Feasibility of the Held-Karp LP relaxation of the TSP *

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Abstract

A careful study of the points of the Subtour Elimination Polytope could lead to a better approximation algorithm for the Travelling Salesman Problem or at least a better understanding of how good a lower bound on the optimal value of the Travelling Salesman Problem we obtain by optimizing over the Subtour Elimination Polytope. In this paper we study the graph-theoretic structure of the support graph of points of the Subtour Elimination Polytope and we introduce the concept of block-toughness. This is a stronger requirement than Chvátal's toughness criteria for a graph. We extend both block-toughness and toughness to give a necessary condition for a graph to be a possible support graph of a Subtour Elimination point. We finish our paper with some numerical results.

1 Hamiltonian and SEP-feasible Graphs

A *Hamilton cycle* in a graph G is a cycle in G which contains all the vertices of G. We say that a graph is *Hamiltonian* if it has a Hamilton cycle. The problem of determining whether or not a graph has a Hamilton

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cycle is one of the most famous NP-complete [12] problems. There have been many attempts to find a good characterization of Hamiltonian graphs. There are several well-known sufficient conditions [10, 16, 8] for a graph to be Hamiltonian. In this paper, we would like to explore further the graph theoretical structure of Hamiltonian graphs.

Concurrently, we would like to explore the nature of a certain polytope. Let $G = (V, E)$ be a graph. The *Subtour Elimination Polytope* (henceforth abbreviated to SEP) associated with G is the subset of $\mathbb{R}^E$ which obeys the following constraints.

$$\begin{aligned} x(\delta(v)) &= 2 \quad \text{for all } v \in V \\ x(\delta(S)) &\geq 2 \quad \text{for all } \emptyset \subset S \subset V \\ x_e &\geq 0 \quad \text{for all } e \in E \end{aligned}$$

This polytope was first introduced by Dantzig, Fulkerson, and Johnson [9] in their landmark paper. In that paper, the SEP was used as the feasible region of a linear programming problem used to solve an instance of the famous Travelling Salesman Problem (TSP) - the problem of finding a minimum cost Hamilton cycle in a graph with edge-costs.

Since every point, x, of the SEP associated with a graph G is an assignment of real values to the edges of G, we can define the *support graph* of x which is the subgraph of G induced by the edges with positive x-values. The support graph of a point, specifically an extreme point, of the SEP gives us the graph-theoretic structure of the constraints of the form $x_e \geq 0$ which are not met with equality. This can be helpful in understanding how these constraints interact in the cobasis of an extreme point and can give us some insight into the nature of extreme points of the SEP. By studying the structure of extreme points of the SEP we hope to better understand the relationship between the SEP and the TSP which could lead to better lower bounds on the optimal value of the TSP or even an efficient combinatorial approximation algorithm for the TSP.

If the SEP associated with a graph G is non-empty then we say that G is *SEP-feasible*. As with Hamiltonian graphs, in this paper, we would like to futher explore the graph theoretical structure of SEP-feasible graphs.

For a graph $G = (V, E)$, and a Hamilton cycle C, we define the characteristic vector of C to be $x \in \mathbb{R}^E$ where

$$x_e = \begin{cases} 1 & \text{if } e \in C \\ 0 & \text{otherwise} \end{cases} .$$

It is straightforward to see that the characteristic vector of any Hamilton cycle of a graph G obeys the constraints of the SEP associated with G. Hence if G is Hamiltonian then G is also SEP-feasible. Conversely, any necessary condition for a graph to be SEP-feasible is also a necessary condition for a graph to be Hamiltonian.

Now, since we know the linear constraints which define the SEP on a given graph G, we can apply Farkas' Lemma [11] to find another characterization of SEP-feasible graphs.

Lemma 1. *G is not SEP-feasible if and only if there exist variables y and d which satisfy*

$$\begin{aligned} \sum_{v \in V} y_v + \sum_{\emptyset \subset S \subset V} d_S &> 0 \\ y_u + y_v + \sum_{\substack{\emptyset \subset S \subset V \\ uv \in \delta(S)}} d_S &\leq 0 \text{ for each } uv \in E \\ d_S &\geq 0 \text{ for each } \emptyset \subset S \subset V. \end{aligned}$$

We will use Lemma 1 for several of our proofs to prove that a certain graph is not SEP-feasible or not Hamiltonian.

2 Graph Toughness

One simple characteristic of both Hamiltonian and SEP-feasible graphs is stated in Proposition 2. Although this result is fairly straightforward, we will provide a proof which illustrates the effectiveness of Lemma 1.

Proposition 2. *If a graph G is Hamiltonian or SEP-feasible then it is 2-vertex-connected.*

Proof. Let $G = (V, E)$ be a graph which is SEP-feasible. Suppose, for a contradiction, that G has a cut vertex, u. Let $Q_1, \ldots, Q_s$ be the components of $G - u$ where $s \geq 2$. Define

$$y_v = \begin{cases} -1 & \text{if } v = u \\ 0 & \text{otherwise} \end{cases}$$

and

$$d_S = \begin{cases} 1 & \text{if } S = Q_i \text{ for some } 1 \leq i \leq s \\ 0 & \text{otherwise} \end{cases} .$$

We can compute

$$\sum_{v \in V} y_v + \sum_{\emptyset \subset S \subset V} d_S = s - 1.$$

Any edge of $\delta(Q_i)$ for some $1 \leq i \leq s$ must have one endpoint in Q_i and the other endpoint must be u. Thus we see that the conditions in Lemma 1 are met and therefore G is not SEP-feasible. Hence we have a contradiction so G must be 2-vertex-connected.

Now, if G is Hamiltonian then it is also SEP-feasible so, from above, G is 2-vertex-connected. □

More general necessary conditions have been proven as well. Let $k(G)$ denote the number of connected components of a graph G. We say that a graph, $G = (V, E)$ is *t-tough* if for every $T \subset V$ we have that

$$|T| \geq t \star k(G - T).$$

The concept of t-toughness was first introduced and explored by Chvátal [6]. In his paper, Chvátal was interested in using toughness to find necessary and sufficient conditions for a graph to be Hamiltonian. He noted that every Hamiltonian graph must be 1-tough and conjectured that any graph that was t-tough for some $t > \frac{3}{2}$ is necessarily Hamiltonian. However, Bauer, Broersma, and Veldman [2] later found a 2-tough non-Hamiltonian graph. As well, Bauer, Hakimi, and Schmeichel [3] proved that deciding whether or not a graph was t-tough is an NP-hard problem.

Other authors also picked up on this sort of structure in graphs. Chvátal [7] later introduced the notion of a sub-2-factor showed that if a graph is Hamiltonian then it has a sub-2-factor. He further showed that the set of all graphs with a sub-2-factor is a proper subset of the set of all 1-tough graphs. Most recently, Katona [13] introduced the idea of edge-toughness and showed that every Hamiltonian graph is 1-edge-tough. Katona [14] later showed that every 1-edge-tough graph also has a sub-2-factor. Bauer, Broersma, and Schmeichel [1] penned an excellent survey of toughness in graphs.

Returning to 1-toughness, notice that every 1-tough graph is also 2-vertex-connected - removing any vertex in a 1-tough graph leaves exactly one component. Thus 1-toughness is a stronger necessary condition than 2-vertex-connectivity. For our purposes, 1-toughness also plays a role in SEP-feasibility.

Proposition 3. *If G is SEP-feasible then G is 1-tough.*

Proof. Suppose G is not 1-tough, then there exists $\emptyset \subseteq T \subset V$ such that $k(G-T) > |T|$. Let $Q_1, \ldots, Q_s$ be the components of $G-T$. Define

$$y_v = \begin{cases} -1 & \text{if } v \in T \\ 0 & \text{otherwise} \end{cases}$$

and

$$d_S = \begin{cases} 1 & \text{if } S = Q_i \text{ for some } 1 \leq i \leq s \\ 0 & \text{otherwise} \end{cases} .$$

Since $Q_1, \ldots, Q_s$ are the components of $G-T$, any edge of $\delta(Q_i)$ for some $1 \leq i \leq s$ must have exactly one endpoint in T and the other in Q_i. Hence

$$y_u + y_v + \sum_{\substack{\emptyset \subset S \subset V \\ uv \in \delta(S)}} d_S \leq 0 \text{ for every } \; uv \in E.$$

Furthermore,

$$\sum_{v \in V} y_v + \sum_{\emptyset \subset S \subset V} d_S = -|T| + k(G-T) > 0.$$

Therefore, by Lemma 1, G is not SEP-feasible. □

We can impose similar, but stronger, conditions on a graph which are necessary for SEP-feasibility. Given a graph, H, let $b_0(H)$ denote the number

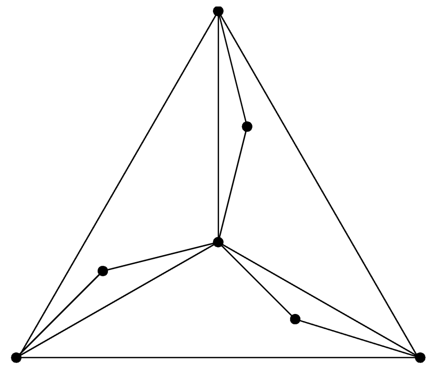

Figure 1: A 1-tough graph that is not 1-block-tough

of blocks of H which contain no cut vertices and let $b_1(H)$ denote the number of blocks of H which contain exactly one cut-vertex. We will say that a graph, $G = (V, E)$, is *t-block-tough* if for every $T \subset V$ we have that

$$|T| \geq t \left(b_0(G - T) + \frac{1}{2} b_1(G - T) \right).$$

Let's define an *end-block* of H to be any block of H which has exactly one cut-vertex.

Proposition 4. *Every t-block-tough graph is t-tough.*

Proof. Since every block graph is a forest, every component of $G - T$ which has a cut-vertex must have at least two end-blocks. Thus

$$b_0(G - T) + \frac{1}{2} b_1(G - T) \geq k(G - T)$$

and so the result follows. □

However, the converse of Proposition 4 does not hold. An example, first presented by Chvátal [6], is shown in Figure 1 of a graph which is 1-tough but not 1-block-tough. Interestingly, it can be seen that this graph is also not SEP-feasible.

Theorem 5. *If G is SEP-feasible then G is 1-block-tough*

Proof. Let x be a feasible point of the SEP associated with G. Suppose, for a contradiction, that $G = (V, E)$ is not 1-block-tough. Then there exists some $T \subset V$ such that

$$|T| < b_0(G - T) + \frac{1}{2} b_1(G - T).$$

If there are $s \geq 2$ end-blocks of $G - T$ which share a common cut-vertex, v, then these end-blocks become separate components in $G - (T \cup \{v\})$. Each of these components is either 2-vertex-connected or contains at least two end-blocks. Thus

$$\begin{aligned}
& b_0(G - (T \cup \{v\}) + \frac{1}{2} b_1(G - (T \cup \{v\})) \\
\geq \;& b_0(G - T) + \frac{1}{2}(b_1(G - T) - s) + s \\
= \;& b_0(G - T) + \frac{1}{2}(b_1(G - T)) + \frac{1}{2}s \\
\geq \;& b_0(G - T) + \frac{1}{2}(b_1(G - T)) + 1 \\
> \;& |T| + 1 \\
= \;& |T \cup \{v\}|.
\end{aligned}$$

Hence, $T \cup \{v\}$ also proves that G is not 1-block-tough and we can replace T by $T \cup \{v\}$. Therefore, we may assume that no two end-blocks of T share a common cut-vertex.

Now let $Q_1, \ldots, Q_s$ be the end-blocks of $G - T$ which contain cut-vertices $v_1, \ldots, v_s$ respectively. Let $R_1, \ldots, R_l$ be the 2-connected components of $G - T$. Define

$$y_v = \begin{cases} -2 & \text{if } v \in T \\ -1 & \text{if } v = v_i \text{ for some } 1 \leq i \leq s \\ 0 & \text{otherwise} \end{cases}$$

and

$$d_S = \begin{cases} 2 & \text{if } S = R_i \text{ for some } 1 \leq i \leq l \\ 1 & \text{if } S = Q_i \text{ for some } 1 \leq i \leq s \\ 1 & \text{if } S = Q_i \backslash \{v_i\} \text{ for some } 1 \leq i \leq s \\ 0 & \text{otherwise} \end{cases}.$$

It is straightforward to check that

$$y_u + y_v + \sum_{\substack{\emptyset \subset S \subset V \\ uv \in \delta(S)}} d_S \leq 0 \text{ for every } uv \in E.$$

Furthermore, notice that

$$\sum_{v \in V} y_v = -2|T| - s$$

and

$$\sum_{\emptyset \subset S \subset V} d_S = 2l + s + s$$

so

$$\begin{aligned}
\sum_{v \in V} y_v + \sum_{\emptyset \subset S \subset V} d_S &= -2|T| - s + 2l + 2s \\
&= 2l + s - 2|T| \\
&= 2b_0(G-T) + b_1(G-T) - 2|T| \\
&= 2(b_0(G-T) + \frac{1}{2}b_1(G-T) - |T|) \\
&> 0.
\end{aligned}$$

Hence, by Lemma 1, G is not SEP-feasible which is a contradiction. □

Unfortunately, not every 1-block-tough graph is SEP-feasible. Bauer, Broersma, and Veldman [2] present a graph which is 1-block-tough but not SEP-feasible. As an immediate corollary of Theorem 5, however, we get that if G is Hamiltonian then it is 1-block-tough. However, we can do even better as we see in Theorem 6.

Theorem 6. *If $G = (V, E)$ is Hamiltonian then for every $T \subset V$,*

$$|T| \geq b_0(G-T) + \sum_{i=1}^{s} \left\lceil \frac{b_1(Q_i)}{2} \right\rceil$$

where $Q_1, \ldots, Q_s$ are the components of $G-T$ which are not 2-vertex-connected.

Proof. Suppose G is Hamiltonian and let x be the characteristic vector of a Hamilton cycle of G. Let $T \subset V$ and let $Q_1, \ldots, Q_s$ be the components of $G - T$ which are not 2-vertex-connected. Let H be an end-block of Q_i for some $1 \leq i \leq s$ and let v be the unique cut-vertex contained in H. Then

$$x(\delta(H)) + x(\delta(H \backslash \{v\})) = x(\delta(v)) + 2x(T, H \backslash \{v\}).$$

However $x(\delta(H)) \geq 2$, $x(\delta(H \backslash \{v\})) \geq 2$, and $x(\delta(v)) = 2$ so $2x(T, H \backslash \{v\}) \geq 2$ and thus $x(T, H \backslash \{v\}) \geq 1$. Therefore, for each end-block of Q_i, there is at least one edge of the Hamilton cycle with one end in T and the other in $H \backslash \{v\}$. Hence $x(\delta(Q_i)) \geq b_1(Q_i)$. Futhermore, since x is the characteristic vector of a Hamilton cycle, $x(\delta(Q_i))$ must be even for every $1 \leq i \leq s$. Thus

$$x(\delta(Q_i)) \geq 2 \left\lceil \frac{b_1(Q_i)}{2} \right\rceil$$

for every $1 \leq i \leq s$.

For any 2-vertex-connected component, J, of $G - T$, we have that $x(\delta(J)) \geq 2$. Thus

$$x(\delta(T)) \geq 2b_0(G - T) + 2 \sum_{i=1}^{s} \left\lceil \frac{b_1(Q_i)}{2} \right\rceil .$$

But $x(\delta(v)) \geq 2$ for every $v \in T$ and hence $x(\delta(T)) \leq 2|T|$. Therefore

$$2|T| \geq 2b_0(G - T) + 2 \left\lceil \frac{b_1(Q_i)}{2} \right\rceil$$

and hence the result follows. □

3 An Application of 1-block-toughness

In order to thoroughly study the nature of the SEP, we have been working to generate all the extreme points of the SEP on the complete graph K_n for small values of n. One very successful method for exhaustive generation of these extreme points has begun with a consideration of potential support graphs of extreme points of the SEP.

We began with what information was previously known about the support graphs of SEP extreme points. Boyd and Pulleyblank [5] proved that every support graph of an SEP extreme point has at most $2n-3$ edges. Later, Benoit and Boyd [4] showed that any extreme point for $n \geq 4$ whose support graph had a vertex of degree 2 could be easily obtained from an extreme point of the SEP on K_{n-1}. Lastly, as per Proposition 2 we limited our attention to graphs that were 2-vertex-connected. Thus, for small values of n, we generated all of the non-isomorphic 2-vertex-connected graphs which had degree at least 3 at every vertex and at most $2n-3$ edges. We used a software program called NAUTY (created by Brendan McKay [15]) to accomplish this task.

In Table 1 we give the results of our search. The second column tells us how many non-isomorphic 2-vertex-connected graphs which had degree at least 3 at every vertex and at most $2n-3$ edges were generated by NAUTY. The last column shows how many of those graphs were 1-block-tough.

n	Graphs produced by NAUTY	1-block-tough Graphs
8	38	28
9	302	250
10	3745	2718
11	54721	43585
12	956444	745407

Table 1: Number of 1-block-tough Graphs

References

[1] D. Bauer, H. J. Broersma, and E. Schmeichel, “Toughness in Graphs - A Survey”, *Graphs and Combinatorics*, **22** (2006), 1 – 35.

[2] D. Bauer, H. J. Broersma, H. J. Veldman, “Not every 2-tough graph is hamiltonian”, *Discrete Applied Mathematics*, **99** (2000), 317 – 321.

[3] D. Bauer, S. L. Hakimi, and E. Schmeichel, “Recognizing tough graphs is NP-hard”, *Discrete Applied Mathematics*, **28** (1990), 191 – 195.

[4] G. Benoit and S. Boyd, “Finding the exact integrality gap for small Travelling Salesman Problems”, submitted to: *Mathematics of Operations Research*, 12 pages.

[5] S. C. Boyd and W. R. Pulleyblank, "Optimizing over the subtour polytope of the travelling salesman problem", *Mathematical Programming*, **49** (1991), 163 – 187.

[6] V. Chvátal, "Tough Graphs and Hamiltonian Circuits", *Discrete Mathematics*, **5** (1973), 215 – 228.

[7] V. Chvátal, "Hamiltonian cycles", in: *The Traveling Salesman Problem* (E. L. Lawler, J. K. Lenstra, A. H. G. Rinnooy Kan, and D. B. Shmoys eds.), John Wiley & Sons Ltd., New York (1985), 403 – 429.

[8] V. Chvátal and P. Erdös, "A note on Hamiltonian circuits", *Discrete Mathematics*, **2** (1972), 111 – 113.

[9] G. Dantzig, R. Fulkerson, and S. Johnson, "Solution of a large-scale traveling-salesman problem", *Operations Research*, **2** (1954), 393 – 410.

[10] G. A. Dirac, "Some theorems on abstract graphs", *Proceedings of the London Mathematical Society*, **1** (1952), 68 – 81.

[11] J. Farkas, "Theorie der einfachen Ungleichungen", *Journal für die reine und angewandte Mathematik*, **124** (1902), 1 – 27.

[12] R. M. Karp, "Reducibility among combinatorial problems", in: *Complexity of Computer Computations* (R. E. Miller and J. W. Thatcher, eds.), Plenum Press, New York (1972), 85 – 103.

[13] Gyula Y. Katona, "Toughness and edge-toughness", *Discrete Mathematics*, **164** (1997), 187 – 196.

[14] Gyula Y. Katona, "A large set of non-Hamiltonian graphs", *Discrete Applied Mathematics*, **115** (2001), 99 – 115.

[15] B. McKay, "NAUTY User's Guide (Version 1.5)", *Technical Report TR-CS-90-02*, Department of Computer Science, Australia National University, 1990.

[16] W. T. Tutte, "A Theorem on Planar Graphs", *Transactions of the American Mathematical Society*, **82** (1956), 99 –116.