Operations for Generating SEP Extreme Points *

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Abstract

A careful study of the points of the Subtour Elimination Polytope could lead to a better approximation algorithm for the Travelling Salesman Problem or at least a better understanding of how good a lower bound on the optimal value of the Travelling Salesman Problem we obtain by optimizing over the Subtour Elimination Polytope. In this paper we look at two simple operations which we can use to generate many new extreme points of the Subtour Elimination Polytope on n vertices from our knowledge of the extreme points of the Subtour Elimination Polytope on n' vertices where $n' < n$. We also look at what structures we can observe in an extreme point that tell us that the extreme point can be obtained by the application of one of the operations.

Let $K_n = (V, E)$ be the complete graph on n vertices. The *Subtour Elimination Polytope* (henceforth abbreviated to SEP) is the subset of $\mathbb{R}^E$ which obeys the following constraints.

$$x(\delta(v)) = 2 \text{ for all } v \in V \tag{1}$$

$$x(\delta(S)) \geq 2 \text{ for all } \emptyset \subset S \subset V \tag{2}$$

$$x_e \geq 0 \text{ for all } e \in E \tag{3}$$

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The contraints of the form (1) above are called the *vertex equalities*. The constraints of the form (2) are called the *cut constraints* and (3) are called the *zero-edge inequalities* or the *nonnegativity constraints*.

Given a point x of the SEP, a constraint is said to be *tight* if it holds with equality for x. If a constraint of the form (2) given by $x(\delta(S)) \geq 2$ is tight then we call $\delta(S)$ a *tight cut* and S a *tight set*. We also call an edge $e \in E$ a *1-edge* if $x_e = 1$. We define $E_x = \{e \in E \mid x_e > 0\}$ and we then define the *support graph* of x to be the subgraph $G_x = (V, E_x)$.

Lemma 1. *Let x be a point of the SEP-polytope. Let S be a tight set of x such that $|S| \geq 2$ and there exists an edge $uv \in \delta(S)$ such that $x_{uv} = 1$ and let $v \in S$. Then $S \backslash \{v\}$ is a tight set of x and*

$$x(\delta(S)) = x(\delta(S\backslash\{v\})) - x(\delta(v)) + 2x_{uv}.$$

Proof. Suppose, for a contradiction, that there exists some $w \in \overline{S}\backslash\{u\}$ such that $x_{vw} > 0$. Thus by the vertex equality, $x(\delta(v)) = 2$, we know that $x(v, S\backslash\{v\}) < 1$. However, $x(\delta(S\backslash\{v\})) \geq 2$ so $x(S\backslash\{v\}, \overline{S}) > 1$. However, since $x_{uv} = 1$ and $x_{vw} > 0$ we know that $x(v, \overline{S}) > 1$. Now

$$\begin{aligned} x(\delta(S)) &= x(\delta(\overline{S})) \\ &= x(v, \overline{S}) + x(S\backslash\{v\}, \overline{S}) \\ &> 1 + 1 \\ &= 2 \end{aligned}$$

which contradicts the fact that $x(\delta(S)) = 2$. Therefore, $x(v, \overline{S}) = 1$ and so $x(v, S\backslash\{v\}) = 1$ because of the vertex equality $x(\delta(v)) = 2$. As well, we have that $x(S\backslash\{v\}, \overline{S}) = 1$ since $x(\delta(\overline{S})) = 2$. Thus $x(\delta(S\backslash\{v\})) = x(v, S\backslash\{v\}) + x(S\backslash\{v\}, \overline{S}) = 2$ and the result follows. □

Please note that for Proposition 2, we only consider tight sets in the cobasis of size at least 3 and at most $n - 3$.

Proposition 2. *Let x be an extreme point of the SEP-polytope on n vertices. Then there is a cobasis, $\mathcal{B}$, for x such that the tight sets corresponding to the tight cut constraints in $\mathcal{B}$ form a Laminar set and no such tight cut contains a 1-edge.*

Proof. Begin with a cobasis consisting of all the valid equalities of the form $x_{uv} = 0$ and $x_{uv} = 1$. Add to this as many vertex equalities as possible (of course the rest will be implied). Build up a cobasis for x by then adding tight cut constraints for tight sets of size between 3 and $n-3$. We can use the uncrossing lemma to ensure that the tight sets of the tight cut constraints in the cobasis form a Laminar set.

Suppose that there is a tight cut in the cobasis which contains a 1-edge. Among all such tight cuts, choose one, call it $\delta(S)$, whose tight set is inclusionwise minimal within the Laminar set. Let uv be a 1-edge of $\delta(S)$ where $v \in S$.

Suppose, for a contradiction, that $S\backslash\{v\}$ crosses another tight set, T, in the Laminar set induced by our cobasis. Then T and S are not disjoint and it cannot be that $S \subset T$. Hence, due to the Laminar nature of our tight sets in the cobasis, we know that $T \subset S$. However, since T crosses $S\backslash\{v\}$ it must be that $v \in T$. On the other hand, $u \notin S$ and $T \subset S$ so $u \notin T$. Thus $uv \in \delta(T)$ and we contradict the fact that S is a minimal subset of our Laminar set which induces a tight cut in our cobasis containing a 1-edge. Hence, if we replace S by $S\backslash\{v\}$ we still have a Laminar set.

By Lemma 1 we know that $x(\delta(S)) = 2$ can be replaced in our cobasis by $x(\delta(S\backslash\{v\})) = 2$ since $x_{uv} = 1$ is already in the cobasis and $x(\delta(v)) = 2$ is implied by the constraints in our cobasis. Clearly $x(\delta(S\backslash\{v\})) = 2$ cannot already be in the cobasis since this constraint is implied by $x(\delta(S)) = 2$, $x(\delta(v)) = 2$, and $x_{uv} = 1$.

Therefore, we obtain a new cobasis which has all the same properties as the old - namely that the tight sets of the tight cut constraints in the cobasis form a Laminar set. Furthermore, this new cobasis is the same as the old except that one of the tight sets has been replaced with a smaller one. If we repeatedly choose a tight cut containing a 1-edge and whose tight set is inclusionwise minimal within the Laminar set we can generate cobasis after cobasis. Each repetition reduces the size of one of the tight sets, so we cannot repeat this action indefinitely. Therefore, we must arrive at a cobasis, $\mathcal{B}$, where no tight cut in the cobasis contains a 1-edge. Additionally, we have preserved the Laminar property of these bases at every iteration so the tight sets corresponding to tight cut constraints in $\mathcal{B}$ form a Laminar set. □

Lemma 3. *Let x be a feasible point of the SEP-polytope on n vertices and* *let G_x be the support graph of x. Let S be a tight set of x such that $G_x[S]$ is a cycle. If $x_e < 1$ for every $e \in \gamma(S)$ then every tight set of x either contains S or is disjoint from S.*

Proof. Let $T \neq S$ be a tight set of x which intersects S.

Suppose, for a contradiction, that S properly contains T. Since T is a tight set, $G_x[T]$ is connected and hence is a path. However, due to the vertex equalities, $x(\delta(T)) = 2$ if and only if $x(\gamma(T)) = |T| - 1$. But $G_x[T]$ is a path and so contains exactly $|T| - 1$ edges. Hence $x_e = 1$ for every $e \in \gamma(T)$. This contradicts the fact that $x_e < 1$ for every $e \in \gamma(S)$. Therefore S cannot properly contain a tight set.

Suppose, for a contradiction, that S does not properly contain S and T does not properly contain S. Then $S - T$ is either a single vertex or a tight set. As noted above, $S - T$ cannot be a tight set since it is a tight set properly contained in S so it must be that $S - T$ is a single vertex. However, since $T - S$ is nonempty we also have that $S \cap T$ is either a tight set or a single vertex. Again, since $S \cap T$ is a tight set properly contained in S it must be that $S \cap T$ is a single vertex. But $S = (S - T) \cup (S \cap T)$ so $|S| = 2$. This contradicts the fact that $G_x[S]$ is a cycle and hence must contain at least three vertices.

The only remaining possibility is that that $S \subset T$.

Therefore any tight set of x either contains S or is disjoint from S. □

Lemma 4. *Let x be an extreme point of the SEP-polytope on n vertices and let G_x be the support graph of x. Let S be a tight set of x such that $G_x[S]$ is an odd cycle. If $x_e < 1$ for every $e \in \gamma(S)$ then no vertex of $\overline{S}$ can be adjacent to two distinct vertices of S in G_x.*

Proof. Let $uv \in \delta(S)$ such that $v \in S$.

Suppose, for a contradiction, that there exists some $w \in S$ such that $w \neq v$ but $uw \in \delta(S)$. Choose any $\epsilon > 0$ and let P be the unique even $\{v, w\}$-path in $G_x[S]$. Let $e_1, \ldots, e_k$ be edges encountered as we travel along

P from v to w. Define $e_{k+1} = wu$ and $e_{k+2} = uv$. Define x' such that

$$x'_e = \begin{cases} x_e + \epsilon & \text{if } e = e_i \text{ for some even } i \\ x_e - \epsilon & \text{if } e = e_i \text{ for some odd } i \\ x_e & \text{otherwise} \end{cases} .$$

Since $e_1, \ldots, e_k, e_{k+1}, e_{k+2}$ form an even cycle in G_x and since we alternated whether or not we added or subtracted ϵ to the x-values around this cycle to obtain the x'-values, the vertex equalities all hold for x'. Futhermore, since $x_{e_1}, \ldots, x_{e_{k+2}} > 0$ we know that x' does not violate any of the tight zero-edge equalities of x. Lastly, since the edges $e_1, \ldots, e_{k+2}$ form a cycle in G_x, every tight cut of x must contain an even number of these edges. However, by Lemma 3, no tight cut can contain the edges $e_1, \ldots, e_k$. Thus every tight cut of x either contains none of the edges of the cycle or it contains only the edges e_{k+1} and e_{k+2}. In either case, if T is a tight set of x then it is also a tight set of x'.

Thus all the inequalities which define the SEP-polytope and which are tight for x are also tight for x'. However, x is an extreme point and hence is the unique solution to its set of tight inequalities so we have a contradiction. Therefore every vertex of $\overline{S}$ can be adjacent to at most one vertex of S. □

Let x be a feasible point of the SEP-polytope with support graph G_x and let S be a tight set of x. We let $x \downarrow S$ to denote the set of edge values induced on G_x/S (where S is identified to a single vertex, v) as follows.

$$(x \downarrow S)_e = \begin{cases} x(u, S) & \text{if } e = uv \\ x_e & \text{otherwise} \end{cases}$$

Lemma 5. *Let x be a feasible point of the SEP-polytope on n vertices and let G_x be the support graph of x. If S is a tight set of x then $x \downarrow S$ is a feasible point of the SEP-polytope on $n - |S| + 1$ vertices.*

Proof. Clearly $x'_e \geq 0$ for every edge e so the edge inequalities all hold for x'. Let us call v the vertex obtained by the contraction of S. If $w \neq v$ is a vertex of G_x/S then $x'(\delta(w)) = x(\delta(v)) = 2$. Also $x'(\delta(v)) = x(\delta(S)) = 2$ so all of the vertex equalities hold for x'. Now let T be any subset of the vertices of G_x/S where $2 \leq |T| \leq n - |S| - 1$. If $v \notin T$ then $x'(\delta(T)) = x(\delta(T)) \geq 2$. If $v \in T$ then $x'(\delta(T)) = x(\delta(S \cup (T \backslash \{v\})) \geq 2$. Hence all of the cut constraints hold for x' and therefore x' is a feasible point of the SEP-polytope. □

Theorem 6. *Let x be an extreme point of the SEP-polytope on n vertices and let G_x be the support graph of x. Let S be a tight set of x such that $G_x[S]$ is an odd cycle. If $x_e < 1$ for every $e \in \gamma(S)$ then $x \downarrow S$ is an extreme point of the SEP-polytope on $n - |S| + 1$ vertices.*

Proof. Let $x' = x \downarrow S$. By Lemma 5, x' is a feasible point of the SEP-polytope on $n - |S| + 1$ vertices.

By Lemma 4, every edge of G_x/S corresponds to a unique edge of G_x. Thus contracting S in x results simply in removing the edges of $\gamma(S)$ and relabelling the edges of $\delta(S)$. By Lemma 3, no tight cut of x contains any edges of $\gamma(S)$ and furthermore, every tight set of x is either disjoint from S or contains S. Notice too, from Lemma 5 that every tight set of x' corresponds to a tight set of x. Additionally, by Lemma 3, every tight set of x corresponds to a unique tight set of x' except for S which corresponds to v. Hence, the tight cut constraints of x' are just relabellings of tight cut constraints of x and vice versa (with the exception that $x(\delta(S)) = 2$ gets mapped to $x'(\delta(v)) = 2$. Furthermore, every vertex equality $x(\delta(w)) = 2$ for each $w \notin S$ gets relabelled to the vertex equality $x'(\delta(w)) = 2$.

Let m be the number of edges in G_x. Then G_x/S has $m - |S|$ edges. If we take any cobasis of x which contains all of the tight zero-edge inequalities and remove the tight zero-edges inequalities then we get a linearly independent set of vertex equalities and tight cuts constraints of size m. If we then remove any vertex equalities which correspond to vertices in S we get a linearly independent set of constraints of size at least $m - |S|$. By contracting S we get a relabelling of the variables in these constraints. This new set of constraints is therefore linearly independent and is feasible for x'. Furthermore, these constraints only contain variables corresponding to the edges of G_x/S. However, G_x/S only has $m - |S|$ edges so this set of constraints, along with the tight zero-edge inequalities of x', must form a cobasis for x'. Therefore, x' is an extreme point. □

Let x be a feasible point of the SEP-polytope on n vertices and let G_x be the support graph of x. Let v be a vertex of G_x. Suppose we can partition the edges of G_x incident to v into k non-empty parts, $(E_1, \ldots, E_k)$ where

$k \geq 3$ is odd and for each $0 \leq i \leq k-1$ we have that

$$\sum_{j=0}^{\frac{1}{2}(k-3)} x(E_{i+2j+2}) < 1$$

(where all indices are taken modulo k). Then we define $x \uparrow_v (E_0, \ldots, E_{k-1})$ as follows. Remove v from G_x and add k new vertices, $v_0, \ldots, v_{k-1}$. Let $S = \{v_0, \ldots, v_{k-1}\}$

$$(x \uparrow_v (E_0, \ldots, E_{k-1}))_e = \begin{cases} x_e & \text{if } e \in \gamma(\overline{S}) \\ x_{uv} & \text{if } e = uv_i \in \delta(S) \\ \sum_{j=0}^{\frac{1}{2}(k-3)} x(E_{i+2j+2}) & \text{if } e = v_i v_{i+1} \\ 0 & \text{otherwise} \end{cases}.$$

Lemma 7. *Let let x be a feasible point of the SEP-polytope on n vertices and let G_x be the support graph of x. Let $k \geq 3$ be an odd integer and let v be a vertex of G_x. If the edges incident to v in G_x can be partitioned into k non-empty parts, $(E_0, \ldots, E_{k-1})$, such that for each $0 \leq i \leq k-1$ we have that*

$$\sum_{j=0}^{\frac{1}{2}(k-3)} x(E_{i+2j+2}) < 1$$

(where all indices are taken modulo k) then $x \uparrow_v (E_0, \ldots, E_{k-1})$ is a feasible point of the SEP-polytope on $n+k-1$ vertices. Furthermore, if S is the set of the new vertices added to $G_x - v$ then S is a tight set of $x \uparrow_v (E_0, \ldots, E_{k-1})$ and every tight set of $x \uparrow_v (E_0, \ldots, E_{k-1})$ either contains S or is disjoint from S.

Proof. Let $x' = x \uparrow_v (E_0, \ldots, E_{k-1})$ and let $G_{x'}$ be the support graph of x' with vertex set V'.

Clearly the nonnegativity constraints hold.

If $w \in V' \backslash S$ then $x'(\delta(w)) = x(\delta(w)) = 2$. As well, for each $0 \leq i \leq k-1$,

$$\begin{aligned} x'(\delta(v_i)) &= x'_{v_{i-1}v_i} + x'_{v_i v_{i+1}} + \sum_{w \in V' \backslash S} x'_{v_i w} \\ &= \sum_{j=0}^{\frac{1}{2}(k-3)} x(E_{i+2j+1}) + \sum_{j=0}^{\frac{1}{2}(k-3)} x(E_{i+2j+2}) + x(E_i) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{k-1} x(E_j) \\
&= x(\delta(v)) \\
&= 2.
\end{aligned}
$$

Therefore the vertex equalities hold.

Let $T \subset V'$ such that $2 \leq |T| \leq n+k-3$. Notice that $x = x' \downarrow S$ so if $S \cap T = \emptyset$ then $x'(\delta(T)) = x(\delta(T)) \geq 2$. If $S \subseteq T$ then $x'(\delta(T)) = x(\delta((T \backslash S) \cup \{v\})) \geq 2$. If $T \subset S$ then $G_{x'}[T]$ is a collection of paths since $G_{x'}[S]$ is a cycle. However, $\sum_{j=0}^{\frac{1}{2}(k-3)} x(E_{i+2j+2}) < 1$ for each $0 \leq i \leq k-1$ and hence $x'_e < 1$ for every $e \in \gamma(T)$. Thus $x'(\gamma(T)) < |T|-1$ and so, by the vertex equalities, we get that $x'(\delta(T)) > 2$.

The only case that remains occurs when $S \cap T \neq \emptyset$, $S - T \neq \emptyset$, and $T - S \neq \emptyset$. Furthermore, since $\delta(\overline{T}) = \delta(T)$, we may assume that these properties hold for $\overline{T}$ as well. In particular, we may assume that $S \cup T \subset V'$. Now, by simple comparison of the variables, we have that

$$x'(\delta(T)) = x'(\delta(S-T)) + x'(\delta(T-S)) - x'(\delta(S)) + 2x'(S \cap T, \overline{S \cup T}).$$

But $S - T \subset S$ so $x'(\delta(S-T)) > 2$ as noted above. We also have that $(T-S) \cap S = \emptyset$ so from above, $x'(\delta(T-S)) \geq 2$. Trivially, $S \subseteq S$ so from above, $x'(\delta(S)) = x(\delta(v)) = 2$. Lastly, due to the nonnegativity of the variables, $x'(S \cap T, \overline{S \cup T}) \geq 0$. Putting all these pieces together, we get that

$$x'(\delta(T)) > 2 + 2 - 2 + 0$$

and therefore, $x'(\delta(T)) > 2$. Hence, in all cases, $x'(\delta(T)) \geq 2$ and x' obeys the cut constraints. Therefore x' is a feasible point of the SEP-polytope.

Notice that in the above discussion, that $x'(\delta(S)) = 2$ and if T is not disjoint from S and T does not contain S then $x'(\delta(T)) > 2$. Thus, the result follows. □

Theorem 8. *Let let x be an extreme point of the SEP-polytope on n vertices and let G_x be the support graph of x. Let $k \geq 3$ be an odd integer and let v be a vertex of G_x. If the edges incident to v in G_x can be partitioned into k non-empty parts, $(E_0, \ldots, E_{k-1})$, such that for each $0 \leq i \leq k-1$ we have* *that*

$$\sum_{j=0}^{\frac{1}{2}(k-3)} x(E_{i+2j+2}) < 1$$

(where all indices are taken modulo k) then $x \uparrow_v (E_0, \ldots, E_{k-1})$ *is an extreme point of the SEP-polytope on* $n + k - 1$ *vertices.*

Proof. Let $x' = x \uparrow_v (E_0, \ldots, E_{k-1})$ and let $G_{x'}$ be the support graph of x' with vertex set V'. By Lemma 7 we know that x' is a feasible point of the SEP-polytope on $n+k-1$ vertices. Let $S = \{v_0, \ldots, v_{k-1}\}$ be the set of new vertices added to $G_x - v$. Notice that $G_{x'}[S]$ is an odd cycle where $x'_e < 1$ for every $e \in \gamma(S)$. In fact, $x = x' \downarrow S$.

Suppose that

$$x' = \frac{1}{2}y + \frac{1}{2}z$$

where y and z are feasible points of the SEP-polytope on $n + k - 1$ vertices. Since S is a tight set of x', it must also be a tight set of y and z. Thus by Lemma 5 $y \downarrow S$ and $z \downarrow S$ are feasible points of the SEP-polytope on n vertices. Additionally $x = \frac{1}{2}(y \downarrow S) + \frac{1}{2}(z \downarrow S)$. Since x is an extreme point we have that $y \downarrow S = x$ and $z \downarrow S = x$. Futhermore, the support graphs of y and z, call them G_y and G_z respectively, must each be subgraphs of $G_{x'}$. Hence every vertex of $\overline{S}$ is adjacent to at most one vertex of S in both G_y and G_z. Therefore, y and z all have the same edge values as x' on every edge except possibly those of the odd cycle of $G_{x'}[S]$.

Now, for any $0 \leq i \leq k-1$ we have that

$$\begin{aligned} y_{v_i v_{i+1}} &= y(\gamma(S)) + \sum_{j=0}^{\frac{1}{2}(k-3)} y(v_{2j+i+2}, \overline{S}) - \sum_{j=0}^{\frac{1}{2}(k-3)} y(\delta(v_{2j+i+2})) \\ &= (k-1) + \sum_{j=0}^{\frac{1}{2}(k-3)} x'(v_{2j+i+2}, \overline{S}) - \sum_{j=0}^{\frac{1}{2}(k-3)} 2 \\ &= (k-1) + \sum_{j=0}^{\frac{1}{2}(k-3)} x(E_{2j+i+2}) - 2(\frac{1}{2}(k-3)+1) \\ &= (k-1) + x'_{v_i v_{i+1}} - (k-1) \\ &= x'_{v_i v_{i+1}}. \end{aligned}$$

By symmetry, we have that

$$z_{v_iv_{i+1}} = x'_{v_iv_{i+1}}.$$

Hence $y = z = x'$ and therefore x' is an extreme point of the SEP-polytope. □

Lemma 9. *Let x be an extreme point of the SEP-polytope on $n \geq 4$ vertices and let G_x be the support graph of x. Let u and v be vertices of G_x such that $x_{uv} = 1$. Then there exists at most one vertex, w, of G_x such that $x_{uw} > 0$ and $x_{vw} > 0$.*

Proof. Suppose, for a contradiction, that $x_{uw} = 1$. Then $x(\gamma(\{u, v, w\})) > 2$ since $x_{vw} > 0$. However, since x is a feasible point of the SEP-polytope on $n \geq 4$ vertices, we know that $x(\gamma(S)) \leq 2$ for any distinct triple, S, of the vertices of G_x. Thus we know that $x_{uw} < 1$ and, by symmetry, $x_{vw} < 1$.

By Lemma 2, we know that there is a cobasis, $\mathcal{B}$, for x such that no tight cut in the cobasis contains a 1-edge. Specifically, no tight cut of $\mathcal{B}$ contains the edge uv. Now, $\{uv, uw, vw\}$ is a cycle of G_x so any cut of G_x must contain an even number of edges of this cycle. Hence every tight cut either contains no edges in this cycle or it contains precisely uw and vw. Also, since $x_{uw} < 1$ and $x_{vw} < 1$ we know that $x_{uw} = 1$ and $x_{vw} = 1$ are not constraints in $\mathcal{B}$. Therefore, the only possible constraints of $\mathcal{B}$ which contain exactly one of x_{uw} and x_{vw} are the vertex equalities $x(\delta(u)) = 2$ and $x(\delta(v)) = 2$. Notice that these vertex equalities may not even be in $\mathcal{B}$.

Suppose, for a contradiction, that there is another vertex of G_x, say $z \neq w$ such that $x_{uz} > 0$ and $x_{vz} > 0$. Then by the above reasoning, the only possible constraints of $\mathcal{B}$ which contain exactly one of x_{uz} and x_{vz} are the vertex equalities $x(\delta(u)) = 2$ and $x(\delta(v)) = 2$. Choose some $\epsilon > 0$ and define x' as follows.

$$x'_e = \begin{cases} x_e + \epsilon & \text{if } e = uw \text{ or } e = vz \\ x_e - \epsilon & \text{if } e = vw \text{ or } e = uz \\ x_e & \text{otherwise} \end{cases}$$

Since $0 < x_{uw}, x_{vw}, x_{uz}, x_{vz} < 1$, x' obeys all the same zero-edge and 1-edge constraints as x. Also x' clearly obeys all the vertex equalities. Lastly, as noted above, every tight cut in $\mathcal{B}$ contains

1. none of x_{uw}, x_{vw}, x_{uz}, and x_{vz},

2. x_{uw} and x_{vw} but not x_{uz} or x_{vz},

3. x_{uz} and x_{vz} but not x_{uw} or x_{vw}, or

4. all of x_{uw}, x_{vw}, x_{uz}, and x_{vz}.

Therefore, every tight cut of x in $\mathcal{B}$ is also a tight cut of x'. Hence x' is also a solution to the system of constraints in $\mathcal{B}$ which contradicts the fact that x is an extreme point with cobasis $\mathcal{B}$. □

Theorem 10. *Let x be an extreme point of the SEP-polytope on $n \geq 4$ vertices and let G_x be the support graph of x. Let u and v be vertices of G_x such that $x_{uv} = 1$. If there exists a vertex, w, of G_x such that $x_{uw} > 0$ and $x_{vw} > 0$ then $x \downarrow \{u, v\}$ is an extreme point of the SEP-polytope on $n - 1$ vertices.*

Proof. Let $x' = x \downarrow \{u, v\}$ and let $G_{x'}$ be the support graph of x'. Since $x_{uv} = 1$, by the vertex equalities we get that $x(\delta(\{u, v\})) = 2$ and so $\{u, v\}$ is a tight set of x. Hence, by Lemma 5, x' is a feasible point of the SEP-polytope on $n - 1$ vertices.

As noted above in the proof for Lemma 9, there is a cobasis, $\mathcal{B}$ for x such that the only constraints of $\mathcal{B}$ containing exactly one of x_{uw} or x_{vw} are the vertex equalities $x(\delta(u)) = 2$ and $x(\delta(v)) = 2$. In this cobasis (which contains all the 1-edges of x by Lemma 2) we will replace the constraint $x_{uv} = 1$ with the equivalent constraint (since all the vertex equalities are implied by $\mathcal{B}$) $x(\delta(\{u, v\})) = 2$ to obtain a new cobasis $\mathcal{B}'$ of x. Notice, that this constraint contains both x_{uw} and x_{vw}. Furthermore, $x(\delta(u)) = 2$ and $x(\delta(v)) = 2$ are the only possible constraints in $\mathcal{B}'$ which contain x_{uv}. Hence, if we remove $x(\delta(u)) = 2$ and $x(\delta(v)) = 2$ (if they are present) from $\mathcal{B}'$, we get a set, $\mathcal{S}$, of linearly independent constraints, none of which contain x_{uv} and such that every constraint which contains x_{uw} also contains x_{vw} and vice versa.

Hence if G_x has m edges then $\mathcal{B}'$ has m constraints which are tight 1-edge constraints, vertex equalities, and tight cut constraints. Hence $\mathcal{S}$ contains at least $m - 2$ such constraints. Furthermore, to obtain $G_{x'}$ from G_x we simply contract uv and identify the edges uw and vw. Thus $G_{x'}$ has $m - 2$ edges. Therefore, $\mathcal{S}$ along with all the tight zero-edge constraints of x' forms a linearly independent set of size $\frac{1}{2}(n-1)(n-2)$ and thus is a cobasis for x'. Hence, x' is an extreme point of the SEP-polytope on $n - 1$ vertices. □

Let x be a feasible point of the SEP-polytope on n vertices and let G_x be the support graph of x. If zw is an edge of G_x and we can partition the edges, apart from zw, which are incident to z in G_x into two parts, E_1 and E_2, such that $0 \leq x(E_1), x(E_2) \leq 1$ then we define $x \uparrow_z (zw, E_1, E_2)$ by deleting the vertex z and adding two new vertices u and v where

$$(x \uparrow_z (zw, E_1, E_2))_e = \begin{cases} 1 & \text{if } e = uv \\ 1 - x(E_1) & \text{if } e = uw \\ 1 - x(E_2) & \text{if } e = vw \\ x_{qz} & \text{if } e = qu \text{ and } qz \in E_1 \\ x_{qz} & \text{if } e = qv \text{ and } qz \in E_2 \\ x_e & \text{if } e \text{ is an edge of } G_x - z \\ 0 & \text{otherwise} \end{cases}.$$

Please see Figure 1 for a pictorial representation of this operation.

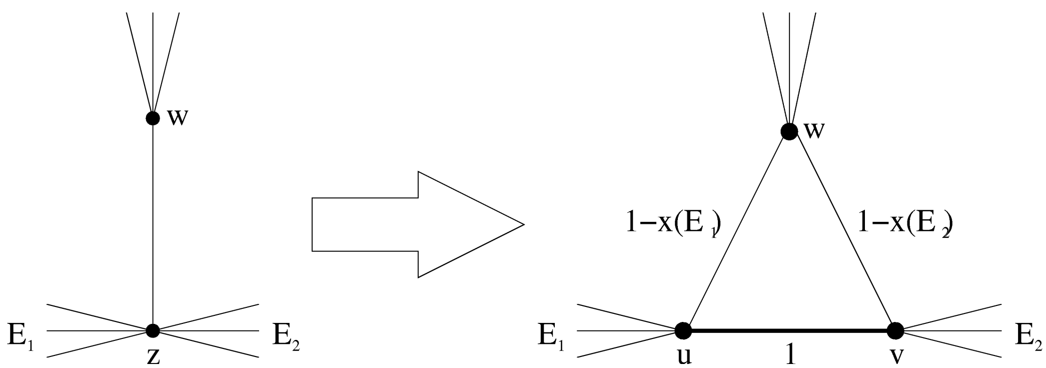

Figure 1: An edge-splitting operation

Lemma 11. *Let x be a feasible point of the SEP-polytope on n vertices and let G_x be the support graph of x. If zw is an edge of G_x and we can partition the edges, apart from zw, which are incident to z in G_x into two parts, E_1 and E_2, such that $0 \leq x(E_1), x(E_2) \leq 1$ then $x \uparrow_z (zw, E_1, E_2)$ is a feasible point of the SEP-polytope on $n+1$ vertices.*

Proof. Let $x' = x \uparrow_z (zw, E_1, E_2)$ and let $G_{x'}$ be the support graph of x'. Let u and v be the vertices added to $G_x - z$ as described above.

Since $0 < x(E_1), x(E_2) < 1$, clearly $x'_e \geq 0$ for every edge e of $G_{x'}$. Also, $x'(\delta(q)) = x(\delta(q)) = 2$ for every vertex $q \notin \{u, v, w\}$ of $G_{x'}$. Since x is feasible, $x(\delta(z)) = 2$ so $x(E_1) + x(E_2) + x_{zw} = 2$. Now

$$\begin{aligned} x'(\delta(w)) &= x(\delta(w)) - x_{zw} + (1 - x(E_1)) + (1 - x(E_2)) \\ &= 4 - x_{zw} - x(E_1) - x(E_2) \\ &= 2. \end{aligned}$$

Additionally, $x'(\delta(u)) = 1 + x(E_1) + 1 - x(E_1) = 2$ and similarily $x'(\delta(v)) = 2$. Thus all the vertex equalities hold for x'.

Let S be any subset of the vertices of $G_{x'}$ where $2 \leq |S| \leq n-1$. If $u, v \notin S$ then $x'(\delta(S)) = x(\delta(S)) \geq 2$. If $u, v \in S$ then $x'(\delta(S)) = x(\delta(S \backslash \{u, v\})) \geq 2$. If $u \in S$ but $v \notin S$ then

$$x'(\delta(S)) = x'(\delta(S \backslash \{u\})) - x'(\delta(u)) + 2x'(u, \overline{S}).$$

However, as noted above, $x'(\delta(S \backslash \{u\})) = x(\delta(S \backslash \{u\})) \geq 2$ and $x'(\delta(u)) = 2$. Furthermore $v \in \overline{S}$ so $x'(u, \overline{S}) \geq x'_{uv} = 1$. Thus $x'(\delta(S)) \geq 2$. By symmetry, if $v \in S$ and $u \notin S$ then $x'(\delta(S)) \geq 2$. Hence, in all cases, $x'(\delta(S)) \geq 2$ and therefore x' is a feasible point of the SEP-polytope on $n + 1$ vertices. □

Theorem 12. *Let x be an extreme point of the SEP-polytope on n vertices and let G_x be the support graph of x. If zw is an edge of G_x and we can partition the edges, apart from zw, which are incident to z in G_x into two parts, E_1 and E_2, such that $0 \leq x(E_1), x(E_2) \leq 1$ then $x \uparrow_z (zw, E_1, E_2)$ is an extreme point of the SEP-polytope on $n + 1$ vertices.*

Proof. Let $x' = x \uparrow_z (zw, E_1, E_2)$ and let $G_{x'}$ be the support graph of x'. Let u and v be the vertices added to $G_x - z$ to obtain $G_{x'}$ as described above. By Lemma 11, x' is a feasible point of the SEP-polytope on $n + 1$ vertices.

Let a and b be feasible points of the SEP-polytope on $n+1$ vertices such that $x' = \frac{1}{2}a + \frac{1}{2}b$. Every tight cut of x' is also a tight cut of a and b. Thus $\{u, v\}$ is a tight set of both a and b. Hence $a \downarrow \{u, v\}$ and $b \downarrow \{u, v\}$ are both feasible points of the SEP-polytope on n vertices by Lemma 5. Notice that $x' \downarrow \{u, v\} = x$ so since x is an extreme point, we know that $a \downarrow \{u, v\} = b \downarrow \{u, v\} = x$. Thus $a_e = b_e = x'_e$ for every edge e of $G_{x'}$ except for possibly the edges uv, uw, or vw. However, since $\{u, v\}$ is a tight set of a

and b we know that $a_{uv} = b_{uv} = 1 = x'_{uv}$. By our vertex equalities we know that

$$\begin{aligned} a_{uw} &= 2 - \sum_{q \notin \{u,w\}} a_{uq} \\ &= 2 - \sum_{q \notin \{u,w\}} x'_{uq} \\ &= 2 - 1 - x(E_1) \\ &= 1 - x(E_1) \\ &= x'_{uw} \end{aligned}$$

Similarily, $a_{vw} = x'_{vw}$ and so $a = x'$. By symmetry, $b = x'$ and therefore, x' is an extreme point of the SEP-polytope on $n+1$ vertices. ☐

Theorem 13. *Let x be an extreme point of the SEP-polytope on $n \geq 4$ vertices. Let G_x be the support graph of x and let F be the set of all 1-edges of x. Suppose that no two edges of F are adjacent in G_x. Suppose further that $G_x - F$ has a bipartite component with vertex set C and let $uv \in F$ such that $u \in C$. If $v \notin C$ or if u and v are in different parts of the bipartition of $(G_x - F)[C]$ then $x \downarrow \{u, v\}$ is an extreme point of the SEP-polytope on $n-1$ vertices.*

Proof. Let $x' = x \downarrow \{u, v\}$ and let $G_{x'}$ be the support graph of x'. If u and v are adjacent to a vertex w in G_x then we can use Theorem 10 to show that x' is an extreme point of the SEP-polytope on $n-1$ vertices. Hence we may assume that u and v have no common neighbour in G_x.

Let $\mathcal{S}$ be the set of all vertex equalities, tight 1-edge constraints, and tight cut constraints for x and let m be the number of edges of G_x. Since x is an extreme points, the rank of $\mathcal{S}$ is m. Thus by Lemma 1, we can remove all of the cut constraints of $\mathcal{S}$ which contain the variable x_{uv} to obtain a set of valid constraints which also has rank m. Also,

$$x_{uv} = \frac{1}{2}(x(\delta(u)) + x(\delta(v)) - x(\delta(\{u, v\})))$$

so we can also remove the constraint $x_{uv} = 1$ from $\mathcal{S}$ without affecting the rank. Hence, let $\mathcal{S}'$ denote the resulting set of constraints, which have rank m.

Let (A, B) be the bipartition of $(G_x - F)[C]$ such that $u \in A$ and so we have that

$$\sum_{w \in A} x(\delta(w)) - 2 \sum_{w,z \in A} x_{wz} - \sum_{w \in A, z \notin C} x_{wz} = \sum_{w \in B} x(\delta(w)) - 2 \sum_{w,z \in B} x_{wz} - \sum_{w \in B, z \notin C} x_{wz}.$$

If $v \in B$ then

$$\begin{aligned} x(\delta(u)) &= -\sum_{w \in A\setminus\{u\}} x(\delta(w)) + 2\sum_{w,z \in A} x_{wz} + \sum_{w \in A, z \notin C} x_{wz} \\ &\quad + \sum_{w \in B} x(\delta(w)) - 2\sum_{w,z \in B} x_{wz} - \sum_{w \in B, z \notin C} x_{wz}. \end{aligned}$$

If $v \notin C$ then

$$\begin{aligned} x(\delta(u)) &= -\sum_{w \in A\setminus\{u\}} x(\delta(w)) + 2\sum_{w,z \in A} x_{wz} + \frac{1}{2}x(\delta(u)) + \frac{1}{2}x(\delta(v)) - \frac{1}{2}x(\delta(\{u,v\})) \\ &\quad + \sum_{w \in A\setminus\{u\}, z \notin C} x_{wz} + \sum_{w \in B} x(\delta(w)) - 2\sum_{w,z \in B} x_{wz} - \sum_{w \in B, z \notin C} x_{wz} \\ \frac{1}{2}x(\delta(u)) &= -\sum_{w \in A\setminus\{u\}} x(\delta(w)) + 2\sum_{w,z \in A} x_{wz} + \frac{1}{2}x(\delta(v)) - \frac{1}{2}x(\delta(\{u,v\})) \\ &\quad + \sum_{w \in A\setminus\{u\}, z \notin C} x_{wz} + \sum_{w \in B} x(\delta(w)) - 2\sum_{w,z \in B} x_{wz} - \sum_{w \in B, z \notin C} x_{wz} \\ x(\delta(u)) &= -2\sum_{w \in A\setminus\{u\}} x(\delta(w)) + 4\sum_{w,z \in A} x_{wz} + x(\delta(v)) - x(\delta(\{u,v\})) \\ &\quad +2\sum_{w \in A\setminus\{u\}, z \notin C} x_{wz} + 2\sum_{w \in B} x(\delta(w)) - 4\sum_{w,z \in B} x_{wz} - 2\sum_{w \in B, z \notin C} x_{wz} \end{aligned}$$

Notice that in either case, all of the constraints corresponding to terms on the right hand side of this equation are in $\mathcal{S}'$ and so we can remove the constraint $x(\delta(u)) = 2$ from $\mathcal{S}'$ without changing the rank. Thus, if we remove both $x(\delta(u)) = 2$ and $x(\delta(v)) = 2$ from $\mathcal{S}'$ to obtain a new set of constraints, $\mathcal{S}''$ then $\mathcal{S}''$ has no constraints containing the variable x_{uv} and has rank $m-1$. Now, since u and v have no common neighbour in G_x, there is a bijective correspondance between the edges of $G_{x'}$ and the edges of $G_x - uv$. Thus $\mathcal{S}''$ induces a set of constraints of rank $m-1$ for x'. Since $G_{x'}$ has precisely $m-1$ edges, we know that x' is an extreme point. $\square$

Let x be a feasible point of the SEP-polytope on n vertices and let G_x be its support graph. Let z be a vertex of G_x such that the edges of G_x incident to z can be partitioned into two parts, E_1 and E_2, where $x(E_1) = 1$ and $x(E_2) = 1$. Consider removing z and replacing it with two vertices, u and v. Then we define $x \uparrow_z (E_1, E_2)$ such that

$$(x \uparrow_z (E_1, E_2))_e = \begin{cases} 1 & \text{if } e = uv \\ x_{qz} & \text{if } e = qu \text{ and } qz \in E_1 \\ x_{qz} & \text{if } e = qv \text{ and } qz \in E_2 \\ x_e & \text{if } e \text{ is an edge of } G_x - z \\ 0 & \text{otherwise} \end{cases}.$$

Theorem 14. *Let x be an extreme point of the SEP-polytope on n vertices with support graph G_x. If z is a vertex of G_x such that the edges of G_x incident to z can be partitioned into two parts, E_1 and E_2, where $x(E_1) = 1$ and $x(E_2) = 1$ then $x \uparrow_z (E_1, E_2)$ is an extreme point of the SEP-polytope on $n+1$ vertices.*

Proof. Let $x' = x \uparrow_z (E_1, E_2)$, let $G_{x'}$ be the support graph of x', and let u and v be the two vertices of $G_{x'}$ which have been added to $G_x - z$. Let m be the number of edges of G_x and let $\mathcal{S}_x$ be the set of all vertex equalities and tight cut constraints of x. Thus $\mathcal{S}_x$ has rank m.

If we replace z with u and v in every tight set of x containing z and also in the vertex equality, $x(\delta(z)) = 2$ we see how $\mathcal{S}_x$ induces a new set of tight constraints, $\mathcal{S}'$, which are valid for x'. Additionally, $\mathcal{S}'$ has rank m. However, no constraint in $\mathcal{S}'$ contains the variable x'_{uv}. Thus, if we add the vertex equality $x'(\delta(u)) = 2$ to $\mathcal{S}'$ we get a set of valid constraints for x' which have rank at least $m+1$. However, $G_{x'}$ has precisely $m+1$ edges so x' must be an extreme point of the SEP-polytope on $n+1$ vertices. □

Corollary 15. *Let x be an extreme point of the SEP-polytope on n vertices. Let G_x be the support graph of x and let F be the set of 1-edges of x. If x cannot be obtained via the vertex splitting or edge splitting operations from an extreme point of the SEP-polytope on $n-1$ vertices then the vertex equalities of $G_x - F$ have rank $n-1$ or n.*

Proof. If $G_x - F$ has more than one component and at least one of them is bipartite, then we see from the proof of Theorem 13 that we can contract a 1-edge of x to get an extreme point, x', of the SEP-polytope on $n-1$ vertices. However, we can use Theorem 14 to obtain x from x' via a vertex splitting operation or use Theorem 12 to obtain x from x' via an edge splitting operation. Hence we may assume that either every component of $G_x - F$ is non-bipartite or that $G_x - F$ is connected and bipartite.

If every component of $G_x - F$ is non-bipartite, then the vertex equalities are all linearly independent, and hence have rank n. If $G_x - F$ has a single bipartite component, then the vertex equalitites have rank $n-1$. □