The cobasis structure of the extreme points of the SEP polytope *

Sylvia Boyd and Paul Elliott-Magwood
University of Ottawa, Ottawa, Canada
email: sylvia@site.uottawa.ca

Abstract

A careful study of the points of the Subtour Elimination Polytope could lead to a better approximation algorithm for the Travelling Salesman Problem or at least a better understanding of how good a lower bound on the optimal value of the Travelling Salesman Problem we obtain by optimizing over the Subtour Elimination Polytope. In this paper, we look at the structure of the tight cut constraints of an extreme point of the Subtour Elimination Polytope. We introduce the idea of a Laminar Basis Tree to compactly store these tight cuts. Although a given extreme point may have many different cobases, we can limit our attention to those whose Laminar Basis Trees have special properties.

Let $K_n = (V, E)$ be the complete graph on n vertices. The *Subtour Elimination Polytope* (henceforth abbreviated to SEP) is the subset of $\mathbb{R}^E$ which obeys the following constraints.

$$
\begin{aligned}
x(\delta(v)) &= 2 \text{ for all } v \in V && (1)\\
x(\delta(S)) &\geq 2 \text{ for all } \emptyset \subset S \subset V && (2)\\
x_e &\geq 0 \text{ for all } e \in E && (3)
\end{aligned}
$$

*This research was partially supported by grants from the Natural Sciences and Engineering Research Council of Canada

The contraints of the form (1) above are called the *vertex equalities*. The constraints of the form (2) are called the *cut constraints* and (3) are called the *zero-edge inequalities* or the *nonnegativity constraints*.

Given a point x of the SEP, a constraint is said to be *tight* if it holds with equality for x. If a constraint of the form (2) given by $x(\delta(S)) \geq 2$ is tight then we call $\delta(S)$ a *tight cut* and S a *tight set*. We also call an edge $e \in E$ a *1-edge* if $x_e = 1$. We define $E_x = \{e \in E \mid x_e > 0\}$ and we then define the *support graph* of x to be the subgraph $G_x = (V, E_x)$.

Lemma 1. *Let G be a connected graph with n vertices and m edges. The rank of the tight edge inequalities and the vertex equalities is $\frac{1}{2}n(n-1)-m+n-1$ if G is bipartite and $\frac{1}{2}n(n-1)-m+n$ otherwise.*

Proof. We would like to build up a linearly independent set of constraints for G. The set of edge inequalities is linearly independent since the edge inequalities each correspond to a distinct edge variable. When we add all the vertex equalities to the edge inequalities corresponding to the edges that are in $\overline{G}$, we can then use the edge inequalities to eliminate all the variables corresponding to edges that are in $\overline{G}$. Thus the problem can be reduced to that of finding the rank of the vertex equalities in G.

Suppose that the set of vertex equalities of G is linearly dependent. Then there is a non-zero linear combination of the vertex equalities which sums to zero. Now if uv is an edge of G then the edge variable, x_{uv}, appears in exactly two vertex equalities - namely the equalities corresponding to u and v. Hence if the coefficient of the vertex equality corresponding to u is non-zero then so is that corresponding to v. Furthermore, the coefficent corresponding to v is just the negative of that of u. Thus all the neighbours of a vertex with a non-zero coefficient must also have a non-zero coefficient and since G is connected, this means that all the coefficients are non-zero. Since every vertex equality must be present in order for our set to be linearly dependent, we know that the rank of the vertex equalities in G must be exactly $n-1$. Furthermore, by the same reasoning, the coefficients all have the same absolute value. As well, the neighbours of any vertex with a positive coefficient are all have negative coefficients and vice versa. Hence the set of vertices of G with positive coefficients is an independent set and the set of vertices of G with negative coefficients is an independent set. Therefore, G is bipartite.

Conversely, if G is bipartite with bipartition (V_1, V_2) then

$$\sum_{v\in V_1} x(\delta(v)) - \sum_{v\in V_2} x(\delta(v)) = 0$$

and thus the set of vertex equalities of G is linearly dependent and, as noted above, has rank $n-1$.

Therefore G is not bipartite if and only if the tight edge equalities along with the vertex equalities have full rank, namely $\frac{1}{2}n(n-1)-m+n$. Otherwise the rank of this set of constraints is $\frac{1}{2}n(n-1)-m+n-1$. □

Corollary 2. *If G is a connected graph with a cobasis consisting entirely of vertex equalities then either G is a tree or G is a 1-tree with an odd cycle.*

Proof. If G is bipartite then by Lemma 1 the rank of the vertex equalities is exactly $n-1$. However, the vertex equalities form a cobasis for G so G has $n-1$ edges. Furthermore, G is connected and so G must be a tree. If G is not bipartite then by Lemma 1 the rank of the vertex equalities is exactly n. Again, G must have n edges and must be a 1-tree since G is connected. However, since G is not bipartite, the unique cycle of G must be an odd cycle. □

Let x be an extreme point of the SEP-polytope on n vertices. Let G_x be the support graph of x and let m be the number of edges of G_x. Consider building a cobasis for x starting with the set of the tight edge inequalities and the vertex equalities. Lemma 1 tells us that this set has full rank if G_x is not bipartite. Thus we can extend this set to a cobasis for x by adding precisely $m-n$ tight cut inequalities. If G is bipartite then we can remove any vertex equality from our set and build up a cobasis from there by adding precisely $m-n+1$ tight cut inequalities.

By a result similar to the one found in [1] we may assume that the vertex sets inducing the tight cut constraints that we are adding to form the cobasis form a Laminar set. The maximum size of a Laminar set on n elements is $2n-1$. However any such Laminar set contains all the singletons, the set consisting of all n elements, and the complement of a maximal proper subset. These sets cannot induce cut sets in our cobasis. Hence the maximum number of cuts which are in our cobasis is $n-3$. This type of counting argument was used in [2] when examining the extreme points of the Asymmetric Subtour Elimination Problem polytope.

There is a way of compactly storing the Laminar set of tight cuts in our cobasis by means of a labelled tree which we will call the *Laminar Basis Tree* or LBT for short. The LBT has one node for each tight cut in our Laminar set plus one extra root node. The node corresponding to a set S in our Laminar set will be labelled with the set of vertices of G_x which are in S but not in any proper subset of S in the Laminar set. The root node will be labelled with the set of all vertices of G_x that are in no set in the Laminar set. Two nodes will be adjacent if for the corresponding sets, one is the minimal set which contains the other. The root node will be adjacent to all the sets which are not contained in any other. As an example, Figure 1 shows a Laminar set and Figure 2 shows the corresponding Laminar Basis Tree.

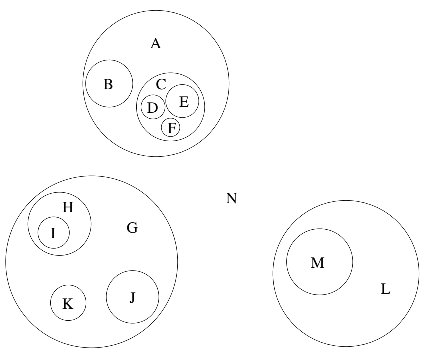

Figure 1: A Laminar Set

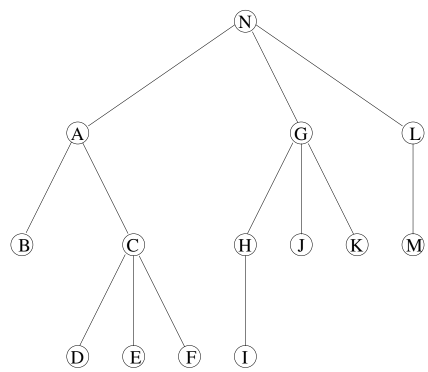

Figure 2: The LBT

Notice now, that if we take a maximal subset in our Laminar set and replace it with its complement, we get another Laminar set which is also a cobasis (along with the tight edge inequalities and the vertex equalities) for x. This occurs because our new Laminar set induces the exact same cuts as the old one. In fact, the constraints in the cobasis are exactly the same. For example, if we take the Laminar set in Figure 1 and we replace A with $\overline{A}$ we get the Laminar set shown in Figure 3 whereas Figure 4 shows the corresponding Laminar Basis Tree.

Compare the Laminar Basis Trees in Figure 2 and Figure 4. They are

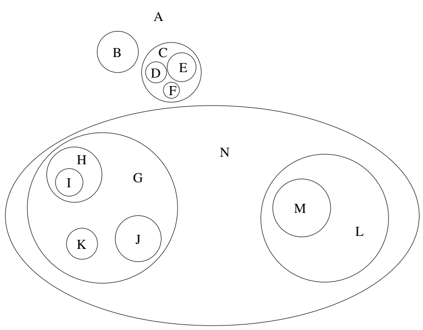

Figure 3: A new Laminar Set

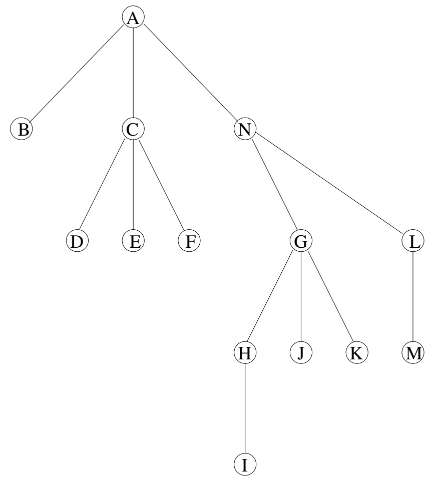

Figure 4: The new LBT

identical, the only difference being which node is the root. By repeating this process of replacing a maximal subset in the Laminar set with its complement, we can make any node the root in the corresponding Laminar Basis Tree. However, these changes to the Laminar set do not actually change the constraints in the cobasis. Hence, for a given cobasis, we can construct an LBT which we will consider to be unrooted. We now proceed with some properties of LBT's.

Proposition 3. *Let T be a Laminar Basis Tree for a support graph with n vertices and let n_1 and n_2 denote the numbers of nodes of T of degree 1 and 2 respectively. Then*

1. *T has at most $n-2$ nodes,*
2. *the total number of labels on the nodes of T is n,*
3. *every node of T of degree 1 must contain at least two labels,*
4. *every node of T of degree 2 must contain at least one label, and*
5. *$2n_1 + n_2 \leq n$.*

Proof. As noted above, the number of tight cuts in our Laminar set contained in the cobasis is at most $n-2$. Since the number of nodes in the LBT is the same as the number of subsets in the corresponding Laminar set, T has at most $n-2$ nodes. Secondly, since every label corresponds to a distinct vertex in the support graph, there are exactly n labels. Thirdly, any node of T of degree 1 is a subset in the corresponding Laminar set and hence, corresponds to a cut constraint. Any subset inducing a cut constraint must have at least two vertices and thus the corresponding node must have at least two labels. Fourthly, any node of T of degree 2 (we may assume that this node is not the root, since we can easily change the root as described above) corresponds to a subset in the laminar set which contains a single maximal proper subset. Since these subsets are different, there must be at least one label on the node. Lastly, since each node of T of degree 1 must have at least two labels, each node of degree 2 must have at least one label, and the total number of labels is n we have that $2n_1 + n_2 \leq n$. □

Proposition 4. *Let x be an SEP extreme point with support graph G_x and let S be the set of labels corresponding to some node, v, of a Laminar Basis Tree, T, for some cobasis of x. Then each component of $G_x[S]$ is either a tree or a 1-tree with an odd cycle. Futhermore $G_x[S]$ has at most $deg_T(v)$ components.*

Proof. No tight cut in our cobasis contains an edge of $G_x[S]$. Consider reducing the right hand sides of the vertex equalities for the vertices of S by the total flow, in x, on edges of $\delta(S)$ incident to each vertex. Hence, the x-values assigned to the edges of a component of $G_x[S]$ are completely determined by the tight edge inequalities of $G_x[S]$ and these new vertex equalities of $G_x[S]$. Thus, by a similar argument as in Corollary 2, the component must be a tree or a 1-tree with an odd cycle.

Now let W denote the tight set of x corresponding to the node v in the tree. Let t denote the number of tight sets in the cobasis properly contained in W. Let c denote the number of components in $G_x[S]$. Since W is a tight set and so are its t proper tight sets from the cobasis, the amount of flow entering the components of $G_x[S]$ is at most $2t+2$. However, since x is an SEP extreme point, each component of $G_x[S]$ needs at least 2 units of flow entering it and there is no edge between any two such components. Hence $2c \leq 2t+2$ or $c \leq t+1$. However, $\deg_T(v) = t+1$ so the result holds. □

Proposition 5. *Let x be an SEP extreme point with support graph G_x. Then for any tight set, S, $G_x[S]$ is connected. If $G_x[S]$ is not 2-vertex-connected then for any cut vertex, v, of $G_x[S]$, $G_x[S] - v$ has exactly two components, V_1 and V_2. Furthermore, V_1 and V_2 are tight sets and $N_{G_x}(v) \subseteq V_1 \cup V_2$.*

Proof. Suppose $G_x[S]$ is not connected. Let $V_1, V_2, \ldots, V_k$ be the components of G_x where $k \geq 2$. Then

$$\begin{aligned} x(\delta(S)) &= x(\delta(V_1)) + x(\delta(V_2)) + \ldots + x(\delta(V_k)) \\ &\geq 2k \\ &\geq 4 \end{aligned}$$

which contradicts the fact that S is a tight set of x. Therefore, $G_x[S]$ must be connected.

Now suppose $G_x[S]$ has a cut vertex, v, and let $V_1, V_2, \ldots, V_k$ be the components of $G_x[S] - v$ where $k \geq 2$. Then the amount of flow in x between v and $V_1 \cup V_2 \cup \ldots \cup V_k$ is the total amount of flow in x leaving each of $V_1, V_2, \ldots, V_k$ minus the amount of flow leaving S. Thus

$$\begin{aligned} x(\delta(v)) &\geq x(\delta(V_1)) + x(\delta(V_2)) + \ldots + x(\delta(V_k)) - x(\delta(S)) \\ &\geq 2k - 2 \end{aligned}$$

However, $x(\delta(v)) = 2$ and $k \geq 2$ so it must be that $k = 2$ and V_1 and V_2 are tight sets. Furthermore, we have that $x(\delta(v)) = x(\delta(V_1)) + x(\delta(V_2))$ so $N_{G_x}(v) \subseteq V_1 \cup V_2$. □

Theorem 6. *Let x be an SEP extreme point with support graph G_x. For any subset, S corresponding to a leaf node of an Laminar Basis Tree of x, $G_x[S]$ is*

1. *a 1-path of odd length,*
2. *an odd cycle, or*
3. *a lollipop graph containing an odd cycle as shown in Figure 5.*

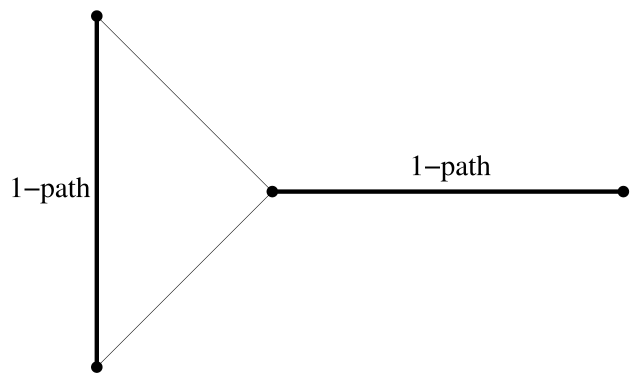

Figure 5: A lollipop graph with an odd cycle

Proof. By Proposition 4 we know that every component of $G_x[S]$ is either a tree or a 1-tree with an odd cycle. However, S is also a tight set so by Proposition 5, $G_x[S]$ has exactly one component. If $G_x[S]$ is a tree then it cannot have a vertex of degree 3 or more since then the removal of this vertex would create at least 3 components, contradicting Proposition 5. Thus $G_x[S]$ must be a path. Furthermore, every internal vertex of this path is a cut vertex so it's neighbours in G_x must also be in S. Hence, every edge of the path has an x-value of 1. If $G_x[S]$ is an even path with vertices $v_1, v_2, \ldots, v_k$ where k is odd then

$$x(\delta(S)) = \sum_{\substack{i=1 \\ i \text{ odd}}}^{k} x(\delta(v_i)) - \sum_{\substack{i=2 \\ i \text{ even}}}^{k-1} x(\delta(v_i)).$$

Thus the cut constraint induced by S is linearly dependent with the vertex equalities and so they cannot be in a cobasis together. This contradicts the fact that S is a tight set in our LBT. Therefore, $G_x[S]$ must be an odd path.

Now suppose that $G_x[S]$ is a 1-tree with an odd cycle. By removing the edges of this unique cycle, we obtain a forest on the vertices of S. By the same reasoning as above, each of the trees in the forest must be a path such that the x-values of all the edges on the path are 1. Hence, the endpoint of each path which is not on the cycle must contribute exactly one unit of flow to $x(\delta(S))$. Since S is a tight set, there can be at most two such paths. If there are exactly two paths then there is no flow between a vertex on the cycle and $\overline{S}$. Let u and v denote the two vertices on the odd cycle of $G_x[S]$ which have degree 3. Every other vertex on the cycle has degree 2 in G_x and hence is incident to precisely two edges with x-values of 1. Since the cycle in $G_x[S]$ is odd, there must be at least one such vertex. Furthermore, v is adjacent to such a vertex, call it w, and so $x_{vw} = 1$. But v is also incident to an edge on the two paths described above which has an x-value of 1. But $x(\delta(v)) = 2$ so v has degee 2 in $G_x[S]$ which is a contradiction. Therefore, by removing the edges of the cycle, we get at most one path. Hence, $G_x[S]$ is either an odd cycle or a lollipop graph with an odd cycle. If $G_x[S]$ is a lollipop graph, let v be its unique vertex of degree 3. By the same reasoning as above, every other vertex of the cycle is incident to exactly two edges with x-values of 1 and so G_x must be exactly a lollipop graph as depicted in Figure 5. □

Corollary 7. *Let x be an SEP extreme point with support graph G_x. There is a cobasis for x such that for any subset, S, corresponding to a leaf node of the Laminar Basis Tree of x, $G_x[S]$ is either a single edge or an odd cycle.*

Proof. If $G_x[S]$ is the lollipop graph shown in Figure 5 then let v be the unique vertex of degree 3. Let V_1 and V_2 be the components of $G_x[S] - v$. Then

$$x(\delta(S)) = x(\delta(V_1)) + x(\delta(V_2)) - x(\delta(v)).$$

Hence, we can replace S in the cobasis with V_1 and V_2. Only one of these sets is needed in the cobasis and $G_x[V_1]$ and $G_x[V_2]$ are paths. As noted in Theorem 6, the path that is added to the cobasis in place of S must be an odd path. Furthermore, since S is a minimal set in our Laminar set and both V_1 and V_2 are properly contained in S, replacing S with either V_1 or V_2 will result in another Laminar cobasis. We continue in this way until, for any subset, S, corresponding to a leaf node of the current LBT, $G_x[S]$ is either an odd cycle or an odd path.

If $G_x[S]$ is an odd path with vertices $v_1, v_2, \ldots, v_k$ then notice that

$$x(\delta(S)) = x(\delta(\{v_1, v_2\})) - \sum_{\substack{i=3 \\ i \text{ odd}}}^{k-1} x(\delta(v_i)) + \sum_{\substack{i=4 \\ i \text{ even}}}^{k} x(\delta(v_i)).$$

Thus we can replace S with $\{v_1, v_2\}$ in our cobasis and obtain a new Laminar Basis. We continue in this way until, for any subset, S, corresponding to a leaf node of the current LBT, $G_x[S]$ is either an odd cycle or a single edge. $\square$

For any graph H let $\kappa(H)$ denote the number of components of H.

Theorem 8. *Let x be an SEP extreme point with support graph G_x. There is a cobasis for x such that all the tight cut constraints in the cobasis form a Laminar set and for each cut constraint $x(\delta(S)) = 2$ in the cobasis, $G_x[S]$ is 1-tough.*

Proof. Let $\mathcal{B}$ be the laminar set of tight sets in our cobasis for x. If $G_x[S]$ is not 1-tough for every $S \in \mathcal{B}$ then choose $S \in \mathcal{B}$ where S is a tight set such that $G_x[S]$ is not 1-tough and S is of minimum cardinality with respect to this property. Hence, let $T \subset S$ be such that $|T| < \kappa(G_x[S] - T)$ and let $V_1, \ldots, V_k$ be the components of $G_x[S] - T$. Notice that since $\kappa(G_x[S] - T)$ is integer,

$$|T| + 1 \leq \kappa(G_x[S] - T).$$

Now,

$$x(\delta(T)) + x(\delta(S)) = \sum_{i=1}^{k} x(\delta(V_i)) + 2x(T, \overline{S})$$

but S is a tight set so

$$x(\delta(T)) + 2 = \sum_{i=1}^{k} x(\delta(V_i)) + 2x(T, \overline{S}).$$

In addition, $x(\delta(V_i)) \geq 2$ for all $1 \leq i \leq k$ and hence

$$x(\delta(T)) + 2 \geq 2k + 2x(T, \overline{S}).$$

However, $x(\delta(T)) \leq 2|T|$ so

$$2|T| + 2 \geq 2k + 2x(T, \overline{S}).$$

But k is the number of components of $G_x[S] - T$ and thus $k = \kappa(G_x[S] - T)$, hence

$$|T| + 1 \geq \kappa(G_x[S] - T) + x(T, \overline{S}).$$

However, from above, $|T| + 1 \leq \kappa(G_x[S] - T)$ and so we must have that

- $|T| + 1 = \kappa(G_x[S] - T)$,
- $x(T, \overline{S}) = 0$,
- $x(\delta(T)) = 2|T|$ and hence T is an independent set of G_x, and
- V_i is a tight set for each $1 \leq i \leq k$.

Thus,

$$x(\delta(S)) = \sum_{i=1}^{k} x(\delta(V_i)) - \sum_{v \in T} x(\delta(v))$$

where $V_1, \ldots, V_k$ are tight sets. Hence we can extend $\mathcal{B} \backslash \{S\}$ to a cobasis of x by adding one of the tight sets among $V_1, \ldots, V_k$ to obtain a new cobasis.

Suppose, for a contradiction, that there exists some $R \in \mathcal{B}$ such that $R \subset S$ and R crosses some V_i where $1 \leq i \leq k$. Assumer, without loss of generality, that R crosses V_1. Then R must intersect T since otherwise, $R \subset S \backslash T$ and R contains vertices from at least two different components of $G_x[S \backslash T]$. Thus $G_x[R]$ is not connected, contradicting the fact that R is a tight set.

From [1] we have that if U and W are tight sets which cross then $U - W$ and $W - U$ are also tight sets where $x(U - W, U \cap W) = x(W - U, U \cap W) = 1$. Hence if R crosses l components of $G_x[S] - T$ then $x(\delta(R)) \geq l$. However, since R is a tight set, it must be that $l \leq 2$.

Case 1: $l = 2$

Suppose, without loss of generality, that R also crosses V_2 and let r be the number of components of $G_x[S] - T$ which intersect R. From the above note, we see that if $l = 2$ then $x(R, V_1 - R) = 1$ and $x(R, V_2 - R) = 1$. Furthermore, from [1] we get that $x(V_1 - R, R - V_1) = 0$ and $x(V_2 - R, R - V_2) = 0$. Thus the edges of $\delta(R)$ are edges whose endpoints are either both in V_1 or both in V_2. Since $V_1 - R$ and $V_2 - R$ are tight sets, we deduce that

$$\begin{aligned} x(\delta(R \cap T)) &= 2(r - 2) + 2 \\ 2|R \cap T| &= 2r - 2 \\ |R \cap T| &= r - 1. \end{aligned}$$

But $G_x[R] - (R \cap T)$ has r components, and by the minimality of S, $G_x[R]$ must be 1-tough so $|R \cap T| \geq r$ which is a contradiction. This completes Case 1.

Case 2: $l = 1$

Suppose, without loss of generality, that R intersects $V_1, \ldots, V_r$. Then

$$\begin{aligned}
x(\delta(R \cap T)) &= x(R \cap T, V_1) + \sum_{i=2}^{r} x(R \cap T, V_i) + \sum_{i=r+1}^{k} x(R \cap T, V_i) \\
&= 1 \sum_{i=2}^{r} (x(\delta(V_i)) - x(V_i, \overline{S}) - x(V_i, T - R)) + \sum_{i=r+1}^{k} x(R \cap T, V_i) \\
&= 1 \sum_{i=2}^{r} (2 - x(V_i, \overline{S}) - x(V_i, T - R)) + \sum_{i=r+1}^{k} x(R \cap T, V_i) \\
&= 1 + 2(r-1) - \sum_{i=2}^{r} x(V_i, \overline{S}) - \sum_{i=2}^{r} x(V_i, T - R)) + \sum_{i=r+1}^{k} x(R \cap T, V_i).
\end{aligned}$$

But

$$\begin{aligned}
x(\delta(R)) &= x(R \cap V_1, V_1 - R) + \sum_{i=2}^{r} x(V_i, \overline{S}) \\
&\quad + \sum_{i=2}^{r} x(V_i, T - R)) + \sum_{i=r+1}^{k} x(R \cap T, V_i) \\
2 &= 1 + \sum_{i=2}^{r} x(V_i, \overline{S}) + \sum_{i=2}^{r} x(V_i, T - R)) \\
&\quad + \sum_{i=r+1}^{k} x(R \cap T, V_i) \\
- \sum_{i=2}^{r} x(V_i, \overline{S}) - \sum_{i=2}^{r} x(V_i, T - R)) &= -1 + \sum_{i=r+1}^{k} x(R \cap T, V_i)
\end{aligned}$$

Hence,

$$x(\delta(R \cap T)) = 1 + 2(r-1) - 1 + \sum_{i=r+1}^{k} x(R \cap T, V_i) + \sum_{i=r+1}^{k} x(R \cap T, V_i)$$

$$\begin{aligned} &= 2r-2+2\sum_{i=r+1}^{k} x(R\cap T, V_i) \\ 2|R\cap T| &= 2r-2+2\sum_{i=r+1}^{k} x(R\cap T, V_i) \\ |R\cap T| &= r-1+\sum_{i=r+1}^{k} x(R\cap T, V_i) \end{aligned}$$

However, by the minimality of S, $G_x[R]$ is 1-tough. But $G_x[R]-(R\cap T)$ has r components so $|R\cap T|\geq r$. Thus,

$$\sum_{i=r+1}^{k} x(R\cap T, V_i) \geq 1.$$

But,

$$\begin{aligned} x(\delta(R)) &\geq x(R\cap V_1, V_1-R)+\sum_{i=r+1}^{k} x(R\cap T, V_i) \\ 2 &\geq 1+\sum_{i=r+1}^{k} x(R\cap T, V_i) \\ \sum_{i=r+1}^{k} x(R\cap T, V_i) &\leq 1. \end{aligned}$$

Therefore,

$$\sum_{i=r+1}^{k} x(R\cap T, V_i) = 1.$$

As a result, $\sum_{i=2}^{r} x(V_i, \overline{S}) = 0$ and $\sum_{i=2}^{r} x(V_i, T-R)) = 0$. Thus,

$$x(\delta(R)) = x(\delta(V_1-R)) + \sum_{v\in R\cap T} x(\delta(v)) - \sum_{i=1}^{r} x(\delta(V_i)).$$

Therefore, we know that $(\mathcal{B}\backslash\{S,R\})\cup\{V_1-R, V_1,\ldots,V_k\}$ contains a cobasis for x. This completes Case 2.

We will proceed in this way, replacing each $R \in \mathcal{B}$ where $R \subset S$ and R crosses V_i for some $1 \leq i \leq k$ with $V_i - R$. Once we have completed all these replacements, we may have two tight sets of our set of tight sets which cross. However these two tight sets must be contained in some V_i where $1 \leq i \leq k$. We can use the uncrossing theorem from [1] to obtain a new set of tight sets where these tight sets no longer cross.

Notice that if R does not cross any V_i and R is not a subset of some V_i then it is never changed and remains in the current set of tight sets. Furthermore, since all the replaced sets are contained in some V_i, R does not cross any of those sets either. Thus we have a Laminar set of tight sets. All that is left to do is remove some of the tight sets to obtain a cobasis. We will remove S and choose to remove only tight sets that are subsets of S to obtain a new cobasis $\mathcal{B}'$.

If the minimum cardinality tight set of $\mathcal{B}'$ which induces a subgraph which is not 1-tough is smaller than $|S|$ then we know that this tight set is a subset of S. We will repeat this entire process. Eventually we will arrive at a Laminar Basis which has fewer tight sets which induce subgraphs which are not 1-tough than $\mathcal{B}$. By again repeating, we will arrive at a cobasis of x which is Laminar and for which the tight sets all induce 1-tough subgraphs. $\square$

References

[1] G. Cornuéjols, J. Fonlupt, and D. Naddef, "The Traveling Salesman Problem on a Graph and Some Related Integer Polyhedra", *Mathematical Programming* 33, 1 - 27, 1985.

[2] S. Vempala and M. Yannakakis, "A Convex Relaxation for the Asymmetric TSP", *Proceedings of the tenth annual ACM-SIAM symposium on Discrete algorithms*, 975 - 976, 1999.