Performance Analysis of Coded V-BLAST with Optimum Power and Rate Allocation

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Motivation

- **Why V-BLAST?**
  - MIMO: high spectral efficiency
  - V-BLAST is a practical (not too complex) approach
  - MMSE BLAST achieves full MIMO capacity
  - SIC implements the chain rule of MI

- **V-BLAST challenges**
  - Error propagation effect degrades performance
  - Ordering: high computational complexity

- **V-BLAST improvements**
  - Optimum power/rate allocation

- **Coded V-BLAST**
  - Most of prior work: uncoded BLAST
  - Uncoded systems are rare; powerful (capacity-approaching) codes exist

- Performance analysis: challenging but insights
**V-BLAST: launch multiple bit streams**

Interference cancellation: cancel out the interference from already detected symbols.

Interference nulling (ZF): project out the interference from yet to be detected symbols.

Optimal ordering procedure: symbol with highest after-processing SNR is detected first - excluded.
System Model

- Generic multi-stream transmission or ZF V-BLAST (no ordering)
- Instantaneous power and/or rate allocation
- Capacity-achieving temporal codes
- Channel model

\[ r = \mathbf{H} \mathbf{\Lambda} \mathbf{s} + \xi = \sum_{i=1}^{m} h_i \sqrt{\alpha_i} s_i + \xi \]

- System capacity: capacity of the extended channel (with the V-BLAST transmission/processing architecture)
- Outage probability: probability that the system cannot support a target rate \( mR \),

\[ P_{out} = \mathbb{P} [ C < mR ] \]
Instantaneous vs. Average Optimization

- Total power constraint: $\sum_{i=1}^{m} \alpha_i = m$
- Target rate: $mR$

Power/rate allocation strategies:

$$\overline{\alpha}_C = \arg \max_{\alpha(\gamma_0)} \mathbb{E} [C(\alpha)]$$
$$\overline{\alpha}_{out} = \arg \min_{\alpha(\gamma_0)} \mathbb{P} [C(\alpha) < mR]$$
$$\alpha_C = \arg \max_{\alpha(\gamma_0, H)} C(\alpha)$$
$$\alpha_{out} = \arg \min_{\alpha(\gamma_0, H)} \mathbb{P} [C(\alpha) < mR]$$

Theorem (Instantaneous vs. Average Optimization)

$$\mathbb{P} [C(\overline{\alpha}_C) < mR] \geq \mathbb{P} [C(\overline{\alpha}_{out}) < mR]$$
$$\geq \mathbb{P} [C(\alpha_{out}) < mR] = P^*_{out}$$
$$= \mathbb{P} [C(\alpha_C) < mR]$$
Unoptimized system

- All $\alpha_i = 1$ (uniform power allocation)
- All per-stream target rates $= R$ (uniform rate allocation)

\[ C_u = m \min_i C_i, \quad C_i = \ln(1 + \alpha_i g_i \gamma_0) \]

\[ P_{out}^u = \mathbb{P}[C_u < mR] = 1 - \prod_{i=1}^{m} (1 - \mathbb{P}[C_i < R]) \]

- $g_i = |h_{i\perp}|^2$ is the $i$–th stream power gain $^1$.

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Optimization strategies

Instantaneous Rate Allocation (IRA)

- All $\alpha_i = 1$
- Per-stream rates are adjusted to match the per-stream capacities $C_i$

$$C_{IRA} = \sum_{i=1}^{m} C_i, \quad C_i = \ln(1 + g_i \gamma_0)$$

$$P_{out}^{IRA} = \mathbb{P} \left[ \sum_i C_i < mR \right]$$
Instantaneous Power Allocation (IPA) ²

\[ C_{IPA} = m \max_{\alpha} \min_i \ln(1 + \alpha_i g_i \gamma_0), \quad \text{s.t.} \sum_i \alpha_i = m, \alpha_i \geq 0 \]

Theorem (IPA)

\[ C_{IPA} = \begin{cases} 
  m \ln(1 + \bar{g} \gamma_0), & \text{all } g_i > 0 \\
  0, & \text{otherwise}
\end{cases} \]

where \( \bar{g} \) is the harmonic mean per-stream gain,

\[ \bar{g} = \left( \frac{1}{m} \sum_i g_i^{-1} \right)^{-1} \]

\( C_{IPA} \) is achieved by the channel inversion: \( \alpha_i = \bar{g} / g_i \)

Optimization strategies

Instantaneous Power and Rate Allocation (IPRA)

\[ C_{IPRA} = \max_{\alpha} \sum_i \ln(1 + \alpha_i g_i \gamma_0), \quad \text{s.t.} \quad \sum_i \alpha_i = m, \quad \alpha_i \geq 0 \]

- Power allocation subject to \( \sum_{i=1}^{m} \alpha_i = m \)
- Per-stream rates match per-stream capacities \( \ln(1 + \alpha_i g_i \gamma_0) \)
- Conventional waterfilling (WF): not optimal in V-BLAST!
Fractional waterfilling (FWF) = conventional WF on all subsets of streams:

1. Begin
2. Select a set of active Tx antennas (streams)
3. Do WF
4. Go to 2 until all combinations are done
5. Select the best active set
6. End
Performance analysis: Capacities

Proposition (Capacity bounds)

\[
\ln (1 + m\gamma_0 g_{\text{max}}) \leq C_{WF} \leq m \ln (1 + \gamma_0 g_{\text{max}})
\]
\[
\ln (1 + m\gamma_0 g_{\text{max}} \perp) \leq C_{WF} \leq m \ln (1 + \gamma_0 g_{\text{max}} \perp)
\]
\[
\ln (1 + \gamma_0 g_{\text{max}} \perp) \leq C_{IRA} \leq m \ln (1 + \gamma_0 g_{\text{max}} \perp)
\]
\[
C_u = m \ln (1 + \gamma_0 g_{\text{min}} \perp)
\]

Proposition (Instantaneous capacities)

\[
C_u \leq C_{IRA} \leq C_{WF} \leq C_{FWF}
\]
\[
C_u(\gamma_0) \leq C_{IPA}(\gamma_0) \leq C_u(m\gamma_0)
\]
\[
C_{IRA}(\gamma_0) \leq C_{WF}(\gamma_0) \leq C_{IRA}(m\gamma_0)
\]
Proposition (Outage probabilities, any fading)

\[ P_{out}^{FWF} \leq P_{out}^{WF} \leq P_{out}^{IRA} \leq P_{out}^{u} \]

\[ P_{out}^{u}(m\gamma_{0}) \leq P_{out}^{IPA}(\gamma_{0}) \leq P_{out}^{u}(\gamma_{0}) \]

\[ P_{out}^{IRA}(m\gamma_{0}) \leq P_{out}^{WF}(\gamma_{0}) \leq P_{out}^{IRA}(\gamma_{0}) \]
Performance analysis: Diversity gains

Diversity gain $d$:

$$P_{out} \approx \frac{c}{\gamma_0^d} \quad \text{or} \quad d = - \lim_{\gamma_0 \to \infty} \frac{\ln P_{out}}{\ln \gamma_0}.$$ 

Proposition (Diversity gains)

In the low outage regime, 

$$d_u = n - m + 1 = d_{IPA}$$

$$\leq d_{IRA} = \sum_{i=1}^{m} (n - m + i) = d_{WF}$$

$$\leq d_{FWF} = nm$$

The equality is achieved for $m = 1$ only, i.e. only the FWF achieves the full MIMO channel diversity $nm$ for $m > 1$. 
Figure: $2 \times 2$ V-BLAST in i.i.d. Rayleigh fading channel at $R = 3$ [nat/s/Hz]
Theorem: Outage probabilities, wideband

\[ P_{\text{out}}^u = 1 - \prod_{i=1}^{m} (1 - F_{n-m+i}(x)) \approx \frac{x^{n-m+1}}{(n-m+1)!}, \]

\[ P_{\text{out}}^{\text{IRA}} \approx F_{d_{\text{IRA}}} (mx) \approx \frac{1}{d_{\text{IRA}}!} (mx)^{d_{\text{IRA}}}, \]

\[ P_{\text{out}}^{\text{WF}} \approx \prod_{i=1}^{m} F_{n-m+i}(x) \approx \frac{x^{d_{\text{IRA}}}}{\prod_{i=1}^{m}(n-m+i)!} \]

\[ P_{\text{out}}^{\text{FWF}} \approx F_{m}^{n}(x) \approx \frac{x^{nm}}{(n!)^{m}} \]

where the second approximation in each case holds at the low outage regime, \( x = R/\gamma_0 \ll 1 \). \( F_k(x) = 1 - e^{-x} \sum_{l=0}^{k-1} x^l / l! \) is the outage probability of \( k \)-th order MRC.

\(^3\)to the best of our knowledge, it is the first time when the WF/FWF outage probability is found in a closed form.
Example (wideband)

Figure: $2 \times 2$ V-BLAST in i.i.d. Rayleigh fading channel at $R = 0.1$ [nat/s/Hz]
In the low SNR regime, \( m \gamma_0 \max_i |h_i|^2 \ll 1 \),

\[
C_u \approx m \gamma_0 \min_i |h_i\perp|^2 \\
C_{IRA} \approx \gamma_0 \sum_{i=1}^{m} |h_i\perp|^2 \\
C_{WF} \approx m \gamma_0 \max_i |h_i\perp|^2 \\
C_{FWF} \approx m \gamma_0 \max_i |h_i|^2
\]

**Corollary**

*FWF significantly outperforms the WF, \( C_{WF} \ll C_{FWF} \), when \( \max_i |h_i\perp| \ll \max_i |h_i| \) and their performance is close otherwise.*
Example (fixed channel)

Figure: 2 × 2 V-BLAST with the FWF and the conventional WF for two channel realizations:
(a) "good": $h_1 = [1 \ 1]^T, h_2 = [0 \ 1]^T$
(b) "bad": $h_1 = [1 \ 10]^T, h_2 = [0 \ 1]^T$
Conclusion

- Optimum power/rate allocation for coded V-BLAST
- IPA: within a bounded SNR gain ($\leq m$) of $U$
- IRA: extra diversity gain
- WF: within a bounded SNR gain ($\leq m$) of IRA
- Conventional WF is not optimal in V-BLAST!
- Fractional waterfilling algorithm (FWF)
  - maximizes capacity/minimizes the outage probability
  - significantly outperforms the other strategies (achieves the full channel diversity)
- Closed-form solutions + performance analysis
- Also good for generic multi-stream transmission (e.g. OFDM, MAC, SIC equalizers)