Bit Error Rate is Convex at High SNR

Sergey Loyka\textsuperscript{1}, Victoria Kostina\textsuperscript{2}, Francois Gagnon\textsuperscript{3}

\textsuperscript{1}School of Information Technology and Engineering  
University of Ottawa, Canada 
sergey.loyka@ieee.org

\textsuperscript{2}Department of Electrical Engineering  
Princeton University, USA  
vkostina@princeton.edu

\textsuperscript{3}Department of Electrical Engineering  
Ecole de Technologie Superieure, Montreal, Canada  
francois.gagnon@etsmtl.ca
MOTIVATION

• Importance of convexity\(^1\)
  o strong analytical structure (convex problem ~ linear problem)
  o optimization problems: global solution
  o powerful numerical/analytical techniques
  o significant insight (even if no solution)

• BER/SER is in the core of digital communications
  o important performance objective
  o subject to analysis/optimization/design

• Various applications/insights/generalizations

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\(^1\) see e.g. S. Boyd, L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.
Convex Analysis

- strong analytical structure
- analytical/numerical techniques
- algorithms
- global solutions
- convergence

bridge

Digital Communications
SYSTEM MODEL: AWGN CHANNEL

Baseband, discrete-time AWGN channel

\[ \mathbf{r} = \mathbf{s} + \xi \]

\( s, \mathbf{r} \) are the Tx and Rx symbols,

\( s \in \{ s_1, s_2, \ldots, s_M \} \), a set of \( M \) symbols (constellation/codebook)

\( \xi \sim \mathcal{CN}(0, \sigma_0^2 \mathbf{I}) \) is AWGN,

\[
p_\xi(x) = \left( \frac{1}{2\pi\sigma_0^2} \right)^{\frac{n}{2}} e^{-\frac{|x|^2}{2\sigma_0^2}}
\]

Fading channel can be considered as well.
**MAXIMUM LIKELIHOOD (ML) DETECTOR**

The maximum likelihood (ML) = minimum distance detector,

\[ \hat{s} = \arg \min_s |y - s| \]

Symbol error rate (SER\(_i\) / SER),

\[ P_{ei} = \text{SER}_i = \Pr[\hat{s} \neq s_i | s = s_i], \quad P_e = \text{SER} = \sum_{i=1}^{M} P_{ei} \Pr[s = s_i] \]

Bit error rate (BER):

\[ \text{BER} = \sum_{i=1}^{M} \sum_{j \neq i} h_{ij} \frac{\log_2 M}{\Pr\{s = s_i\} \Pr\{s_i \rightarrow s_j\}} \]
**PRIOR RESULTS**: **CONVEXITY OF SER**

**Theorem 1:**

- **$n \leq 2$:** SER is **convex in low dimensions**, (any constellation/coding), $d^2 \text{SER} / d\text{SNR}^2 > 0$,
- **$n > 2$:**
  - SER is **convex at high SNR**, $\text{SNR} \geq (n + \sqrt{2n})/d_{\text{min}}$
  - **concave at low**, $\text{SNR} \leq (n - \sqrt{2n})/d_{\text{max}}$
  - odd number of inflection points in-between.

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**RECENT IMPROVEMENTS**

**Theorem 1a:** High/low SNR bounds can be strengthened as

- **High SNR:** \[ \text{SNR} \geq \frac{(n-2)}{d_{\text{min}}^2} \]
- **Low SNR:** \[ \text{SNR} \leq \frac{(n-2)}{d_{\text{max}}^2} \]

and no further improvement is possible.

For spherical decision regions: sufficient *and* necessary.

**Extensions:**

- not only ML, any detector such that \( \text{Sphere}(R = d_{\text{min}}) \in \Omega_i \) (simply-bounded)
- any unimodal noise power density (e.g. Laplace)
**BEAUTIFUL** BOUNDS ON SER DERIVATIVES IN SNR

\[- \frac{c_n}{\gamma} \leq P'_e|\gamma| \leq 0, \quad c_1 = \frac{1}{\sqrt{2\pi e}}, \quad c_2 = \frac{1}{e}\]

\[
\frac{\beta_l}{\gamma^2} \leq P''_e|\gamma| \leq \frac{\beta_u}{\gamma^2}, \quad n = 2: \quad 0 \leq P''_e|\gamma| \leq \left(\frac{2}{e\gamma}\right)^2
\]

$c_n, \beta_l, \beta_u$ are constants (depend on dimensionality, but not const. geometry).
MORE PRIOR RESULTS

• SER(noise power):
  ▪ convex at high SNR
  ▪ concave at low
  ▪ odd number of inflex. points in-between

• Fading channels: average SER is
  ▪ convex in low dimensions (mild condition)
  ▪ “Fading is Never Good” in low dimensions
  ▪ convex at high SNR (Rayleigh/Rice/polynomial at 0)
NUMEROUS ADVANTAGES/APPLICATIONS

- Strong analytical structure of convex problems\(^3\)
- Efficient numerical algorithms, convergence, global solution
- Examples\(^4\) (many more in the literature):
  - BLAST optimization
  - Time/power sharing in decrease/increase SER
  - \(n=1,2\): “fading is never good”
  - convexity: power/time sharing is no good (Tx)
  - concavity: power/time sharing is no good (jammer)

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NEW RESULTS: PEP IS CONVEX AT HIGH SNR

Theorem 3:

• Any $n$: the pairwise error probability (PEP) $\Pr\{s_i \rightarrow s_j\}$ is 
  \textit{convex at high SNR}, $\gamma \geq (n + \sqrt{2n}) / d_{\min,i}^2$,

• $n = 1, 2$: \textit{concave at low SNR}, $\gamma \leq (n + \sqrt{2n}) / (d_{ij} + d_{\max,j})^2$, inflections (odd) in-between,

• $n > 2$: \textit{convex at low SNR}, $\gamma \leq (n - \sqrt{2n}) / (d_{ij} + d_{\max,j})^2$, inflections (even) in-between.
Main Result: BER is convex at high SNR

Theorem 4:
• For any constellation and bit mapping (also coding), BER is convex at high SNR (under ML detection),

\[ \text{SNR} \geq \frac{n + \sqrt{2n}}{d_{\text{min}}^2} \rightarrow \frac{d^2 \text{BER}}{d\text{SNR}^2} > 0 \]
**Convexity of PEP/Ber in Noise Power**

**Theorem 5:**
- The PEP is convex in noise power at high SNR (low noise).

**Corollary 5.1:**
- For any constellation and bit mapping (also coding), BER is convex at high SNR,

\[
\text{SNR} = \frac{1}{\sigma_0^2} \geq \frac{n+2 + \sqrt{2(n+2)}}{d_{\min}^2} \quad \rightarrow \quad \frac{d^2\text{BER}}{d\left(\sigma_0^2\right)^2} > 0
\]
CONCLUSIONS

• Importance of convexity/concavity

• SER is convex in low dimensions (n=1,2) or high SNR
  o dimensionality is critical, but not geometry
  o in terms of both SNR and noise power
  o universal bounds on derivatives

• BER/PEP is convex at high SNR
  o any constellation/bit mapping/coding
  o in SNR and noise power

• Various applications/advantages
  o “fading is never good” in low dimensions