opt(\textit{multi-SNDP}_{LP} \ on \ G^+) \\
&= \frac{1}{r_l} \ opt(\textit{multi-SNDP}_{LP} \ on \ G).
\end{aligned}
$$

If r_m is even, then $\lceil \frac{r_m}{2} \rceil = \lfloor \frac{r_m}{2} \rfloor = \frac{r_m}{2}$, and, using Proposition 4 and Lemma 3 we have,

$$
\begin{aligned}
cost(K) = cost(K^+) &= \frac{r_m}{2} \cdot opt(M \ on \ G^+) + \frac{r_m}{2} \cdot opt(J \ on \ G^+) \\
&\leq \frac{r_m}{2} \cdot \frac{2}{r_l} \ opt(\textit{multi-SNDP}_{LP} \ on \ G) + \frac{r_m}{2} \cdot \frac{1}{r_l} \ opt(\textit{multi-SNDP}_{LP} \ on \ G) \\
&= \frac{3}{2} \frac{r_m}{r_l} \ opt(\textit{multi-SNDP}_{LP} \ on \ G). \quad (22)
\end{aligned}
$$

Therefore, on G,

$$
opt(\text{multi-SNDP}) \leq cost(K) \leq \frac{3}{2} \frac{r_m}{r_l} \ opt(\text{multi-SNDP}_{LP}),
$$

i.e.,

$$
\frac{opt(\text{multi-SNDP})}{opt(\text{multi-SNDP}_{LP})} \leq \frac{3}{2} \frac{r_m}{r_l}. \quad (23)
$$

Thus, when r_m is even, the integrality gap for the linear programming relaxation of multi-SNDP is at most $\frac{3}{2} \frac{r_m}{r_l}$.

Similarly, if r_m is odd, then $\lfloor \frac{r_m}{2} \rfloor = \frac{r_m - 1}{2}$, $\lceil \frac{r_m}{2} \rceil = \frac{r_m + 1}{2}$, and we have

$$
\begin{aligned}
cost(K) = cost(K^+) &= \frac{r_m + 1}{2} \cdot opt(M \ on \ G^+) + \frac{r_m - 1}{2} \cdot opt(J \ on \ G^+) \\
&\leq \frac{r_m + 1}{2} \cdot \frac{2}{r_l} \ opt(\textit{multi-SNDP}_{LP} \ on \ G) + \frac{r_m - 1}{2} \cdot \frac{1}{r_l} \ opt(\textit{multi-SNDP}_{LP} \ on \ G) \\
&= \left(\frac{3r_m + 1}{2r_l} \right) \ opt(\textit{multi-SNDP}_{LP} \ on \ G) \\
&= \left(\frac{3}{2} \frac{r_m}{r_l} + \frac{1}{2r_l} \right) \ opt(\textit{multi-SNDP}_{LP} \ on \ G). \quad (24)
\end{aligned}
$$

13

Therefore, on G,

$$opt(\text{multi-SNDP}) \leq cost(K) \leq \left(\frac{3}{2}\frac{r_m}{r_l} + \frac{1}{2r_l}\right) opt(multi\text{-}SNDP_{LP}),$$

i.e.,

$$\frac{opt(\text{multi-SNDP})}{opt(\text{multi-SNDP}_{LP})} \leq \frac{3}{2}\frac{r_m}{r_l} + \frac{1}{2r_l}. \tag{25}$$

Thus, the integrality gap for the linear programming relaxation of multi-SNDP when r_m is odd is at most $\frac{3}{2}\frac{r_m}{r_l} + \frac{1}{2r_l}$. $\blacksquare$

From Theorem 4, we obtain the following two corollaries.

Corollary 3 *Given a complete graph $G = (V, E)$ with non-negative edge costs $c \in \mathbb{R}^E$ and vertex connectivity requirements $r_i \in \mathbb{N}_{\geq 0}$, $i \in V$. The cost of K is less than or equal to $\frac{3}{2}\frac{r_m}{r_l}$ of the optimal solution value for the LP relaxation of multi-SNDP if r_m is even, and less than or equal to $\frac{3}{2}\frac{r_m}{r_l} + \frac{1}{2r_l}$ of the optimal solution value for the LP relaxation of multi-SNDP if r_m is odd.*

Proof: By equations (22) and (24). $\blacksquare$

Corollary 4 *Given a complete graph $G = (V, E)$ with non-negative edge costs $c \in \mathbb{R}^E$ and vertex connectivity requirements $r_i \in \mathbb{N}_{\geq 0}$, $i \in V$. The cost of K is less than or equal to $\frac{3}{2}\frac{r_m}{r_l}$ of the optimal solution value for multi-SNDP if r_m is even, and less than or equal to $\frac{3}{2}\frac{r_m}{r_l} + \frac{1}{2r_l}$ of the optimal solution value for multi-SNDP if r_m is odd.*

Proof: We know that $opt(\text{multi-SNDP}_{LP}) \leq opt(\text{multi-SNDP})$. Combining this with equations (22) and (24) we obtain the result. $\blacksquare$

The constructive proof of Theorem 4, along with Corollary 4, yields the following approximation algorithm for multi-SNDP:

Input: A complete graph $G = (V, E)$ with non-negative edge costs $c \in \mathbb{R}^E$ and vertex connectivity requirements $r_i \in \mathbb{N}_{\geq 0}$, $i \in V$.
Output: A r_m-edge connected spanning multi-subgraph (i.e., subgraph with multiple copies of the edges allowed) of G, having cost within $\frac{3}{2}\frac{r_m}{r_l}$ of the optimal solution value for multi-SNDP, r_m even, and $\frac{3}{2}\frac{r_m}{r_l} + \frac{1}{2r_l}$, r_m odd.

A0. Form the complete graph G^+ with vertex set V^+, respective vertex connectivity requirements from G, and edge costs c_{ij}^+, $i, j \in V^+$, the cost of a minimum-cost i-j path in G.

A1. Find a minimum cost spanning tree, $M = (V, F)$, of G^+ and take $\lceil \frac{r_m}{2} \rceil$ copies of its edges. Let this form the edge set E'. Let $V^{odd} \subset V$ be the set of vertices of G^+ having odd degree in M.

14

A2. Find a minimum cost T-join $J \subseteq E$, on G^+, where $T = V^{odd}$, i.e., M has a corresponding V^{odd}-join. This is accomplished by finding a minimum cost "pairing" of the odd degree vertices of M using shortest paths from G^+. Take $\lfloor \frac{r_m}{2} \rfloor$ copies of the edges in J, and let this form the edge set J'.

A3. Combine the minimum cost spanning trees from (A1) and the T-joins from (A2), i.e., form the multi-subgraph $K^+ = (V, E' \cup J')$.

A4. Form a multi-subgraph, K, of G from K^+ by adding the Steiner vertices of G to K^+ and by replacing the edges of K^+ by the minimum-cost paths in G that gave rise to their edge costs.

Theorem 5 *Given a complete graph $G = (V, E)$ with non-negative edge costs $c \in \mathbb{R}^E$ and vertex connectivity requirements $r_i \in \mathbb{N}_{\geq 0}$, $i \in V$. The above algorithm yields, in polynomial time, a feasible multi-subgraph K which is r_m-edge connected and which has cost less than or equal to $\frac{3}{2} \frac{r_m}{r_l}$ of the optimal solution value for multi-SNDP when r_m is even, and cost less than or equal to $\frac{3}{2} \frac{r_m}{r_l} + \frac{1}{2r_l}$ of the optimal solution value for multi-SNDP when r_m is odd. Thus, it is a $\frac{3}{2} \frac{r_m}{r_l}$-approximation algorithm for multi-SNDP, for r_m even, and a $(\frac{3}{2} \frac{r_m}{r_l} + \frac{1}{2r_l})$-approximation algorithm for multi-SNDP, for r_m odd.*

Proof: From Theorem 4, K is a r_m-edge connected spanning multi-subgraph of G. By Corollary 4, it has cost less than or equal to $\frac{3}{2} \frac{r_m}{r_l} opt(\text{multi-SNDP})$ when r_m is even, and cost less than or equal to $(\frac{3}{2} \frac{r_m}{r_l} + \frac{1}{2r_l}) opt(\text{multi-SNDP})$ when r_m is odd. From the discussion in Section 2, this algorithm has worst case running time $O(n^3)$. ∎

Jain [18] gave both an approximation guarantee of 2 for multi-SNDP, and an integrality gap of 2 for the LP relaxation of multi-SNDP. Given distinct vertex connectivity requirements $\rho_o = 0 < \rho_1 < \rho_2 < \ldots < \rho_q$, Goemans and Bertsimas' [16] improved tree heuristic for multi-SNDP has approximation guarantee, $\sum_{j=1}^{q} \frac{f(\rho_j - \rho_{j-1})}{\rho_j}$, where $f(c) = \frac{3}{2} c$ when c is even, and $f(c) = \frac{3c+1}{2}$ when c is odd. Although not indicated, the proof of this implicitly yields an integrality gap of $\sum_{j=1}^{q} \frac{f(\rho_j - \rho_{j-1})}{\rho_j}$.

For r_m even, our approximation guarantee (and the integrality gap of the LP relaxation) for multi-SNDP is strictly less than 2 (and is thus an improvement on Jain's result when applied to multi-SNDP) when $\frac{r_m}{r_l} < \frac{4}{3}$. For r_m odd, our approximation guarantee (and the integrality gap of the LP relaxation) for multi-SNDP is strictly less than 2 when $\frac{r_m}{r_l} + \frac{1}{3r_l} < \frac{4}{3}$.

When $r_m = r_l = k$ (i.e., when we are looking at the special case of multi-kEC), our approximation guarantee and integrality gap, and those of Goemans and Bertsimas are the same – equaling $\frac{3}{2}$ for k even, and $\frac{3}{2} + \frac{1}{2k}$ for k odd. When $r_m > r_l$, there are certain instances where our approximation guarantee and integrality gap is an improvement both on the bounds of Goemans and Bertsimas and on the bounds of Jain, hence providing for those instances a better approximation guarantee for multi-SNDP and integrality gap for the LP relaxation of multi-SNDP than has been previously known. It is difficult to

determine all the instances where our bound is an improvement on the best known bounds, however, for an example of a family of such instances, see the family in Section 3.1.

Thus, our integrality gap for the LP relaxation of multi-SNDP and our approximation algorithm for multi-SNDP are an improvement on the best known bounds for special cases of multi-SNDP.

3.1 A Family of Examples

In this section, we present a family of examples for multi-SNDP where the approximation guarantee for multi-SNDP of Theorem 5, and the integrality gap of the LP relaxation of multi-SNDP of Theorem 4, are both less than Jain's result and less than that of Goemans and Bertsimas (and hence are an improvement on the best known bounds). Let $G = (V, E)$ be a complete graph with non-negative edge costs $c \in \mathbb{R}^E$ and vertex connectivity requirements $r_i \in \mathbb{N}_{\geq 0}$, $i \in V$. Let $\mathcal{F}$ be the family of multi-SNDP problems such that:

(i) $1 < r_l < r_m$ if r_m is even; and $1 < r_l < r_m - 1$ if r_m is odd

(ii) $\frac{r_m}{r_l} < \frac{4}{3}$ if r_m is even; and $\frac{r_m}{r_l} + \frac{1}{3r_l} < \frac{4}{3}$ if r_m is odd

(iii) all the connectivity requirements are present between r_l and r_m (i.e., let the distinct connectivity requirements be $r_l, r_l + 1, r_l + 2, \ldots, r_m - 1, r_m$).

Recall that our bound (both approximation guarantee and integrality gap) for multi-SNDP is at most $\frac{3}{2}\frac{r_m}{r_l}$ when r_m is even, and $\frac{3}{2}\frac{r_m}{r_l} + \frac{1}{2r_l}$ when r_m is odd. We now show that on the family $\mathcal{F}$ this is an improvement on the best known bounds for multi-SNDP. First, we will show that condition (ii) ensures that our bound is strictly less than 2 for the family $\mathcal{F}$, and hence an improvement on Jain's bound [18]. This is seen by noting that $\frac{3}{2}\frac{r_m}{r_l} < 2$ if and only if $\frac{r_m}{r_l} < \frac{4}{3}$; and $\frac{3}{2}\frac{r_m}{r_l} + \frac{1}{2r_l} < 2$ if and only if $\frac{r_m}{r_l} + \frac{1}{3r_l} < \frac{4}{3}$.

Secondly, we will show that conditions (i)-(iii) ensure that our bound is strictly less than that of Goemans and Bertsimas [16]. Note that when $r_m = r_l$, our bound and that of Goemans and Bertsimas is the same, and less than that of Jain. Thus we only examine instances where $r_l < r_m$.

Notice that when $r_l = 1$ and r_m is even, condition (ii) implies that $r_m = 0$, which is a contradiction with $r_l < r_m$. When $r_l = 1$ and r_m is odd, condition (ii) implies that $r_m < 1$, which is a contradiction with $r_l < r_m - 1$. Thus we state that $r_l > 1$. Note that for both r_m even and r_m odd we have, by condition (ii), that $\frac{r_m}{r_l} < \frac{4}{3}$, and thus $\frac{1}{r_m} > \frac{3}{4r_l}$. Furthermore, notice that $r_l < r_l + 1 < r_l + 2 < \ldots < r_m - 1 < r_m$ implies that $\frac{1}{r_l} > \frac{1}{r_l+1} > \frac{1}{r_l+2} > \ldots > \frac{1}{r_m-1} > \frac{1}{r_m}$. To show that conditions (i)-(iii) ensure that our bound is strictly less than that of Goemans and Bertsimas for problems in the family $\mathcal{F}$, we consider two cases.

Case 1: r_l is odd, and r_m is even or odd. Notice that r_l is odd and $r_l < r_m$ automatically implies that $r_l < r_m - 1$ when r_m is odd. The bound of Goemans

and Bertsimas [16] for problems in $\mathcal{F}$ with r_l odd is

$$
\begin{aligned}
\frac{\frac{1}{2}[3r_l+1]}{r_l} + \frac{2}{r_l+1} + \frac{2}{r_l+2} + \ldots + \frac{2}{r_m} \quad &> \quad \frac{3r_l+1}{2r_l} + \frac{2}{r_m} + \frac{2}{r_m} + \ldots + \frac{2}{r_m} \\
&= \quad \frac{3r_l+1}{2r_l} + (r_m - r_l)\frac{2}{r_m}.
\end{aligned}
$$

For both r_m even and r_m odd we have $\frac{1}{r_m} > \frac{3}{4r_l}$, and thus

$$
\begin{aligned}
\frac{\frac{1}{2}[3r_l+1]}{r_l} + \frac{2}{r_l+1} + \frac{2}{r_l+2} + \ldots + \frac{2}{r_m} \quad &> \quad \frac{3r_l+1}{2r_l} + (r_m - r_l)(2)\left(\frac{3}{4r_l}\right) \\
&= \quad \frac{3r_l+1}{2r_l} + \frac{3r_m - 3r_l}{2r_l} \\
&= \quad \frac{3}{2}\frac{r_m}{r_l} + \frac{1}{2r_l} \\
&> \quad \frac{3}{2}\frac{r_m}{r_l}.
\end{aligned}
$$

Since our bound is at most $\frac{3}{2}\frac{r_m}{r_l}$ when r_m is even, and $\frac{3}{2}\frac{r_m}{r_l} + \frac{1}{2r_l}$ when r_m is odd, it follows that Goemans and Bertsimas' bound is always greater than our bound for problems in family $\mathcal{F}$ with r_l odd and r_m even or odd.

Case 2a: r_l is even, and r_m is even. The bound of Goemans and Bertsimas [16] for problems in $\mathcal{F}$ with r_l even is

$$
\begin{aligned}
\frac{\frac{3}{2}3r_l}{r_l} + \frac{2}{r_l+1} + \frac{2}{r_l+2} + \ldots + \frac{2}{r_m} \quad &> \quad \frac{3}{2} + \frac{2}{r_m} + \frac{2}{r_m} + \ldots + \frac{2}{r_m} \\
&= \quad \frac{3}{2} + (r_m - r_l)\frac{2}{r_m}.
\end{aligned}
$$

Applying $\frac{1}{r_m} > \frac{3}{4r_l}$, we have

$$
\begin{aligned}
\frac{\frac{3}{2}3r_l}{r_l} + \frac{2}{r_l+1} + \frac{2}{r_l+2} + \ldots + \frac{2}{r_m} \quad &> \quad \frac{3}{2} + (r_m - r_l)(2)\left(\frac{3}{4r_l}\right) \\
&= \quad \frac{3r_l}{2r_l} + \frac{3r_m - 3r_l}{2r_l} \\
&= \quad \frac{3}{2}\frac{r_m}{r_l}.
\end{aligned}
$$

Since our bound is at most $\frac{3}{2}\frac{r_m}{r_l}$ when r_m is even, it follows that Goemans and Bertsimas' bound is always greater than our bound for problems in family $\mathcal{F}$ with r_l even and r_m even.

Case 2b: r_l is even, and r_m is odd. Notice that if r_l is even, r_m is odd, and $r_l = r_m - 1$, then our bound is not strictly less than that of Goemans and Bertsimas: Assume it is, i.e., assume that $\frac{3}{2}\frac{r_m}{r_l} + \frac{1}{2r_l} < \frac{\frac{3}{2}r_l}{r_l} + \frac{2}{r_m} = \frac{3}{2} + \frac{2}{r_m}$. Substituting in $r_l = r_m - 1$, we get $\frac{3r_m+1}{2(r_m-1)} < \frac{3r_m+4}{2r_m}$. Simplifying, this leads to

17

$0 < -8$, which is a contradiction. Thus, we must have that $r_l < r_m - 1$ when r_m is odd.

The bound of Goemans and Bertsimas [16] for problems in $\mathcal{F}$ with r_l even is

$$
\begin{aligned}
\frac{\frac{3}{2}3r_l}{r_l} + \frac{2}{r_l+1} + \frac{2}{r_l+2} + \ldots + \frac{2}{r_m} \quad & > \quad \frac{3}{2} + \frac{2}{r_m-1} + \frac{2}{r_m-1} + \ldots + \frac{2}{r_m-1} + \frac{2}{r_m} \\
& = \quad \frac{3}{2} + (r_m - r_l - 1)\frac{2}{r_m-1} + \frac{2}{r_m}
\end{aligned}
$$

Since our bound is at most $\frac{3}{2}\frac{r_m}{r_l} + \frac{1}{2r_l}$ when r_m is odd, it follows that Goemans and Bertsimas' bound is always greater than our bound for problems in family $\mathcal{F}$ with r_l even and r_m odd provided that

$$
\frac{3}{2} + (r_m - r_l - 1)\frac{2}{r_m-1} + \frac{2}{r_m} \quad > \quad \frac{3}{2}\frac{r_m}{r_l} + \frac{1}{2r_l}. \tag{26}
$$

Re-arranging and simplifying, this is true provided that

$$
7r_m^2 r_l - 3r_m r_l - 4r_m r_l^2 - 4r_l - 3r_m^3 + 2r_m^2 + r_m \quad > \quad 0.
$$

Notice that $r_m > r_l + 1$, r_l even, and r_m odd imply that $r_m = r_l + n$, where $n \in \mathbb{N}$ is odd and $n \geq 3$. Substituting in $r_m = r_l + n$, where $n \geq 3$ is odd, we obtain that the above inequality is equivalent to

$$
(n-1)r_l^2 + (-2n^2 + n - 3)r_l + (-3n^3 + 2n^2 + n) \quad > \quad 0, \quad n \geq 3, \ n \text{ odd.} \tag{27}
$$

Note that $r_m < \frac{4}{3}r_l - \frac{1}{3}$ (from condition (ii)) and $r_m = r_l + n$, $n \geq 3$ odd, implies that $r_l > 3n + 1$, i.e., that $r_l \geq 3n + 3$ for $n \geq 3$ odd. Using this, it is easily seen that inequality (27) is true. Thus, inequality (26) is true. Hence, Goemans and Bertsimas' bound is always greater than our bound for problems in family $\mathcal{F}$ with r_l even and r_m odd.

Therefore, Goemans and Bertsimas' bound is always greater than our bound for problems in family $\mathcal{F}$. Hence, for problems in family $\mathcal{F}$, our bound is an improvement on both the bound of Jain and the bound of Goemans and Bertsimas.

4 Application of Multi-SNDP to SNDP and Metric SNDP

As application to SNDP, applying Proposition 2 to Lemmas 2 and 3, we obtain the following:

Corollary 5 *Given a complete graph $G = (V, E)$ with non-negative edge costs $c \in \mathbb{R}^E$, let $G^\triangle = (V, E)$ be the metric completion of G, with metric costs $c^\triangle \in \mathbb{R}^E$. Then*

$$
opt(\text{T-join on } G) \ \leq \ \frac{1}{r_l} \ opt(SNDP_{LP} \text{ on } G^\triangle) \leq \ \frac{1}{r_l} \ opt(SNDP \text{ on } G^\triangle),
$$

and

$$opt(MST \ on \ G) \ \leq \ \frac{2}{r_l} \ opt(SNDP_{LP} \ on \ G^{\triangle}) \leq \ \frac{2}{r_l} \ opt(SNDP \ on \ G^{\triangle}).$$

Applying this to metric-SNDP, we obtain

Corollary 6 *Given a complete graph $G' = (V, E)$ with metric costs,*

$$opt(\textit{T-join}) \ \leq \ \frac{1}{r_l} \ opt(metric \ SNDP_{LP}) \leq \ \frac{1}{r_l} \ opt(metric \ SNDP),$$

and

$$opt(MST \ on \ G'^{+}) \ \leq \ \frac{2}{r_l} \ opt(metric \ SNDP_{LP} \ on \ G') \leq \ \frac{2}{r_l} \ opt(metric \ SNDP \ on \ G').$$

5 Application of Multi-SNDP to Multi-kEC, kEC, and Steiner Multi-kEC

The results for multi-SNDP apply to multi-kEC as follows: Lemmas 2 and 3 hold for multi-kEC. Simply set $r_i = k$ for all $i \in V$. The remainder of the results are similar, except that adjustments need to be made based on the fact that the graph G^+ is no longer required.

For multi-kEC, we need to define one special graph. Let $G = (V, E)$ be a complete graph with non-negative edge costs $c \in \mathbb{R}^E$ and vertex connectivity requirements $r_i \in \mathbb{N}_{\geq 0}$, $i \in V$. Let $M = (V, F)$ be a minimum cost spanning tree of G. Let $V^{odd} \subset V$ be the set of vertices of G having odd degree in M. Let $J \subseteq E$ be a minimum cost T-join on G, where $T = V^{odd}$. Let E' be the edge set consisting of $\lceil \frac{r_m}{2} \rceil$ copies of the edges in F, and J' be the edge set consisting of $\lfloor \frac{r_m}{2} \rfloor$ copies of the edges in J. Let $K = (V, E' \cup J')$. It will be seen that K is the heuristic solution to multi-kEC that is returned by our abbreviated approximation algorithm.

The following proposition is obtained from Proposition 4.

Proposition 5 *Given a complete graph $G = (V, E)$ with non-negative edge costs $c \in \mathbb{R}^E$, $k \geq 2$, $k \in \mathbb{Z}$. The graph $K = (V, E' \cup J')$ is a k-edge connected spanning multi-subgraph of G.*

Theorem 4 (on integrality gap), Corollary 3, and Corollary 4, all apply to multi-kEC. Simply set $r_m = r_l \equiv k$ and $r_i \equiv k$ for all $i \in V$.

When applying the approximation algorithm for multi-SNDP in Section 3 to multi-kEC, $r_m = r_l \equiv k$, and steps A0 and A4 can be omitted. It follows directly from Theorem 5 that this is a $\frac{3}{2}$-approximation algorithm for multi-kEC, for k even, and a $(\frac{3}{2} + \frac{1}{2k})$-approximation algorithm for multi-kEC, for k odd.

Note that the bounds for integrality gap and our approximation algorithm hold as well for Steiner multi-kEC, and, in this case, are strictly less than 2 and

equal to the bounds of Goemans and Bertsimas. Hence, they are equal to the best known values to date.

In 1994, Khuller and Vishkin [20] presented a 2-approximation algorithm for kEC, given a general connected simple graph with non-negative edge costs. Their algorithm works by converting to a directed graph, and does not create multiple copies of edges in the heuristic solution.

When the edge costs are uniform in a general graph, Gabow, Goemans, Tardos, and Williamson [13], gave a rounding algorithm in 2005 that has an approximation guarantee of $1 + \frac{2}{k}$ for kEC when $k > 1$ is even, and $1 + \frac{3}{k}$ when k is odd. Using Jain's iterated rounding algorithm, they improved this guarantee to $1 + \frac{2}{k}$ when k is odd. They also provide tight examples for these guarantees. In addition, they showed that the integrality gap of the LP relaxation of kEC with uniform edge costs is at most $1 + \frac{2}{k}$, both for undirected and directed graphs.

It is interesting to note that, for uniform edge costs in a general multi-graph, Gabow [12] in 2004 gave a $\frac{3}{2}$-approximation algorithm for finding a 3-edge connected (spanning) subgraph.

Setting $r_l = k$, Corollary 5 applies to kEC, and Corollary 6 applies to metric kEC.

6 Application of Multi-2EC to Metric 2EC

Given a complete graph $G' = (V, E)$ with metric edge costs and $|V| \geq 3$, equality holds in Lemma 1 for the special case where $r_i = k = 2$ for all $i \in V$. This is shown by Alexander, Boyd, and Elliott-Magwood [2] in their Lemma 3. In other words, when the edge costs are metric in a complete graph $G' = (V, E)$ with $|V| \geq 3$, we have that

$$opt(\text{multi-}2EC \text{ on } G') \;=\; opt(2EC \text{ on } G').$$

It is important to note that equality in Lemma 1 does not hold in general. In fact, equality does not even hold in general for kEC and multi-kEC for $k > 2$, even for metric edge costs. For example, take the family of complete graphs on $k + 1$ vertices, $k > 2$. Arrange the vertices in a circle. Assign edge costs 1 to all the "outside" edges, and edge cost 2 to all the "inside" edges. These edge costs are metric. The optimal kEC solution is the graph itself, with total cost $1(k + 1) + 2[\frac{(k-2)(k+1)}{2}] = k^2 - 1$. An optimal multi-$k$EC solution, k even, is obtained by taking $\frac{k}{2}$ copies of all the outside edges, with total cost $\frac{k}{2}(k + 1) < k^2 - 1$. An optimal multi-$k$EC solution, k odd, is obtained by taking $\frac{k+1}{2}$ copies of k of the outside edges and $(\frac{k+1}{2}) - 1$ copies of the remaining outside edge, with total cost $(\frac{k+1}{2})(k) + [(\frac{k+1}{2}) - 1] = \frac{1}{2}(k^2 + 2k - 1) < k^2 - 1$. Thus, for this family of graphs with metric edge costs and $k > 2$, $opt(\text{multi-}kEC) < opt(kEC)$. Figure 1 illustrates this family of graphs for $k = 3$.

In 2006, Alexander, Boyd, and Elliott-Magwood showed that the integrality gap of the LP relaxation of multi-2EC (and metric 2EC) is $\leq 3/2$ [2], and
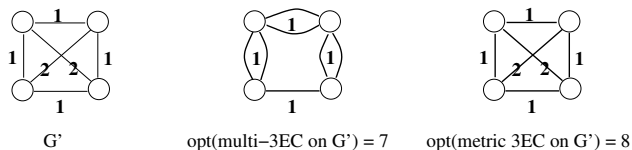
Figure 1: $k = 3$ in the family of graphs.

showed the exact integrality gap for graphs of up to 10 vertices, and a tight lower bound for the integrality gap for graphs of 11 to 14 vertices. In both [3] and [2], it is mentioned that examples exist with an integrality gap ratio which asymptotically approaches $6/5$, thus indicating that the integrality gap for the LP relaxation of multi-2EC (and metric 2EC) is between $6/5$ and $3/2$. In 1998, Carr and Ravi [3] conjectured that the integrality gap is in fact $4/3$, and presented a result that provides some support for this. In 1982, Frederickson and Ja'Ja' presented a $3/2$-approximation algorithm that applies to metric 2EC [10].

Note that when edge costs are metric, $k = 2$, and there are at least three vertices in the graph, Alexander, Boyd, and Elliott-Magwood [2] show that there is always an optimal solution of multi-2EC that is simple. In other words, given a complete graph $G' = (V, E)$, with $|V| \geq 3$ and with metric edge costs $c' \in \mathbb{R}^E$,

$$opt(\text{multi-}2EC \ on \ G') \quad = \quad opt(metric \ 2EC \ on \ G'). \tag{28}$$

Problems multi-2EC and metric 2EC are closely related. This is shown by the following proposition.

Proposition 6 *Given a complete graph $G = (V, E)$, $|V| \geq 3$, with non-negative edge costs $c \in \mathbb{R}^E$, let $G^\triangle = (V, E)$ be the metric completion of G, with metric costs $c^\triangle \in \mathbb{R}^E$. Then*

$$opt(multi\text{-}2EC \ on \ G) \quad = \quad opt(metric \ 2EC \ on \ G^\triangle).$$

Proof: Set $k = 2$ in Corollary 2 and apply equation (28) with $G' = G^\triangle$. ∎

It is interesting to note that, for graphs that are their own metric completion, there is a relationship between the integrality gaps for the linear programming relaxation of multi-2EC and for the linear programming relaxation of metric 2EC.

Lemma 4 *Given a complete graph $G' = (V, E)$, $|V| \geq 3$, with metric edge costs $c' \in \mathbb{R}^E$, if the metric completion $G^\triangle$ is the same as G', i.e., if $G^\triangle = G'$, then the integrality gap for the linear programming relaxation of multi-2EC on G' is less than or equal to the integrality gap for the linear programming relaxation of metric 2EC on G'.*

Proof: By Lemma 1,

$$opt(multi\text{-}2EC \ on \ G') \quad \leq \quad opt(metric \ 2EC \ on \ G').$$

By equation (28), $opt(\text{multi-}2EC \text{ on } G') = opt(metric \ 2EC \text{ on } G')$. Hence, $opt(\text{multi-}2EC_{LP} \text{ on } G') = opt(metric \ 2EC_{LP} \text{ on } G')$. We therefore have that

$$\frac{opt(\textit{multi-}2EC \ \textit{on} \ G')}{opt(\text{multi-}2EC_{LP} \ \textit{on} \ G')} \leq \frac{opt(metric \ 2EC \ \textit{on} \ G')}{opt(metric \ 2EC_{LP} \ \textit{on} \ G')}.$$

∎

The following lemma comes from equation (28) and the proof of Theorem 4. It agrees with a result presented by Alexander, Boyd, and Elliott-Magwood [2].

Lemma 5 *Given a complete graph $G' = (V, E)$, $|V| \geq 3$, with metric edge costs $c' \in \mathbb{R}^E$, the integrality gap for the linear programming relaxation of metric 2EC is at most $\frac{3}{2}$.*

A graph is said to be *k-vertex connected* if it has at least k internally vertex-disjoint paths between every pair of vertices. All k-vertex connected graphs are k-edge connected, but not vice versa. Given a complete graph, $G = (V, E)$, with non-negative edge costs $c \in \mathbb{R}^E$, the *k-vertex connected subgraph problem* (kVC) is the problem of finding a minimum cost k-vertex connected spanning subgraph of G. Problem 2VC is sometimes called the *biconnected augmentation problem*. Problem kVC with metric edge costs will be referred to as *metric kVC*. For a summary of the current known performance guarantees on the different versions of kVC, see [21].

It can be shown that $opt(metric \ 2EC) = opt(metric \ 2VC)$. This is an implicit result of Frederickson and Ja'Ja''s shortcutting method (cf [10]), and is used by Carr and Ravi in [3], and by Monma, Munson, and Pulleyblank in [23].

In 1982, Frederickson and Ja'Ja' examined the biconnected augmentation problem on complete graphs with metric edge costs and $F = \emptyset$ [10], i.e., metric 2VC. Following a similar idea to that of Christofides' $\frac{3}{2}$-approximation algorithm for the metric travelling salesman problem [5], Frederickson and Ja'Ja' presented a $\frac{3}{2}$-approximation algorithm for metric 2EC and metric 2VC.

Given a complete graph $G' = (V, E)$ with metric edge costs, our approximation algorithm for multi-kEC in Section 3 can be modified to get a $\frac{3}{2}$-approximation algorithm for metric 2EC that is similar to that of Frederickson and Ja'Ja''s. The modification is as follows: Set $k = 2$ and let $\hat{H}$ be the heuristic solution obtained from our approximation algorithm (cf Section 3) applied to G'. Note that $\hat{H}$ is a 2-edge connected spanning multi-subgraph. Also note that any multi-edges of $\hat{H}$ consist of exactly two parallel edges. (Otherwise, we can remove at least one copy of the parallel edges without affecting the 2-edge connectivity, thus obtaining a cheaper solution.) We can remove the parallel edges of $\hat{H}$ as follows [10]: For each edge uv in $J' \cap E'$: (a) remove uv from J'; (b) find an adjacent edge ux or vw in E', remove it from E' and replace it with either vx or uw, respectively; (c) insert the remaining edges of J' into E', and call this combined set of edges $\tilde{E}$. Form the simple subgraph $\tilde{H} = (V, \tilde{E})$.

Corollary 7 *Given a complete graph $G' = (V, E)$, $|V| \geq 3$, with metric edge costs $c' \in \mathbb{R}^E$, the approximation algorithm from Section 3 (with $k = 2$), with*

the addition of the above transformation as step (A4), is a $\frac{3}{2}$-approximation algorithm for metric $2EC$.

Proof: Set $k = 2$ and let $\hat{H}$ be the heuristic solution obtained from the approximation algorithm from Section 3 applied to G'. Let $\tilde{H}$ be the resulting subgraph after applying the above transformation to $\hat{H}$.

Note that, by construction, all the vertices of $\hat{H}$ have even degree. Without loss of generality, assume that the edge ux is removed along with uv in the step (A4) above, and xv is added. The degree of u decreases by two, and the degree of x and v remain the same. Since the vertex u was incident with the parallel edge uv and the edge ux, and its degree was even, therefore the degree of u was greater than or equal to four. Thus, decreasing the degree of u by two does not disconnect u from the graph. Hence, the resulting subgraph is connected and still has vertices of all even degree. As well, no new parallel edges are created, since the edge xv is not in $\hat{H}$ (if xv were in E' then (V, E') would contain a cycle (cycle $xuvx$) and thus would not be a minimum spanning tree; and if xv were in J' then J' would not be a T-join for $T = V'$). Repeating for each of the parallel edges of $\hat{H}$, we see that the resulting subgraph $\tilde{H}$ is connected and has vertices of all even degree. Hence, $\tilde{H}$ is 2-edge connected. $\tilde{H}$ is also simple.

Due to metric costs, the transformation of step (A4) is performed at no increase in total edge cost. Thus, $\tilde{H}$ is a 2-edge connected spanning simple subgraph of G' with cost at most $\frac{3}{2}opt(metric\ 2EC)$. Since step (A4) can also be done in polynomial time, this yields a $\frac{3}{2}$-approximation algorithm for metric 2EC. ∎

Note that if, furthermore, we wish the metric 2EC heuristic solution $\tilde{H}$ to be 2-vertex connected, then we apply Frederickson and Ja'Ja"s shortcutting method [10] to obtain at equal cost (due to metric costs) a 2-vertex connected spanning simple subgraph in the following manner: Choose a cut vertex v of $\tilde{H}$ and find edges uv and vw in $\tilde{E}$ such that u and w are in different blocks (maximal two-vertex connected components of the graph). Replace uv and vw by $uw \in E$. The blocks containing u and w will then become one block. Repeat until all the blocks of the subgraph are merged and no cut vertices remain. This forms a $\frac{3}{2}$-approximation algorithm for metric 2VC. The proof of our approximation algorithm for this case provides an alternate, simpler, and more compact proof for Frederickson and Ja'Ja"s result.

It is also worthwhile noting that, given a complete graph $G = (V, E)$ with non-negative edge costs, a $\frac{3}{2}$-approximation algorithm for multi-2EC can be obtained by taking the metric completion $G^{\triangle}$ of G, applying Frederickson and Ja'Ja"s $\frac{3}{2}$-approximation algorithm for metric 2EC [10], and then using the transformation by Goemans and Bertsimas in the proof of their Theorem 3 [16], to transform the heuristic solution on $G^{\triangle}$ back to a heuristic solution on G at no increase in cost and no decrease in edge connectivity.

7 Acknowledgment

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