Performance Analysis of V-BLAST with Optimum Power Allocation

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Abstract—Comprehensive performance analysis of the unordered V-BLAST algorithm with various power allocation strategies is presented, which makes use of analytical tools and resorts to Monte-Carlo simulations for validation purposes only. High-SNR approximations for the optimized average block and total error rates are given. The SNR gain of optimization is rigorously defined and studied using analytical tools, including lower and upper bounds, high and low SNR approximations. The gain is upper bounded by the number of transmitters, for any modulation format and any type of fading. This upper bound is achieved at high SNR by the considered optimization strategies. While the average optimization is less complex than the instantaneous one, its performance is almost as good at high SNR. A measure of robustness of the optimized algorithm is introduced and evaluated, including compact closed-form approximations. The optimized algorithm is shown to be robust to perturbations in individual and total transmit powers. Based on the algorithm robustness, a pre-set power allocation is suggested as a low-complexity alternative to the other optimization strategies, which exhibits only a minor loss in performance over the practical SNR range.

I. Introduction

The V-BLAST algorithm [1] has attracted in recent years significant attention as a signal processing strategy in the MIMO receiver due to its relative simplicity and also the ability to achieve, under certain conditions, the full MIMO capacity. Unfortunately, the algorithm has a few drawbacks as well. The optimal ordering procedure is computationally-demanding, which is a limitation for some applications. Since the successive interference cancellation is used, lower detection steps have on average a smaller SNR and thus produce more errors, which further propagate to higher steps [5]-[7] so that the overall error performance may be not satisfactory, especially if no coding is used.

A popular approach to improve the error performance of the V-BLAST algorithm is to decrease the error rates at lower steps by employing a non-uniform power allocation among the transmitters. Several techniques have been reported that find the transmit (Tx) power allocation that minimizes the instantaneous (i.e. for given channel realization) total error rate (TBER)[1] of the V-BLAST, with or without the optimal ordering [3][4]. The approximate solutions for the instantaneous Tx power allocation have also been found, based on various approximations. In [2], the instantaneous BLER[1] (rather than the TBER) is considered as an optimization criterion, and the optimum Tx power allocation is found numerically for the V-BLAST with two transmitters. Although the instantaneous power allocation techniques proposed in [2]-[4] do demonstrate a few dB performance improvement over the original (unoptimized) V-BLAST, they also add considerably to the system complexity, since new feedback session and power reallocation are needed each time the channel matrix changes; the instantaneous per-stream (transmitter) SNRs also need to be sent to the Tx end.

A less complex approach is to use an average rather than instantaneous optimization, i.e. the optimum power allocation is found based on the average error rate (BLER or TBER) [5],[8]-[10]. Since this ignores the small-scale fading, only occasional feedback sections and power reallocations are required, when the average SNR changes, and only the average SNR needs to be fed back to the Tx end.

Performance evaluation of the optimized systems has been done in [2]-[5],[10] through simulations, by comparing optimized and non-optimized error rate curves, and it was noted that the optimum power allocation gives a few dB gain in terms of the SNR.

In this paper, we present analytical performance evaluation of the optimized system via a rigorous definition of the SNR gain of the optimization and via a measure of robustness, in addition to the traditional error rate analysis. Upper and lower bounds on the SNR gain are given, which hold for any modulation and any type of fading. Specifically, it is shown that the SNR gain cannot exceed m (the number of transmitters). This upper bound is achieved at high SNR. The lower bound is approached by the SNR gain at low SNR. Additional properties of the gain, including compact high and low-SNR approximations, are also given. These results are summarized in Theorems 1-5 and Corollaries in Section V.

The impact of perturbations in the individual and total Tx powers on the performance of the optimized system is studied using a measure of robustness and relying on the generic principles of convex optimization [11]. It is demonstrated that the optimized system is robust to such perturbations, which also indicates that the closed-form approximations for the optimum power allocation can be used without noticeable loss in the performance. Based on this, a pre-set power allocation is suggested as a low-complexity alternative to other optimization strategies. Due to the robustness of the proposed power allocation, it is expected that a significant portion of the

[1] The TBER is defined as the error rate at the output stream to which all the individual sub-streams are merged after the detection (see [7] for more details). Thus, it takes into account the actual number of errors at the transmitted symbol vector. The block error rate (BLER) is defined as the probability to have at least one error at the detected Tx symbol vector [7]. It does not take into account the actual number of errors, but only the fact of their presence.

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theoretically-predicted gain can also be achieved in practice.

Analytical results and conclusions are validated via Monte-Carlo (MC) simulations.

II. SYSTEM MODEL

We employ the following standard baseband discrete-time system model,

$$\mathbf{r} = \mathbf{H}\mathbf{A}\mathbf{s} + \xi = \sum_{i=1}^{m} \mathbf{h}_i \sqrt{\alpha_i} s_i + \xi \quad (1)$$

where $\mathbf{s} = [s_1, s_2, \ldots s_m]^T$ and $\mathbf{r} = [r_1, r_2, \ldots r_n]^T$ are the vectors representing the Tx and Rx symbols respectively, "T" denotes transposition, $\mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2, \ldots \mathbf{h}_m]$ is the $n \times m$ matrix of the complex channel gains between each Tx and each Rx antenna, where $\mathbf{h}_i$ denotes i-th column of $\mathbf{H}$, which are assumed to be i.i.d. Rayleigh fading unless otherwise indicated[2], n and m are the numbers of Rx and Tx antennas respectively, $n \ge m$, ξ is the vector of circularly-symmetric additive white Gaussian noise (AWGN), which is independent and identically distributed (i.i.d.) in each receiver, $\mathbf{A} = diag\left(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_m}\right)$, where α_i is the power allocated to the i-th transmitter. For the regular (unoptimized) V-BLAST, the total power is distributed uniformly among the transmitters, $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 1$. In the optimized system, α_i are chosen to minimize the total BER or the BLER, either average or instantaneous. Since we rely on the BLAST error rate performance analysis in [5],[7]-[9], we also adopt the same basic assumptions.

III. ERROR RATES AND OPTIMUM POWER ALLOCATION

Closed-from expressions for the BLER and TBER of the unordered V-BLAST can be found in [5],[7]-[10]. For the sake of simplicity and completeness, we give below their high-SNR approximations only. The average BLER can be expressed as

$$\overline{P}_B(\boldsymbol{\alpha}) \approx \sum_{i=1}^{m} \overline{P}_{ei} \approx \sum_{i=1}^{m} \frac{C_{2i-1}^{i}}{\left(4\alpha_i \gamma_0\right)^{n-m+i}}, \quad (2)$$

where $\overline{P}_{ei}$ is the average error rate at step i conditioned on no errors at the previous steps, γ_0 is the average SNR, and $C_k^i = k!/(i!(k-i)!)$ are the binomial coefficients. The second equality holds for BPSK modulation, which we assume below[3], unless otherwise indicated. The average TBER with optimum power allocation can be approximated as

$$\overline{P}_{et}(\boldsymbol{\alpha}) = \frac{1}{2m} \sum_{i=1}^{m} (m-i+2) \overline{P}_{ei} \approx \frac{1}{2m} \sum_{i=1}^{m} \frac{(m-i+2) C_{2i-1}^{i}}{\left(4\alpha_i \gamma_0\right)^{n-m+i}}, \quad (3)$$

Based on these approximations, closed-form expressions for optimum power allocation have been obtained [8][9]. When the average BLER is used as the performance metric, the optimum power allocation is

$$\alpha_1^{opt} \approx m - \sum_{i=2}^{m} \alpha_i^{opt}, \ \ \alpha_i^{opt} \approx \frac{b_i}{\left(4\gamma_0\right)^{\frac{i-1}{n-m+i+1}}}, \ i = 2, \ldots, m, \quad (4)$$

[2] some of our results hold true for arbitrary channels

[3] some of our results hold true for arbitrary modulation.

where

$$b_i = \left(\left(n-m+i\right) m^{n-m+2} C_{2i-1}^{i} / n-m+1\right)^{1/(n-m+i+1)} \quad (5)$$

Note that $\alpha_1 \to m$ and $\alpha_2 \ldots \alpha_m \to 0$ as $\gamma_0 \to \infty$, i.e. most of the power goes to 1st transmitter at high SNR. This is consequence of the fact that 1st step error rate dominates the whole error rate.

When the average TBER is used as the performance metric, the optimum power allocation is similar to that for the BLER-based optimization, so that (4) can be used with b_i given by

$$b_i = \left(\frac{C_{2i-1}^{i}\left(n-m+i\right)\left(m-i+2\right) m^{n-m+2}}{\left(m+1\right)\left(n-m+1\right)} \right)^{1/(n-m+i+1)}. \quad (6)$$

The same tendency in power distribution is also observed (most power going to 1st transmitter).

The approximations in (2)-(4) can be used to find the average error rates of the power-optimized V-BLAST. Specifically, based on (2) and (4), we observe that despite of the fact that the transmitters i to m are allocated small amounts of power (and, thus, the corresponding error rates increase compared to the unoptimized system), the 1st step error rate still dominates the overall performance,

$$\overline{P}_{e1}^{opt} \approx \frac{1}{\left(4m\gamma_0\right)^{n-m+1}}, \ \overline{P}_{ei}^{opt} \approx \frac{a_i}{\left(4\gamma_0\right)^{\frac{(n-m+i)(n+m+2)}{n-m+i+1}}}, \quad (7)$$

where $a_i = C_{2i-1}^{i} / b_i^{n-m+i}$, so that $\overline{P}_{e1}^{opt} >> \overline{P}_{e2}^{opt} >> \ldots >> \overline{P}_{em}^{opt}$ at high SNR (the same relationship as for the unoptimized system in [7]). Thus, the average BLER can be approximated as

$$\overline{P}_B^{opt} \approx \overline{P}_{e1}^{opt} \approx 1/\left(4m\gamma_0\right)^{n-m+1}, \quad (8)$$

Comparing (8) to the average BLER without optimization, $\overline{P}_B \approx \overline{P}_{e1} \approx 1/\left(4\gamma_0\right)^{n-m+1}$, we conclude that the power allocation brings $10 \log m$ dB SNR gain in terms of the average BLER.

In a similar way, using (3), the optimized average TBER can be approximated as

$$\overline{P}_{et}^{opt} \approx \frac{m+1}{2m} \overline{P}_{e1}^{opt} \approx \frac{m+1}{2m\left(4m\gamma_0\right)^{n-m+1}}. \quad (9)$$

Comparing (9) to the unoptimized TBER [7] $\overline{P}_{et} \approx \overline{a}_1 \overline{P}_{e1} / m$, where $\overline{a}_1 > 1$ quantifies the effect of error propagation (see [7][12] for details), we conclude that the optimum power allocation brings an SNR gain of

$$m\left(\frac{2\overline{a}_1}{m+1}\right)^{\frac{1}{n-m+1}} < m, \quad (10)$$

i.e. less than that in terms of the average BLER.

IV. ROBUSTNESS OF THE OPTIMUM POWER ALLOCATION

When the optimization algorithm is implemented in a practical system, there are various sources of inaccuracies and perturbations, which may affect its performance but which were ignored in the idealistic analysis above. These may include numerical inaccuracies of the optimization, inaccurate or outdated estimate of the average SNR, which result in inaccuracies in the optimum power allocation coefficients α_i^{opt}. A robust algorithm, which is insensitive to all these factors, is desired from the practical perspective.

In order to estimate the impact of these factors on the system performance, let us introduce the *measure of robustness* δ (sensitivity) of the average error rate (either BLER or TBER) with the optimum power allocation, $\bar{P} = \bar{P}(\boldsymbol{\alpha}^{opt})$, to the changes in system parameter u,

$$\delta = \left| \frac{\Delta \bar{P} / \bar{P}}{\Delta u / u} \right|, \tag{11}$$

where u may represent the total Tx power, $u = \sum_{i=1}^{m} \alpha_i$, or the power allocated to any of the transmitters, $u = \alpha_i$. The measure of robustness (11) is the ratio of the normalized variation in the performance $\Delta \bar{P} / \bar{P}$ to the normalized variation in the system parameter $\Delta u / u$, which causes this performance variation. Note that the use of normalized differences in the definition is essential as it makes the measure to be independent of the scale. The algorithm is robust to variations in the system parameter u if relatively small change in u leads to relatively small change in the error rate $\bar{P}$, i.e. when δ is small or moderate number.

When both the perturbation in the system parameter Δu and in the system performance $\Delta \bar{P}$ are small enough, one can use the derivatives in (11) instead of the finite differences,

$$\delta \approx \delta' = \left| \frac{\partial \bar{P}}{\partial u} \frac{u}{\bar{P}} \right|, \tag{12}$$

so that $\partial \bar{P} / \partial u$ determines the algorithm robustness, and δ' serves as a measure of local robustness. It follows from the Lagrange multiplier technique [11] that $\partial \bar{P} / \partial \alpha_i = \partial \bar{P} / \partial u = -\lambda$, so that

$$\delta \approx \delta' = \lambda u / \bar{P}, \tag{13}$$

where λ is the Lagrange multiplier evaluated at the optimum point [9], which is a part of the optimization problem solution. Thus, the appropriately normalized λ is the measure of local sensitivity[4] of the average error rate to variations in the total or individual Tx power. The normalized variation in the average error rate can be evaluated from the normalized variation in the system parameter using (13),

$$\frac{|\Delta \bar{P}|}{\bar{P}} = \delta \frac{|\Delta u|}{u} \approx \delta' \frac{|\Delta u|}{u}, \tag{14}$$

For the average BLER-based optimization at high SNR, using the results in [8][9], the Lagrange multiplier can be approximated as,

$$\lambda \approx \frac{n-m+1}{m^{n-m+2}} \frac{1}{(4\gamma_0)^{n-m+1}}, \tag{15}$$

For small variations in the system parameter, $u \approx m \approx \alpha_1^{opt}$, so that, using (8)and (15), the robustness measure with respect to the variations in the total or 1st transmitter power is

$$\delta_1' \approx n - m + 1, \tag{16}$$

i.e. equal to the diversity order of the system. The algorithm is locally robust as long as $(n-m)$ is not too large; $\delta_1' \approx 1$ and consequently $|\Delta \bar{P}| / \bar{P} \approx |\Delta u| / u$ if $n = m$. This result is a consequence of the fact that the high-SNR average BLER is dominated by the 1st step BER (see (8)) so that its diversity order and hence the sensitivity to the Tx power is minimum when $n = m$; increasing $(n-m)$ results in increasing diversity order and hence in increasing sensitivity to the Tx power. Thus, the beneficial effect of higher diversity order with more Rx antennas is accompanied by the negative effect of higher sensitivity to variations in system parameters.

The robustness measure with respect to $\alpha_2 ... \alpha_m$ can be approximated as

$$\delta_i' \approx \frac{n-m+1}{m} \frac{b_i}{(4\gamma_0)^{\frac{i-1}{n-m+1+i}}} \ll 1, \quad i = 2...m, \tag{17}$$

Thus, the algorithm is also robust in terms of $\alpha_2, \ldots, \alpha_m$ at high SNR. Furthermore, higher steps exhibit better robustness since, comparing (16) and (17), $\delta_1' > \delta_i'$, $i = 2...m$. It should be noted that this robustness of the algorithm is an unexpected by-product, which was not a goal of the original design.

For the TBER-based optimization, (16) and (17), and hence the conclusions above also hold true; b_i is given by (6) in this case. As an example, Fig. 1 shows the average TBER versus α_1 for 2x2 V-BLAST. When α_1 is far away from α_1^{opt}, the slope of the curves is quite steep and determined by the diversity order of the dominating step; thus, allocating too little power to the 1st Tx increases the 1st step BER, making it dominant, whereas giving too much power to the 1st Tx boosts the 2nd step BER. Note that the slope is steeper in the domain of the dominating 2nd step BER, apparently because of its higher diversity order. But as the power allocation algorithm attempts to balance these two extremes and approaches α_1^{opt}, the curves become very flat, confirming local (in the vicinity of α_1^{opt}) insensitivity of the TBER to variations in α_1.

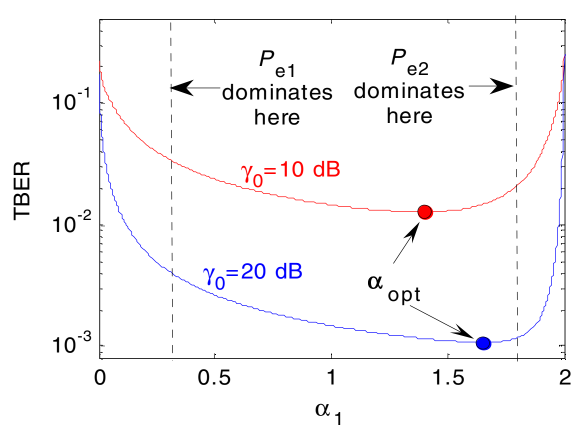

Fig. 1. Average TBER versus α_1 for 2x2 V-BLAST with BPSK modulation.

Thus, small inaccuracies in α^{opt} do not affect the average

[4] An extended discussion of this issue in the general framework of convex optimization can be found in [11].

error rate significantly. This hints at the conclusion that the approximate closed-form $\boldsymbol{\alpha}^{opt}$ will result in almost the same average error rate as the accurate numerical one. Numerical (MC-based) analysis confirms this expectation.

It should also be pointed out that the choice of the optimization criteria (BLER or TBER) does not affect significantly the final result either [8][9]. Thus, BLER or TBER can be used equally well as a performance criterion for optimization.

Small sensitivity of the BLER/TBER to $\boldsymbol{\alpha}$ suggests even further simplification in the optimization algorithm: since $\boldsymbol{\alpha}^{opt}$ changes slowly with the SNR (see (4)), we can pick up only one fixed (pre-set) value of $\boldsymbol{\alpha}$ and still get performance improvement for a wide range of γ_0. Such simplified algorithm does not require any feedback at all, and yet, as Fig. 2 demonstrates, it attains almost the same performance as the dynamically optimized system. In this example, the 3x3 V-BLAST with $\boldsymbol{\alpha} = [2 \quad 0.6 \quad 0.4]^T$ is considered, and its performance is very close to the optimized V-BLAST in the range of $\gamma_0 = 0 \ldots 35$ dB.

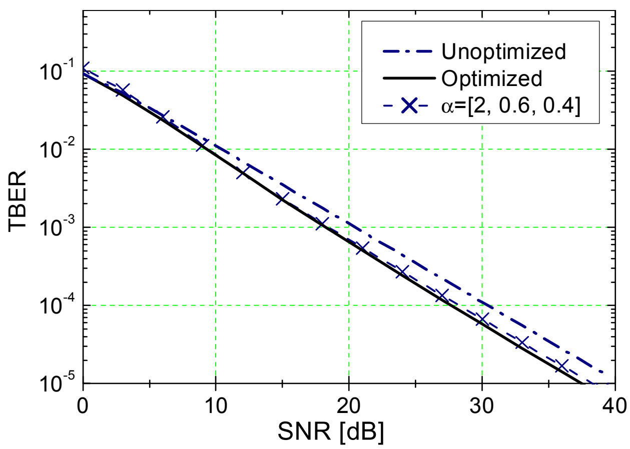

Fig. 2. TBER of 3x3 V-BLAST with BPSK modulation for pre-set (fixed) power allocation.

It should be noted that it is the robustness of the algorithm that is responsible for small difference between instantaneous and average optimization at high SNR observed in [8][9].

The robustness considered above is *local* robustness, i.e. for small variations in the vicinity of the unperturbed values of the system parameter. When variations are not small, the finite differences in (11) cannot be accurately approximated by the derivatives and the approximations in (12), (13), (16), (17) may not be accurate. In such a case, one has to consider a measure of *global* robustness. To this end, let us consider the average error rate of the perturbed system $\bar{P}(\boldsymbol{\alpha}; u+\Delta u)$, where Δu is not necessarily small and u is the total Tx power. Let $\boldsymbol{\alpha}_{\Delta u}^{opt}$ denote the optimum power allocation of the perturbed system, so that the optimum allocation for the unperturbed system is $\boldsymbol{\alpha}_0^{opt} = \boldsymbol{\alpha}_{\Delta u=0}^{opt}$, the optimized average error rate of the unperturbed system is $\bar{P}\left(\boldsymbol{\alpha}_0^{opt}; u\right)$, and the optimized average error rate of the perturbed system is $\bar{P}\left(\boldsymbol{\alpha}_{\Delta u}^{opt}; u+\Delta u\right)$. From the general theory of convex optimization [11], the last two quantities are related by the following global inequality,

$$\Delta P = \bar{P}(\boldsymbol{\alpha}_{\Delta u}^{opt}; u+\Delta u) - \bar{P}(\boldsymbol{\alpha}_0^{opt}; u) \geq -\lambda \Delta u \,, \qquad (18)$$

where λ is evaluated at $\Delta u = 0$, and the equality is achieved for $\Delta u = 0$. It follows that if Δu is positive, i.e. the total Tx power is increased, the optimal value of $\bar{P}$ decreases by *no more* than $\lambda \Delta u$; if Δu is negative, i.e. total Tx power is decreased, the optimal value of $\bar{P}$ is guaranteed to increase by *at least* $\lambda|\Delta u|$. Dividing (18) by $\bar{P}$, one obtains

$$\Delta \bar{P} / \bar{P} \geq -\delta' \Delta u / u \,, \qquad (19)$$

where δ' is given by (13). Therefore, δ', which was introduced as a measure of *local* robustness in (13), also serves as a measure of *global* robustness in (19). Since $\Delta \bar{P}, \Delta u$ may be positive as well as negative (they always have opposite sign), we re-write (19) in the form which includes only positive terms,

$$\frac{\left|\Delta \bar{P} / \bar{P}\right|}{\Delta u / u} \leq \delta',\ \Delta u > 0; \quad \frac{\Delta \bar{P} / \bar{P}}{|\Delta u / u|} \geq \delta',\ \Delta u < 0 \qquad (20)$$

δ' gives upper and lower bounds on the normalized variation in the error rate due to any (not necessarily small) variation Δu in the total Tx power, for positive and negative Δu (i.e. increasing and decreasing the total Tx power), respectively. Thus, when the total Tx power is increased by Δu, the average error rate decreases by not more than $\delta' \bar{P} \Delta u / u$; when the total Tx power is decreased by Δu, the average error rate increases by at least $\delta' \bar{P} |\Delta u| / u$, so that the positive effect never exceeds the negative one. Since the inequality in (19) transforms into the approximate equality for small perturbations (see (14)), these two effects are equal in that case.

Clearly, the Lagrange multiplier λ plays a key role not only in the local, but also in the global robustness. Since it is not known in closed-form for arbitrary SNR, the high-SNR approximation in (15) can be used with reasonable accuracy (for $\gamma_0 > 0$ dB).

V. SNR GAIN OF OPTIMUM POWER ALLOCATION

In this section, we explore some properties of the SNR gain of optimization using mostly analytical techniques, and use numerical results only as the last resort. The analysis and conclusions below are valid for any modulation format, unless otherwise stated.

The *SNR gain* of optimum power allocation is defined as the difference in the SNR required to achieve the same error rate in the optimized and unoptimized systems, i.e. from

$$P\left(\alpha_1^{opt}, \ldots, \alpha_m^{opt}\right) = P(\alpha, \ldots, \alpha) \,, \qquad (21)$$

where P is the performance criterion, i.e.the BLER or the TBER, either instantaneous or average; the left-hand side represents the optimized error rate under the total power constraint $\sum_{i=1}^{m} \alpha_i^{opt} = m$, the right-hand side represents the error rate for the uniform power allocation, $\alpha_i = \alpha$, and the SNR gain is $G = \alpha$. For the average optimization, the average error rate is used in (21). For the instantaneous optimization, one may use both instantaneous and average error rate in (21). In the former case, one obtains the instantaneous gain of the instantaneous optimization, and in the latter case, one obtains the average gain (i.e. in terms of the average error rate) of the instantaneous optimization. To be able to compare the instantaneous and average optimizations below, we use the average gain of the instantaneous optimization G_{inst}. It compares to the gain of the average optimization G_{av} as follows: $G_{inst} \geq G_{av}$, i.e. the instantaneous optimization is at least as good as the average one. We consider below the SNR gain defined in terms of the BLER and the TBER, and also compare the properties of these two different definitions, which share many similarities.

BLER SNR Gain of the Optimum Power Allocation: In this section, we consider the BLER-based optimization strategies and present universal bounds on the BLER SNR gain, either instantaneous or average, which hold for arbitrary modulation and fading. These results are further refined in the case of BPSK modulation and Rayleigh-fading channel.

Theorem 1. The BLER SNR gain of optimum power allocation, either instantaneous or average, for arbitrary modulation and fading, is bounded as follows

$$1 \leq G \leq m \tag{22}$$

Proof. The key to the proof is the fact that the BLER, either instantaneous or average, is a monotonically decreasing function in each argument $\alpha_1,...,\alpha_m$ (see (2) or [8][9] for arbitrary SNR) , and the fact that P_{ei} is a monotonically decreasing function of the SNR. Based on this fact and also on the inequality $P_B\left(\alpha_1^{opt},...,\alpha_m^{opt}\right) \leq P_B\left(1,...,1\right)$, which simply states that the optimized system is at least as good as the unoptimized one, the lower bound in (22) follows. Using the monotonic-decreasing property of the BLER and the fact that $\alpha_i \leq m$, which follows from the total power constraint, one concludes that $P_B\left(\alpha_1^{opt},...,\alpha_m^{opt}\right) \geq P_B\left(m,...,m\right)$. Comparing this inequality with the definition of the gain in (21) in view of the monotonic-decreasing property of the BLER, the upper bound follows. Q.E.D.

Below we explore the small-SNR behavior of the SNR gain in terms of the average BLER, which is related to the lower bound in Theorem 1, for the BLER-based optimization and for a variety of modulation formats.

Theorem 2. Small-SNR behavior of the BLER SNR gain for the average BLER-based optimization is as follows:

$$G_{av} \to G_0 \geq 1 \text{ as } \gamma_0 \to 0, \tag{23}$$

where

$$G_0 = \frac{m\sum_i a_i^2}{\left(\sum_i a_i\right)^2},\ a_i = \left.\frac{\partial \overline{P}_{ei}}{\partial\sqrt{\alpha_i\gamma_0}}\right|_{\alpha_i\gamma_0=0},\ \text{coherent detection}$$

$$G_0 = m\frac{\max_i |b_i|}{\sum_i |b_i|},\ b_i = \left.\frac{\partial \overline{P}_{ei}}{\partial\left(\alpha_i\gamma_0\right)}\right|_{\alpha_i\gamma_0=0},\ \text{noncoherent detection} \tag{24}$$

and a_i, b_i are the coefficients in 1st term of MacLaurin's series expansion of $\overline{P}_{ei}$. The equality in (23) is achieved, i.e. $G_0 = 1$, if and only if all a_i or all b_i are equal, for coherent and non-coherent detection respectively.

Sketch of the proof: expand the BLER in MacLaurin's series and use it to find the optimum power allocation and the corresponding BLER; the gain is found via (21) (detailed proof is given in the extended version of this paper [12]).

This result is valid for a variety of modulation formats for which the error rate admits MacLaurin's series expansion in SNR or $\sqrt{\text{SNR}}$ about SNR=0. In most cases, the strict inequality in (23) holds, i.e. there is an SNR gain of optimization even at very low SNR, since different $\overline{P}_{ei}$ exhibit different behavior so that the expansion coefficients are also different. As an example, for coherent BPSK and non-coherent BFSK, $G_0 = 0.17$ dB and 0.79 dB respectively, for 2x2 system.

Corollary 2.1. The result in (23), (24) also applies to the instantaneous gain of the instantaneous optimization, in which case P_{ei} should be used in (24) instead of $\overline{P}_{ei}$, and the coefficients a_i, b_i and hence the gain depend on the channel realization, as long as the derivatives in (24) exist and are not all equal to zero simultaneously.

Corollary 2.2. The average SNR gain of the instantaneous optimization is also lower bounded by G_0 in (24), $G_{inst} \geq G_0 \geq 1$, because of $G_{inst} \geq G_{av}$.

Thus, we conclude that (23) holds for a variety of scenarios for BLER-based optimization.We now show that the upper bound in (22) is achieved at high SNR.

Theorem 3. High-SNR behavior of the average BLER SNR gain, for both instantaneous and average optimizations using either BLER or TBER as an objective, with BPSK modulation in Rayleigh fading channel, is as follows:

$$G \to m \text{ as } \gamma_0 \to \infty \tag{25}$$

Proof. From the high SNR approximation of the average BLER (8) and corresponding approximation of the unoptimized BLER, $\overline{P}_B \approx \overline{P}_{e1} = \overline{P}_{n-m+1}^{MRC}\left(\gamma_0\right)$ [7], the first step dominates for both unoptimized and optimized systems. Using this in (21), one obtains $G_{av} \to m$ as $\gamma_0 \to \infty$. Using (22) and $G_{inst} \geq G_{av}$, this also holds for the average gain of the instantaneous optimization, $G_{inst} \to m$ as $\gamma_0 \to \infty$. Q.E.D.

Corollary 3.1. Theorem 3 also extends to any modulation/fading for which the first step error rate dominates the average BLER at high SNR. Based on the diversity order argument, this condition should hold for most modulation formats in Rayleigh-fading channels.

For the average BLER-based optimization with BPSK modulation in a Rayleigh fading channel, a high-SNR approximation of the average BLER SNR gain is given by

$$G_{av} \approx \frac{G_\infty}{\sqrt[n-m+1]{1 + c_{m,n} / \sqrt[n-m+3]{4\gamma_0}}} \tag{26}$$

where $c_{m,n} = [\left(n-m+1\right)b_2^{n-m+3} + 3m^{n-m+2}]/[mb_2^{n-m+2}]$, $G_\infty = m$. This approximation follows along the lines of the proof of Theorem 3. Note that (26) reduces to (25) for $\gamma_0 \to \infty$, as it should be. The convergence to the upper bound in (25) is however slow, since the convergence condition is $\sqrt[n-m+3]{\gamma_0} >> 1$.

It follows from (26) that the BLER SNR gain of the average BLER-based optimization with BPSK modulation is an increasing function of the average SNR in the high-SNR range.

Numerical evidence indicates that this also holds for low to intermediate SNR. This conclusion is further reinforced by the following Theorem.

Theorem 4. Under the total power constraint $\sum_{i=1}^{m} \alpha_i = u$, where u is the total Tx power, the BLER SNR gain of BLER-based optimization in (21), either instantaneous or average, is a monotonically increasing function of u:

$$\frac{\partial G}{\partial u} = -\frac{\lambda}{\partial P_B(\alpha \ldots \alpha)/\partial \alpha} \geq 0 \quad (27)$$

Proof. omitted due to the page limit (see [12]).

TBER SNR Gain of the Optimum Power Allocation: In this section, we adapt the results of the previous section to the SNR gain defined in terms of the average TBER. Due to the page limit, we give the results without proofs ([12] gives detailed proofs).

Theorem 1 holds, provided some additional conditions are satisfied by the average TBER and the optimum power allocation, which are not very restrictive [12].

Theorem 2 still holds for the average TBER SNR gain, with the substitution of $\overline{P}_{ei} \to \overline{P}_{et}$ in (24).

Theorem 3 is no longer valid, i.e. the upper bound m is never attained if the gain is defined in terms of the TBER. Instead, the following holds.

Theorem 5. High SNR behavior of the average TBER SNR gain for the average optimization is as follows:

$$G_{av} \to G_{\infty} \text{ as } \gamma_0 \to \infty, \quad (28)$$

where G_{∞} is given by the left-hand side of (10).

Thus, the improvement in average TBER is less than the upper-bound in (22). The reason for this is the increased power of propagating errors for the optimized system, due to higher power going to lower steps, compared to the unoptimized one. For example, the optimum power allocation algorithm gives most of the power to the 1st Tx trying to avoid the errors at the 1st step. But if the error *does* occur at the 1st step, its amplitude is higher than that for the unoptimized system, which makes the error propagation effect more severe.

VI. EXAMPLES

To illustrate the generic results above and to demonstrated their validity via Monte-Carlo simulations, we consider the 2×2 V-BLAST.

It follows from (25) and (28) that the SNR gain is $G_{av} = 2$ (3 dB) and $G_{av} = 8/5$ (2 dB) at high SNR, for the average BLER and TBER respectively. Thus, while the optimum power allocation is insensitive to the criteria (i.e. either BLER or TBER) [8][9], the SNR gain of optimization does depend on it, though not in a dramatic way. Fig. 3 demonstrates the accuracy of the analytical result in (26). Additionally, we note that the average BLER gain of the instantaneous optimization also tends to 3 dB and attains this bound, but much faster than that of the average optimization.

The SNR gain of the optimum power allocation is almost the same, at high SNR, as that of the optimal ordering procedure (see [6] for details). The computational complexity, however, of the former is much less than that of the latter. Hence, the average power optimization can be used instead of the optimal ordering with roughly the same performance.

Similar conclusions also hold in terms of the TBER SNR gain.

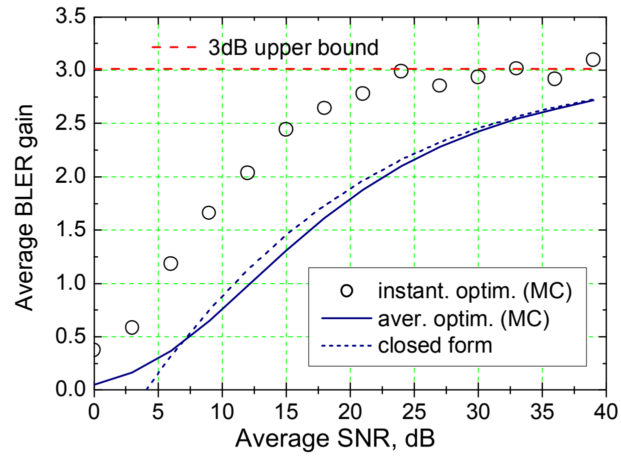

Fig. 3. Average BLER SNR gain vs. SNR for 2x2 V-BLAST with BPSK modulation (MC- Monte-Carlo simulations).

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