On Physically-Based Normalization of MIMO Channel Matrices

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***Abstract*—Various normalizations of the MIMO channel matrix are discussed from a physical perspective. It is demonstrated that the physics of antenna arrays and propagation channel should be taken into account when normalization is chosen, so that SNR has proper physical meaning, the conclusions are physical and correspond to realistic systems. The antenna array geometry and the transmission strategy (coherent/non-coherent) limits the choice of normalization and determines how the capacity and other performance metrics scale with the number of antennas, which is more pronounced for densely-populated antenna arrays. This is especially important for an asymptotic analysis, when the number of antennas increases to infinity. Limitations of such analysis from the physical perspective are pointed out.**

***Index Terms*—Multi-antenna (MIMO) system, channel matrix, normalization, antenna array.**

I. Introduction

While various normalization of the MIMO channel matrix are used in the literature dealing with performance analysis of MIMO systems [1]-[10], the consequences of normalization choice for a particular problem and its implications for the antenna array configuration are not discussed in detail. The standard approach is that the normalization is a matter of convenience and thus can be set up in an ad hoc manner. While this is true to a certain extent, there exist limitations to such a convenience due to the underlying physics of the propagation channel and antenna arrays and also due to the transmission strategy adopted (i.e. coherent/non-coherent).

The purpose of this note is to expose such limitations via a detailed analysis of the MIMO channel matrix normalization employed in communication-theoretic problems from a physical perspective, which establishes a link between the communication-theoretic aspects of the problem on one hand and the antenna array and propagation channel aspects on the other hand. Different problems call for different channel normalizations, which represent adequately the physical behavior of a MIMO system when the number of antennas increases/decreases under a fixed total transmit power. Some antenna physics issues, which are not widely discussed in the communication-theoretic literature but are important, are pointed out.

While the antenna array geometry is known to have a significant impact on the channel correlation and thus on the channel capacity [12], [13], this paper describes a more subtle effect. Specifically, we demonstrate that different channel normalization imply different geometric configurations of the antenna arrays (see Tables 2-4), which has a profound effect on the scaling of SNR and various performance measures that follow from it with the number of antennas. Looking from a different perspective, the gain of an antenna array is different for sparse and densely-populated arrays with the same number of elements, and also for coherent and non-coherent transmission/reception, which has a profound impact on the SNR and other performance metrics under fixed total transmit power. If this effect is not accounted for in the normalization, the scaling of the SNR and the performance metrics with the number of array elements is not correctly reproduced. Thus, not all normalization are physically meaningful. This is especially important for the asymptotic analysis [3]-[8],[10],[11], when the number of antennas (at the transmit (Tx), receive (Rx) or both ends of the link) goes to infinity. In some cases, the asymptotic results may not be justified from the physical perspective (see the constraints in (18),(19) and (19)).

Finally, we point out that the antenna array gain G depends on a number of factors:

- $G = n$, where n is the number of array elements, when i) the element spacing is multiple integer of half a wavelength, ii) coherent combining is used (i.e. full CSI is available), and iii) far-field assumptions are satisfied (see (20));
- $G = 1$ for non-coherent processing (no full CSI; channel distribution information also results in this gain);
- when some of the conditions above are not satisfied, more careful analysis is required.

II. System Model

We employ the standard baseband system model of a frequency flat quasi-static MIMO channel,

$$\mathbf{y} = \mathbf{H}\mathbf{x} \tag{1}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the Tx and Rx symbol vectors, and $\mathbf{H}$ is the non-normalized (includes the propagation path loss etc.) $(n_r \times n_t)$ channel matrix, n_t and n_r are the number of Tx and Rx antennas respectively. The noise contribution is not included in (1) since it is not required for normalization purposes. Each receiver is assumed to have i.i.d. noise of power σ_0^2 and this will be explicitly used in the definition of the signal-to-noise ratio (SNR). Other sources of noise or interference (i.e., multiple access interferences, noise in the form of wavefield impinging on the antennas) are not considered. Unless otherwise indicated, we assume that the BLAST-type transmission (spatial multiplexing) is used, i.e. each Tx transmits independent symbols of the same power: $\langle \mathbf{x}\mathbf{x}^+ \rangle = P_t/n_t \cdot \mathbf{I}$, where P_t is the total Tx power, $\mathbf{I}$ is the identity matrix, $\langle\rangle$ and $^+$ denote expectation and Hermitian conjugation respectively, with full channel state information available at the Rx end. In the analysis below, we assume

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that $\mathbf{H}$ is fixed[1]. The total Rx power (i.e. collected by all Rx antennas from all transmitters) is

$$P_r = \sum_{i=1}^{n_t} \|\mathbf{h}_i\|^2 \frac{P_t}{n_t} = \|\mathbf{H}\|^2 \frac{P_t}{n_t} \tag{2}$$

where $\|\|$ denotes the Frobenius norm, $\|\mathbf{H}\|^2 = \sum_{i,j} |h_{ij}|^2$, h_{ij} and $\mathbf{h}_i$ are the (i,j) entry and i-th column of $\mathbf{H}$, and $\|\mathbf{h}_i\|^2 \frac{P_t}{n_t}$ represents the power contributed by i-th transmitter. In a typical analysis, the MIMO channel capacity is considered for given fixed $\mathbf{H}$ known at the Rx end only [1][2],

$$\begin{aligned} C &= \log\det\left(\mathbf{I} + \frac{P_t}{n_t\sigma_0^2}\mathbf{H}\mathbf{H}^+\right) \\ &= \log\det\left(\mathbf{I} + \frac{\gamma}{n_t}\widetilde{\mathbf{H}}\widetilde{\mathbf{H}}^+\right) \qquad (3) \\ &\approx \log\left(1 + \frac{\gamma}{n_t}\left\|\widetilde{\mathbf{H}}\right\|^2\right) \text{(low SNR)} \qquad (4) \end{aligned}$$

where $\widetilde{\mathbf{H}} = a\mathbf{H}$ is the normalized channel matrix, a is the normalization constant, and γ is the SNR at the receiver. Its meaning is directly related to the way a is defined. From (3),

$$\gamma = \frac{P_t}{|a|^2 \sigma_0^2} \tag{5}$$

The approximation in (4) holds at low SNR and has an intuitively-appealing interpretation as the capacity of an equivalent SISO channel with the power gain $= \left\|\widetilde{\mathbf{H}}\right\|^2 / n_t$, which is clearly related to the normalization of the channel matrix for given SNR γ.

III. Channel Matrix Normalization and SNR

To clarify the meaning of the SNR γ, we consider below some popular normalization. We begin with the following normalization (see e.g. [1][2]),

$$\left\|\widetilde{\mathbf{H}}\right\|^2 = n_t n_r \tag{6}$$

In this case, $|a|^2 = n_t n_r / \|\mathbf{H}\|^2$ and the SNR in (5) can be expressed as

$$\gamma = \frac{P_t \|\mathbf{H}\|^2}{n_t n_r \sigma_0^2} = \frac{P_r}{n_r \sigma_0^2} \tag{7}$$

Thus, in this case, γ is the SNR per receiver from all transmitters, and γ/n_t in (3) is the SNR per receiver per transmitter, i.e. contributed by one "average" transmitter to one "average" receiver. Under this normalization, the equivalent power gain from (4) is $\left\|\widetilde{\mathbf{H}}\right\|^2 / n_t = n_r$ (i.e. "Rx array gain"). Another possible normalization is $\left\|\widetilde{\mathbf{H}}\right\|^2 = n_t$ (see e.g. [8][4]), under which the channel power gain in (4) is $\left\|\widetilde{\mathbf{H}}\right\|^2 / n_t = 1$ (i.e. no Rx array gain).

For convenience, Table 1 summarizes various normalizations and corresponding SNR expressions and their meaning. As this Table demonstrates, adopting a normalization significantly affects the meaning of the SNR. In some cases, γ does not correspond to anything in the real system (i.e. cases 3 and 4). Thus, the normalization should be carefully chosen to ensure meaningful results, especially when something is plotted versus "SNR". The choice of normalization also affects the equivalent channel power gain. It should also be pointed out that, sometimes, (3) is used without $1/n_t$ factor. This also affects the meaning of SNR γ, which can be obtained from Table 1 using γ/n_t instead of γ.

IV. Physics of Antenna Arrays and Normalization

To get some insights into the relationship between normalization of the channel matrix and the physics of antenna arrays, we consider below a uniform linear array (ULA) of isotropic elements (radiators) [14]. Similar conclusions can also be drawn for more complicated (and also more realistic) antenna configurations, but with significant increase in complexity of the analysis [14]-[17]. In the analysis below, we follow the standard approach in the antenna array literature and adopt the far-field assumption, i.e. that the Tx and Rx arrays are separated by sufficiently large distance (see (18)) so that the arriving waves look locally like plane waves [14]-[17].

The normalized magnitude antenna pattern of the broadside[2] ULA of isotropic elements can be expressed as [14],

$$F(\theta) = \left| \frac{\sin\left(\frac{\pi L}{\lambda}\cos\theta\right)}{n \cdot \sin\left(\frac{\pi L}{\lambda n}\cos\theta\right)} \right| \tag{8}$$

where n is the number of elements, $L = nd$ is the array length, d is the element spacing, λ is the wavelength, and θ is the angle measured from the array axis. The array gain[3] is [14],

$$\begin{aligned} G &= \left(\frac{1}{2}\int_0^{\pi} F^2(\theta)\sin\theta d\theta\right)^{-1} \\ &= \frac{n}{1 + \frac{2}{n}\sum_{i=1}^{n-1} i \cdot \text{sinc}((n-i)kd)} \qquad (9) \end{aligned}$$

where $k = 2\pi/\lambda$ is the wave number and $\text{sinc}(x) = \sin x/x$. Following [16], the summation terms in (9) can be interpreted as the normalized mutual resistance between the isotropic elements, $R_{ij} = \text{sinc}((i-j)kd)$. When the element spacing is an integer multiple of half a wavelength, $d = m\lambda/2,\ m = 1, 2, 3...$, the gain is simply $G = n$[4]. This is the standard assumption in the communication and information-theoretic literature. However, we emphasize that for this assumption to hold the following 3 conditions have to be satisfied: 1) the element spacing is $d = m\lambda/2,\ m = 1, 2, 3...$, 2) the far-field assumption holds (see (20)), 3) coherent processing is used

[1] i.e. a given channel realization over the coherence time of the channel; when longer time intervals are of interest, the expectation over $\mathbf{H}$ should also be taken in appropriate expressions.

[2] i.e. the maximum of the radiation is normal to the array axis [17][15].

[3] strictly speaking, (9) gives the directivity of the array, i.e. the ratio of maximum radiation intensity to the radiation intensity averaged over all directions, but for a lossless antenna, which is assumed here, it is identical to the gain [14]. Due to the reciprocity theorem, the directivity, gain and also the antenna pattern are the same at the transmitting and receiving modes [15][17].

[4] This has the following intuition behind it [16]: when $d = m\lambda/2$, $R_{ij} = \delta_{ij}$ ($= 1$ if $i = j$ and 0 otherwise) and the total transmitted power is $P = I^2 \sum_{i,j} R_{ij} = nI^2$, where $I = \sqrt{P/n}$ is the normalized current in each element. This creates the far-field $\sim \sqrt{P/n}$ for each element, which are coherently combined (in the main beam direction), so that the total field is $\sim \sqrt{nP}$, and the received power $\sim nP$, while for the single isotropic element it is $\sim P$, and, thus, the array gain is n.

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TABLE I
CHANNEL MATRIX NORMALIZATIONS AND SNR FOR BLAST-TYPE TRANSMISSION

#	Normalization	Constant	SNR	Equivalent Channel Gain (Low SNR)	Meaning of SNR
1	$\left\|\widetilde{\mathbf{H}}\right\|^2 = n_t n_r$	$\|a\|^2 = \frac{n_t n_r}{\|\mathbf{H}\|^2}$	$\gamma = \frac{P_t \|\mathbf{H}\|^2}{n_t n_r \sigma_0^2} = \frac{P_r}{n_r \sigma_0^2}$	n_r	γ is the SNR per receiver from all transmitters; γ/n_t is the SNR per receiver per transmitter
2	$\left\|\widetilde{\mathbf{H}}\right\|^2 = n_t$	$\|a\|^2 = \frac{n_t}{\|\mathbf{H}\|^2}$	$\gamma = \frac{P_t \|\mathbf{H}\|^2}{n_t \sigma_0^2} = \frac{P_r}{\sigma_0^2}$	1	γ is the total Rx SNR (collected by all receivers) from all transmitters; γ/n_t is the total Rx SNR per transmitter
3	$\left\|\widetilde{\mathbf{H}}\right\|^2 = n_r$	$\|a\|^2 = \frac{n_r}{\|\mathbf{H}\|^2}$	$\gamma = \frac{P_t \|\mathbf{H}\|^2}{n_r \sigma_0^2} = \frac{n_t P_r}{n_r \sigma_0^2}$	$\frac{n_r}{n_t}$	γ/n_t is the SNR per receiver from all transmitters; γ is the SNR per receiver from all transmitters, each with the power P_t (rather than P_t/n_t)
4	$\left\|\widetilde{\mathbf{H}}\right\|^2 = 1$	$\|a\|^2 = \frac{1}{\|\mathbf{H}\|^2}$	$\gamma = \frac{P_t \|\mathbf{H}\|^2}{\sigma_0^2} = \frac{n_t P_r}{\sigma_0^2}$	$\frac{1}{n_t}$	γ/n_t is the total Rx SNR from all transmitters; γ is the total Rx SNR from all transmitters, each with the power P_t (rather than P_t/n_t)

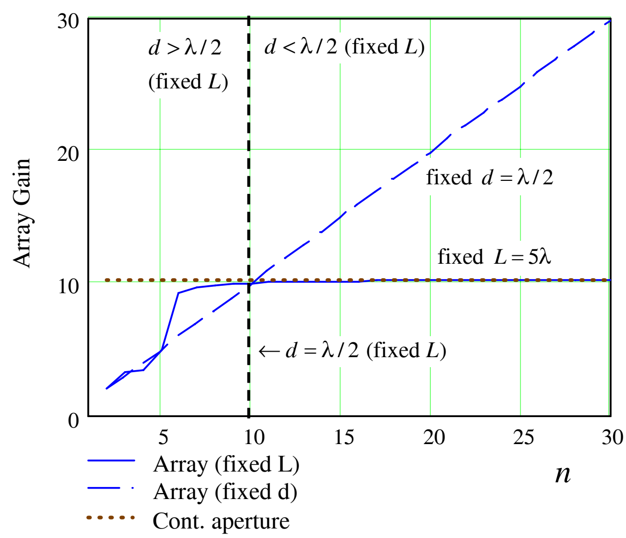

Fig. 1. The array gain G versus the number of elements n for $L = 5\lambda$, and for $d = \lambda/2$. The gain of continuous antenna with $L = 5\lambda$ is also shown. The array gain saturates at about $d = \lambda/2$ for fixed L.

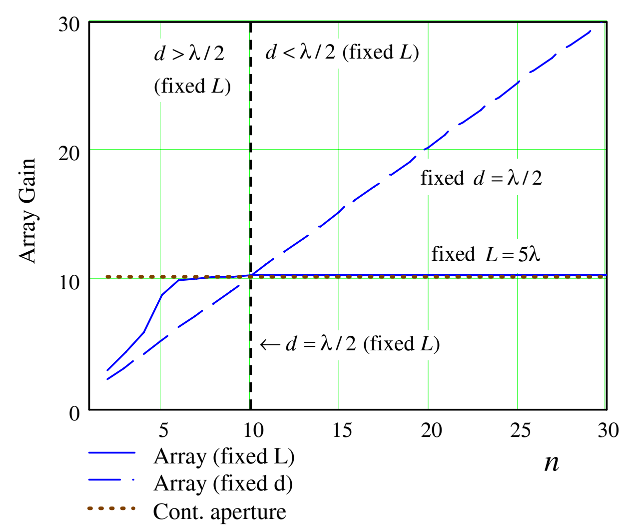

Fig. 2. The gain of the array of short collinear dipoles versus the number of elements. The parameters are the same as in Fig. 1. The gain behaves in almost the same way as in Fig. 1, and the short dipole gain (=3/2) does not affect the array gain when $n >> 1$.

(i.e. full CSI). If any of these 3 conditions is not satisfied, the gain is not equal to n anymore. When the main beam steered away from the broadside, (9) does not apply. In that case, the gain is given by eq. (9.21) in [16], and it is still $G = n$ at $d = m\lambda/2$.

Fig. 1 shows $G(n)$ for fixed element spacing d (adding new elements increases the array length L) and for fixed array length L (adding new elements decreases spacing d). Clearly, they have different tendency for large number of elements n: while $G(n)$ increases without limit for fixed d, it saturates at $G_{\max} = 2L/\lambda = 10$, which is the gain of a continuous linear antenna, for the fixed array length $L >> \lambda$ [14]. The saturation point corresponds to roughly half a wavelength spacing, $d \approx \lambda/2$, i.e. adding more elements at smaller spacing does not increase the array gain. Physically, this is explained by the fact that the power collected by the array of fixed length cannot exceed the power collected by the continuous linear antenna of the same length (when perfectly matched, this continuous antenna can be considered as perfect absorber of electromagnetic power). Consequently, a uniform array with half a wavelength spacing ($d = \lambda/2$) or less is equivalent to continuous linear antenna of the same length, as far as the gain is concerned. This can be shown using (8) in the limit $n \to \infty$,

$$\lim_{n\to\infty} F(\theta) = \left| \frac{\sin\left(\frac{\pi L}{\lambda}\cos\theta\right)}{\frac{\pi L}{\lambda}\cos\theta} \right| = F_{con}(\theta) \tag{10}$$

where $F_{con}(\theta)$ is the continuous antenna pattern [14]. It follows from (9) and (10) that $\lim_{n\to\infty} G = G_{con}$, where $G_{con} = 2L/\lambda$ is the gain of the continuous linear antenna. From (8), $F(\theta)$ approaches closely $F_{con}(\theta)$ when $\frac{\pi L}{n\lambda} < \frac{\pi}{2}$, which corresponds to $d < \lambda/2$. This observation also fits well into the spatial sampling argument [18].

Similar tendencies can also be observed for more complicated antenna configurations (including planar antenna arrays). For example, when array elements are directional with the element pattern $F_e(\theta)$, the overall antenna array pattern follows

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from the pattern multiplication theorem[5][14][15]: $F_A(\theta) = F_e(\theta)F(\theta)$, where $F(\theta)$ is the array factor (array pattern with isotropic elements) in (8). For large array length L, $F(\theta)$ is a dominant factor contributing to the gain, and $G_A \approx G$, where G_A is the array gain with directional elements, and the overall tendency in the gain behavior is the same as before. Fig. 2 shows this for the ULA of short collinear dipoles (see [14][16] for details on the antenna pattern in this case). The only noticeable difference to Fig. 1 is the array gain behavior for $n \leq 7$ at fixed array length L. Note that the short dipole gain of 3/2 does not affect the antenna array gain for $n \geq 2L/\lambda$ ($d \leq \lambda/2$), which is equal to that of the array with isotropic elements, $G_A \approx G$.

It should be noted that we used a simplified model of an antenna array, without mutual coupling of realistic elements[6] and other effects [14][15]. These effects, however, depend heavily on array element design and other factors and, thus, outside of the scope of the present study. The impact of mutual coupling on MIMO system performance can be found, for example, in [24][25].

The antenna gain can be related to the total Rx power and hence the SNR using the standard link budget equation [19],

$$P_r = P_t \frac{G_r G_t}{L_p} \tag{11}$$

where $G_r,\ G_t$ are the Rx and Tx antenna gains, and L_p is the propagation path loss. We note that (11) applies only in the far field, i.e. when the Tx-Rx distance significantly exceeds the antenna size and the wavelength (see (20)), which is assumed below. What happens to the total Rx power P_r when the number of elements n_r increases? The answer is determined by the gain $G_r(n_r)$ (the other factors are independent of n_r). Using the discussion above and especially Fig. 1 and 2, one concludes that the answer depends on the array configuration, i.e. fixed length L or fixed spacing d, and also on specific value of n_r. For fixed spacing $d = m\lambda/2,\ m = 1, 2, 3...$, the gain $G_r = n_r$ and the received power P_r increase linearly with n_r, but this implies increasing length of the Rx array, $L_r = n_r d$, which collects more power with increasing n_r. For fixed length L_r, the power P_r increases with the number of elements n_r up to a certain point, and then saturates at $d \leq \lambda/2$, which corresponds to $n_r \geq 2L_r/\lambda$ and $G_r = G_{\max} = G_{con} = 2L_r/\lambda$, since additional elements do not help to collect more power[7].

Thus, these two array configurations have similar behavior for $n_r < 2L_r/\lambda$, but very different ones for $n_r \geq 2L_r/\lambda$.

Correlating this with the channel matrix normalizations in the previous section, we conclude that $\|\mathbf{H}\|^2$and also $\left\|\widetilde{\mathbf{H}}\right\|^2$ should follow the same tendency as the gain G_r when the number of elements n_r changes, if correct physical behavior of the result is expected under fixed total transmit power or SNR. This argument can be formalized as follows. The total Rx power P_r can be found as a function of $\|\mathbf{H}\|^2$. Using this in the link budget (11), $\|\mathbf{H}\|^2$ can be related to $G_r G_t$ and thus its behavior when n_r, n_t vary can be found. Since the main purpose of the normalization is to remove the distance-dependent large-scale path loss, we further assume that the normalization corresponds to $L_p = 1$. Three different transmission strategies are considered, depending on channel state information (CSI) at each end of the link:

1. *Tx non-coherent / Rx coherent* , or BLAST-type transmission, when each Tx signal is independent of any other one. This type of transmission does not require CSI at the Tx end, but only at the Rx end. In this case, the power P_r is given by (2). While the gain G_r is the standard array gain discussed above, the gain G_t is not because, contrary to a standard antenna array, the Tx signals in this case are not coherent (due to their independence). Using the standard techniques of array pattern analysis with random variations [20]-[22], it is straightforward to show that $G_t = 1$ is this case. Intuitively, this is explained by the fact that any array gain comes from coherent combining (radiation) of signals [14][15] and if coherence is lost, so is the gain. Thus, using (2) and (11), one obtains $\|\mathbf{H}\|^2 = n_t G_r / L_p$ and, for the normalized channel matrix,

$$\left\|\widetilde{\mathbf{H}}\right\|^2 = n_t G_r \tag{12}$$

Thus, for varying n_r, $\left\|\widetilde{\mathbf{H}}\right\|^2$ under physically-justified normalization (which represents correctly the scaling of total Rx power with the number of antennas) exhibits the same behavior as the Rx array gain G_r, and it is always linearly proportional to the number of Tx elements n_t, regardless of the Tx array configuration. Only the Rx array configuration is important in this case[8]. This is summarized in Table 2[9].

2. *Tx non-coherent / Rx non-coherent* (no CSI at either end of the link). An example of such transmission/reception strategy can be found in [26]. In this case, the total Rx power can be shown to be $P_r = \sum_{i=1}^{n_t} \|\mathbf{h}_i\|^2 \frac{P_t}{n_t n_r} = \|\mathbf{H}\|^2 \frac{P_t}{n_t n_r}$, where $\|\mathbf{h}_i\|^2 \frac{P_t}{n_t n_r}$ is the power contributed by i-th transmitter (notice that it is smaller by a factor of n_r compared to (2), i.e. no Rx array gain due to the lack of Rx CSI). Using (11), the normalized channel matrix should satisfy (since

[5]It should be emphasized that the multiplication theorem applies to the antenna pattern but not the gain [14][15], i.e. the total antenna gain is not a product of the element gain and the array gain [[16], p. 710], and the corresponding claim in [[27], p.530] ("…the power gain which is the product of the array gain and the antenna gain…") is incorrect.

[6]recall that mutual coupling between isotropic (ideal) elements is accounted for in (9), via their mutual resistance.

[7]The explanation of the array gain saturation in [[27], p.530] ("…for a fixed-array aperture, increasing the number of elements decreases the space occupied by each antenna and therefore decreases the individual antenna gain" so that the total antenna gain stays the same) is not justified. As it follows from our argument, the array gain saturation is not in general related to decreasing element gain, which is always unity in our model since the elements are isotropic and lossless. This effect is related to the array factor, but not to the element pattern, i.e. it exists even when the element pattern does not change with n. The ultimate cause of this effect is the array factor convergence to that of the corresponding continuous antenna (see (10)), which results in the array gain convergence to that of the continuous antenna as n increases. Fig. 1 and 2 clearly demonstrate this.

[8]This may seem to contradict to the reciprocity theorem of antenna theory [14]-[17] (i.e. the antenna properties in the receiving and transmitting modes are the same, which also extends to the propagation channel), but this is only an imaginary contradiction: the reciprocity theorem does not apply here because, while the Rx array performs coherent combining of the signals, the Tx array radiates non-coherent signals.

[9]Note that the array configurations in the Tables 2-4 apply only to the $d \leq \lambda/2$ ($n \geq 2L/\lambda$) region since smaller n (larger d for fixed-L array) result in similar gain behavior (increase with n) for both configurations, see Fig.1 and 2, so that they cannot be reliably differentiated. We note that the $d \leq \lambda/2$ region becomes important when the space available for the antenna array is limited, i.e. a handset.

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$G_t = G_r = 1$),

$$\left\|\widetilde{\mathbf{H}}\right\|^2 = n_t n_r \tag{13}$$

Thus, the only physically-justified norm in this case is proportional to n_t, n_r, regardless of the array configurations. This is summarized in Table 3.

3. *Tx coherent / Rx coherent* (CSI at both ends of the link). This can be, for example, the join Tx-Rx maximum ratio combing (MRC). In this case, the total Rx power can be expressed as [23]:

$$P_r = \lambda_{\max}(\mathbf{HH}^+)P_t \leq tr(\mathbf{HH}^+)P_t = \|\mathbf{H}\|^2 P_t \tag{14}$$

where $\lambda_{\max}$ is the maximum eigenvalue of $\mathbf{HH}^+$, and the upper bound follows from the fact that the trace is a sum of the eigenvalues, which are positive in this case. The upper bound is achieved when there is only one non-zero eigenvalue, which physically corresponds to ideal beamforming at both Tx and Rx ends of the link, with no multipath (only line-of-sight path) or negligible multipath. Combing the upper-bound in (14) with (11), one obtains $\|\mathbf{H}\|^2 = G_t G_r / L_p$ and, for the normalized channel matrix,

$$\left\|\widetilde{\mathbf{H}}\right\|^2 = G_t G_r \tag{15}$$

The essential difference here to the previous cases in (12) and (13) is that both Tx and Rx antenna array configurations are important[10]. This is summarized in Table 4.

Thus, Tables 2-4 provide physical interpretation of the normalization in terms of the array configurations. The interpretations depend on the transmission strategy. For coherent transmission at both ends (joint Tx-Rx MRC or beamforming), every time $\left\|\widetilde{\mathbf{H}}\right\|^2$ increases with n, corresponding aperture has to increase as well, if $n \geq 2L/\lambda$ or $d \leq \lambda/2$, i.e. a densely populated array. For smaller n (sparser array), it can be both ways since the array gain behavior is somewhat similar for both configurations, as Fig. 1 and 2 indicate; the only difference is that while for fixed d the increase in gain is linear in n, it is non-linear for fixed L. For non-coherent transmission at the Tx end and coherent one at the Rx end, this applies only to n_r, L_r; $\left\|\widetilde{\mathbf{H}}\right\|^2$ has always to increase linearly with n_t. For completely non-coherent transmission, $\left\|\widetilde{\mathbf{H}}\right\|^2$ has always to increase linearly with n_t, n_r, and the array configurations are of no importance. Thus, the array configuration is linked to the normalization only when coherent transmission and/or reception is used (since the array gain depends on the array geometry in that case; for non-coherent processing, the array gain =1, regardless of the array geometry).

V. Limitations of Asymptotic Analysis

These observations are especially important for asymptotic analysis [3]-[8][10][11], when the number of antennas (at either end or both) increases to infinity, as the normalization affects the results, sometimes resulting in no convergence at all [8]. When the number of Rx elements n_r increases to infinity, the total Rx power will be always limited due to the following reasons: i) for fixed length L_r, due to the limited Rx array gain $G_r \leq G_{\max} = 2L_r/\lambda$; ii) for fixed spacing d_r, the gain G_r increases linearly with n_r but, at some point, it saturates, as explained below.

For any array configuration and the number of antennas, the total Rx power P_r always stays finite because it cannot exceed the total Tx power, due to the law of energy conservation, and the latter is assumed to be finite (fixed),

$$P_r \leq P_t \tag{16}$$

Consequently, the total Rx SNR will also stay finite, under fixed P_t. If an adopted normalization violates this rule, the conclusions of the analysis are not physical. This has the following implication for the fixed-d array configuration. From

[10]Note that, in this case, the reciprocity theorem does apply as both Tx and Rx ends of the link rely on coherent signaling.

TABLE II
Channel matrix normalizations for Tx non-coherent/Rx coherent (BLAST-type) transmission and corresponding configurations of densely-populated array ($n \geq 2L/\lambda$ or $d \leq \lambda/2$)

#	Normalization	Array Configuration
1	$\left\|\widetilde{\mathbf{H}}\right\|^2 = n_t n_r$	Fixed d_r; L_r increases linearly with n_r. Arbitrary configuration of the Tx array.
2	$\left\|\widetilde{\mathbf{H}}\right\|^2 = n_t$	Fixed L_r; Arbitrary configuration of the Tx array.
3	$\left\|\widetilde{\mathbf{H}}\right\|^2 = n_r$	Not physical
4	$\left\|\widetilde{\mathbf{H}}\right\|^2 = 1$	Not physical

TABLE III
Channel matrix normalizations for Tx non-coherent/Rx non-coherent transmission and corresponding configurations of array

#	Normalization	Array Configuration
1	$\left\|\widetilde{\mathbf{H}}\right\|^2 = n_t n_r$	Arbitrary configuration of the Tx and Rx arrays.
2	$\left\|\widetilde{\mathbf{H}}\right\|^2 = n_t$	Not physical
3	$\left\|\widetilde{\mathbf{H}}\right\|^2 = n_r$	Not physical
4	$\left\|\widetilde{\mathbf{H}}\right\|^2 = 1$	Not physical

TABLE IV
Channel matrix normalizations for Tx coherent/Rx coherent transmission (joint Tx-Rx MRC or beamforming) and corresponding configurations of densely-populated array ($n \geq 2L/\lambda$ or $d \leq \lambda/2$)

#	Normalization	Array Configuration
1	$\left\|\widetilde{\mathbf{H}}\right\|^2 = n_t n_r$	Fixed d_t, d_r; $L_{t(r)}$ increases linearly with $n_{t(r)}$.
2	$\left\|\widetilde{\mathbf{H}}\right\|^2 = n_t$	Fixed d_t, L_r; L_t increases linearly with n_t.
3	$\left\|\widetilde{\mathbf{H}}\right\|^2 = n_r$	Fixed d_r, L_t; L_r increases linearly with n_r.
4	$\left\|\widetilde{\mathbf{H}}\right\|^2 = 1$	Fixed L_t, L_r.

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(16) and (11), one obtains

$$G_r G_t \leq L_p \tag{17}$$

so that the linear increase of G with n (for fixed-d configuration) holds (see Fig. 1 and 2) as long as this inequality is satisfied. For the BLAST-type and MRC transmissions, this results in,

$$n_r \leq L_p \text{ (BLAST)} \tag{18}$$

$$n_r n_t \leq L_p \text{ (MRC)} \tag{19}$$

If n increases without limit, the fixed-d array gain will ultimately saturate at the points dictated by (17)-(19). This has an implication for the asymptotic analysis: if convergence to the limiting distribution (with a desired accuracy) is achieved within the limits in (17)-(19), the asymptotic results can be applied to a finite real-world system. However, if such a convergence is achieved beyond the limits in (17)-(19), the fundamental physics of energy conservation prevents one from applying such a statistical result to a realistic system.

An additional limitation of the asymptotic analysis is due to the far-field assumption [14]-[17], under which the antenna array model considered above holds true, i.e. the array length L (both, the Tx and the Rx ones) must satisfy the inequalities

$$\frac{2L^2}{\lambda} < R_{\min}, \quad L < R_{\min} \tag{20}$$

where $R_{\min}$ is the minimum of the distances between the Tx and Rx arrays, and also any of those arrays and the scatterers. Using $L = nd$, one obtains another upper bound on the number elements,

$$n < \frac{R_{\min}}{d} \min\left(\sqrt{\frac{\lambda}{2R_{\min}}}, 1\right) \tag{21}$$

If (20), (21) do not hold, (8) does not apply, the array gain at $d = m\lambda/2$, $m = 1, 2, 3...$, is not $G = n$ anymore and some additional assumptions traditionally made (sometimes implicitly) in the array analysis cease to hold (e.g. the array pattern becomes a function of distance) [14]-[17]. In such a case, while it is still possible to introduce some modifications to the model to account for near-field effects, the analysis becomes much more complicated and should be made with extreme care.

VI. Conclusion

Normalization of MIMO channel matrices has been discussed in this paper from a physical perspective, based on the physics of antenna arrays and the link budget considerations. The physically-justified normalization (which represents correctly the scaling of Rx power with the number of antennas) depends on the transmission strategy adopted (coherent/non-coherent) and also on the array geometry. While the latter is well-known to affect the channel correlation and, via this way, the channel capacity [12][13], the present paper demonstrates a more subtle effect: the array geometry also affects significantly the channel matrix normalization and thus has an additional impact on the channel capacity scaling with the number of array elements. These consideration are especially important for an asymptotic analysis ($n \to \infty$) since, under the limited space available for the antennas, the array gain will always saturate (as in Fig. 1, 2). The extension of this work to distributed MIMO/networks is of significant interest.

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