CSI 5166 Some Practice Questions for Test 1 Winter/17

1. Prove that any graph G=(V,E) with n nodes that is a tree has exactly m=n-1 edges.

2. Let T be a tree in which every nodes has degree at most 3 and at least 1. Prove that (the number of nodes of degree 3 in T) = (the number of leaves in T) -2.

3. Write out the entire set of constraints for the ILP for the TSP for K_4 = (V,E) (K_4 is the complete graph on 4 nodes). Use the labels 1, 2, 3, 4 for your nodes. Note that you should write out all of the degree and subtour elimination constraints explicitly (i.e. in the form $x_{12} + x_{13} + x_{14} = 2$, etc.), but you may write any other constraints in a general form. Note that the subtour elimination constraints for sets S and V/S are the same constraint (so don't list such a constraint twice).

4. Give an example of a graph that is 3-edge-connected, but not 3-node-connected.

5. Consider the Windy Salesman Problem (WSP) where the cost to go from a to b can be different than the cost to go from b to a. Consider this problem modeled on a complete weighted directed graph (so all possible arcs, <a,b>, <b,a> are present), and look for a minimum weight directly cycle that visits all of the nodes exactly once. Write an ILP to model this problem.

6. Proof the following:
Theorem 1: Suppose you have a graph G=(V,E) and a subset S of the nodes V. Let K be the subset of the nodes in G which have odd degree. Show that if the size of $S \cap K$ is odd, then the number of edges in the cut $\delta(S)$ must be odd. Hint 1: Use one of the Important Facts from the first classes. Hint 2: Do not use induction.

7. Given a complete weighted graph G=(V,E), n even, define a *perfect triple matching* to be a subset M of the edges of G such that every node of G is incident with exactly 3 edges in M (instead of 1, as for the usual perfect matching problem).
a) Write a simple ILP formulation for the problem of finding a minimum weight triple matching, where we represent such a matching with a 0-1 vector indexed by the edges of G.
b) Use an example to show that it is possible to get a solution for the LP relaxation of your formulation in a) for which the optimal solution have value strictly less than the value of the optimal solution of the ILP.

c) Use Theorem 1 above to show that the odd cut constraints we used for the perfect matching problem in class are also valid for all 0-1 solutions representing triple perfect matchings.

8. Consider the following greedy algorithm for finding a perfect matching in a bipartite graph:

Pick any edge and put it in M.
While M is not perfect, and there are edges in G which are node-disjoint from M
    Pick an edge disjoint from M, and add it to M.
If M is perfect, output M, otherwise no perfect matching exists.

This algorithm is not correct (i.e. it could terminate without finding a perfect matching, even though one exists). Prove this by demonstrating on an example where it fails.

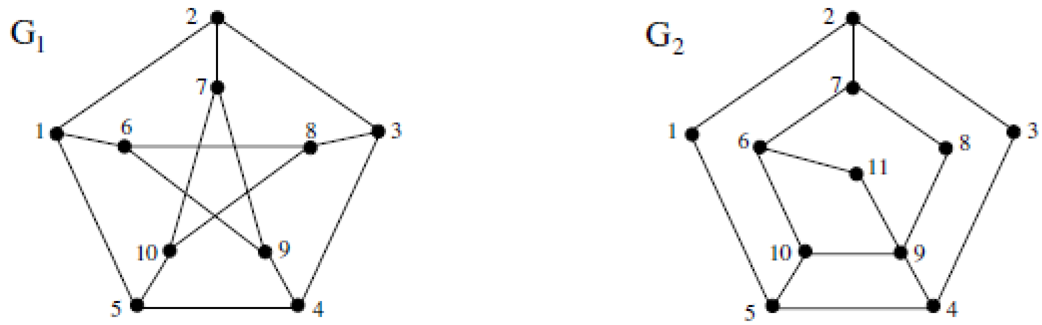
9. The Pen-Up problem solution (for logic circuit drawing) depends on Euler's Theorem. What is the analogous theorem for a directed graph: i.e. given a directed graph D=(V,E) which is strongly connected, we can find a routing along the arcs of D that uses every edge exactly once (no pen-up) if and only if____________?____________________. (You don't need to write a proof, just state the condition).

10. For which of the following degree sequences can we have a simple graph on 5 nodes?
a) 3, 3, 2, 2, 2 b) 4, 3, 3, 2, 2 c) 4, 4, 3, 2, 1 d) 3, 3, 3, 3, 3 e) 3, 3, 3, 3, 2
f) 5, 3, 2, 2, 2

11. Prove that if $G = (X \cup Y, E)$ is bipartite then $m = \Sigma(d(v): v \text{ in } X) = \Sigma(d(v): v \text{ in } Y)$.

12. Prove that if a graph has an odd number of nodes and is regular of degree k at least 1, then it is not bipartite.

13. In the following graphs, find paths of length 9 and 11, and cycles of length 5, 6, 8 and 9, if possible.

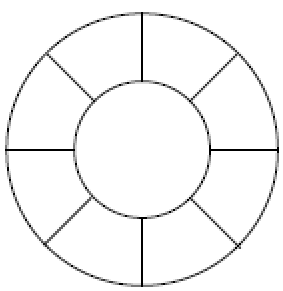


14. A bridge in s a connected graph is an edge which is a cut edge…ie an edge whose removal disconnects the graph. Let uv be a bridge of a connected graph with n > 2. Prove that either u or v is a cut node (i.e. a node whose removal disconnects the graph).

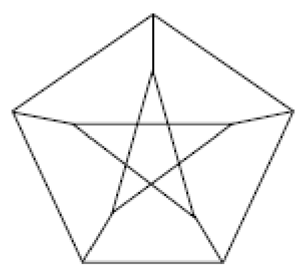
15. Prove that every graph in which each vertex has even degree is bridgeless.

16. Find the minimum number of times that one needs to lift the pencil from the paper to draw each of the figures below without repeating any line, and ending where you started.

1)

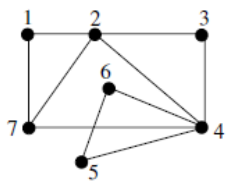


2)

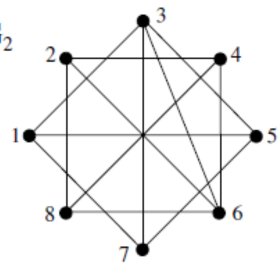


17. What is the edge and node connectivity of the following graphs:

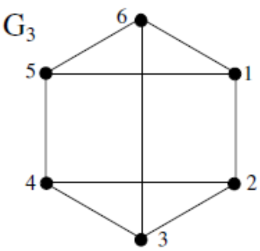
G_1



G_2



G_3



18. For the following graph, write all of the constraints for the LP for the perfect min weight matching problem that also solves the ILP (so with the odd cut constraints added).