STATISTICAL ANALYSIS OF A MEASURED MIMO CHANNEL

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Abstract - **In this paper we discuss an approach to a rigorous statistical analysis of a measured indoor MIMO channel. The channel characteristics such as outage capacity and channel gain distributions, transmit and receive correlations, and the frequency response are statistically analysed under the constraint of limited data available. We also compare the measured channel to the known analytical MIMO channel models in terms of the mean and outage capacity distributions. As the result, we show, in a statistically-rigorous way, that the measured channel is a non-degenerate frequency-selective Rayleigh MIMO channel with significant TX and RX correlations.**

***Index Terms*: wireless communications, MIMO capacity, statistical analysis.**

I. INTRODUCTION

A Multiple-Input-Multiple-Output (MIMO) system is a new paradigm in the modern digital wireless communications. The capacity of a MIMO channel is substantially higher than that of the traditional systems. Telatar in [1] and Foschini *et al* in [2] show that the capacity of an uncorrelated Rayleigh MIMO channel grows proportionally to the channel rank and linearly depends on a number of receive antennas when the number of transmit antennas is asymptotically large.

The outage capacity of a Rayleigh MIMO channel is being studied extensively during last years. The explicit expressions for the distribution of the outage capacity in some particular cases are derived by Telatar [1] and Simon *et al* [3]. In addition Telatar [1] gives an explicit expression for the first moment of the Rayleigh channel outage capacity distribution, and then Smith *et al* [4] give an explicit expression for the corresponding second moment. However, all those expressions are not closed-form and complex for evaluation.

More simple and closed-form expressions are derived by Hachwald *et al* [5], where the authors give the asymptotic distribution of the outage capacity using the results of Telatar's and Foschini's works and the central limit theorem. In particular, Hachwald *et al* [5] show that the distribution is asymptotically Gaussian when the number of transmit and receive antennas goes to infinity. In addition, using Monte-Carlo simulations, Smith *et al* [4] and Hachwald *et al* [5] show that the outage capacity distribution of an uncorrelated Rayleigh MIMO channel converges very fast to Gaussian and becomes "virtually indistinguishable" from the Gaussian distribution when the rank of the channel matrix greater than five [4]. Smith *et al* [4] also notice that the convergence is faster as signal-to-noise-ratio (*SNR*) becomes smaller.

Even though the outage capacity distribution of a theoretical Rayleigh MIMO channel has been well studied, that of a real physical channel remains unknown in many practical scenarios. There are many factors such as channel cross-correlation, "keyholes", mutual coupling between antenna elements and *SNR* variations that can significantly reduce the capacity of a MIMO system. Only relatively few experimental works are available in the literature so far [6].

Unfortunately, all the measured results on MIMO capacity and other channel parameters were not a subject to a rigorous statistical analysis. For instance, the comparison of the measured distributions to the theoretical models was done mostly visually with no strictly defined criteria. As a result, different conclusions were reported in different works. To fill the gap and remove the uncertainty of conclusions, we develop a statistically-sound approach to the analysis of a measured MIMO channel.

II. MIMO CHANNEL CAPACITY

Consider a MIMO system with t transmit and r receive antennas. Assuming a frequency-independent and quasi-static channel, the receiving signal is given by the following matrix model:

$$\mathbf{y} = \mathbf{H} \cdot \mathbf{x} + \mathbf{w} \tag{1}$$

where: $\mathbf{x}$ and $\mathbf{y}$ are vectors representing a transmitting receiving signals respectively; $\mathbf{H}$ is an instant matrix representing the channel; $\mathbf{w}$ is an additive white Gaussian circular-symmetric noise.

The outage capacity C_{out} of a MIMO channel is defined [1] as $P_{out} = \Pr\{C < C_{out}\}$, where C is the instantaneous capacity for a given channel realization:

$$C = \log_2 \det(\mathbf{I} + \rho/t \cdot \mathbf{H} \cdot \mathbf{H}^+) \tag{2}$$

$\mathbf{I}$ is the identity matrix; ρ is a signal to noise ratio determined as the total average power at a receive antenna over the noise power at that receive antenna, and $\mathbf{H}^+$ is a transpose conjugate of matrix $\mathbf{H}$.

When $r, t \to \infty$, the outage capacity distribution of an uncorrelated Rayleigh MIMO channel is Gaussian for $\rho \to 0$ and for $\rho \to \infty$ respectively, as shown in [5].

III. METHODS OF STASTISTICAL ANALYSIS

In general, there are two hypotheses considered against each other in any statistical test: an assumption on some property of the measured data (the null hypothesis H_0) against the possibility that this assumption is not true (the alternative hypothesis H_1) [7]. For this purpose, a test statistics T_n (a function applied on the measured data) is calculated using n observations and compared to some critical value ε. The meaning of ε depends on the meaning of T_n in each particular test. If $|T_n| \le \varepsilon$, H_0 is accepted; otherwise, it is rejected. We should stress, that if H_0 is accepted it does not mean that the measured data possesses the assumed property, it simply means that the test performed did not find any statistically significant difference between the observed and assumed properties. Apparently, there are two probabilities associated with T_n and ε: $\alpha = P\{|T_n| > \varepsilon | H_0\}$ (a significance level) and $\beta = P\{|T_n| \le \varepsilon | H_1\}$.

Unlike the computer-based Monte-Carlo simulations, the common problem of any measurement is a limited number of observations available. Therefore, it is important to choose α and β (test parameters) properly with accordance to the data size, especially when the size is small.

Let us consider T_n of a *monotonically consistent* statistical test. Then the following is true [7]: 1) for any ε $\lim_{n\to\infty} P\{|T_n| > \varepsilon | H_0\} = 0$ and 2) for any given α and ε there is only one n, such that:

$$\alpha = P\{|T_n| > \varepsilon | H_0\} \tag{3}$$

Apparently, as follows from 1) and 2), for any $m > n$, $\alpha > P\{|T_m| > \varepsilon | H_0\}$. Moreover, due to the additive property of the probability measure for any n $P\{|T_n| > \varepsilon | H_0\}$ is a non-increasing function of ε. Therefore, if ε is small, either α or n should be large. This is a general conclusion that is true regardless of any specifics.

On the other hand, let us consider β. The exact value of β depends on the actual distribution of the measured data, which is unknown in most practical cases. However, in general, due to the additive property of the probability measure, β is a non-decreasing function of ε. Thus, if β is low the corresponding α would be high for given n. Therefore, the only way to keep the equality in (3) with smaller α for fixed β would be to increase the size of the acquired data. The relationship given in (3) between n, ε and α is general for any monotonically consistent statistical test, however the exact values of n, ε and α depends on a particular test to be used. Further, we use three statistical tests to analyze the measured MIMO channel: 1) Pearson χ^2 hypothesis test, to check statistical hypothesis on distribution [8]. 2) Generalization of the t-test of correlation coefficients, to check whether the sample correlation is statistically different from zero [8], and 3) Generalization of the F-test (variance ratio test), to check whether two sample variances are statistically identical [8]. It can be shown that all three tests are monotonically consistent. Moreover, since the test statistics distributions of these tests are known [8], (3) for the χ^2 test can be written as:

$$\alpha = 1 - \frac{\gamma(0.5\cdot(K-m-1), 0.5\cdot n\cdot\varepsilon)}{\Gamma(0.5\cdot(K-m-1))} \tag{4}$$

where $\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt$ is the incomplete Gamma function, $\Gamma(a) = \gamma(a,\infty)$ is the Gamma function, K is the number of intervals of the observed data, and m is the number of moments to be estimated. The meaning of ε in (4) is a critical mean relative deviation of the observed histogram from the expected one.

For the generalized t-test, (3) is given by:

$$\alpha = \exp\left\{-0.5\cdot\varepsilon^2\cdot(2n-2)\big/(1-\varepsilon^2)\right\} \tag{5}$$

where ε is a critical sample correlation.

And finally, for the generalized F-test, (3) is:

$$\alpha = 1 - \frac{\Gamma(2n-1)}{\Gamma^2(n-0.5)}\cdot\int_{1-\varepsilon}^{1+\varepsilon}\frac{w^{(n-1.5)}}{(1+w)^{(2n-1)}}dw \tag{6}$$

where ε is a critical deviation of a ratio of two sample variances from one.

To represent the general relationship between n, ε and α, α vs. n in (4) for different ε is plotted in Fig.1. Clearly, decreasing α for given n results in increasing ε, what, in turn, increases β.

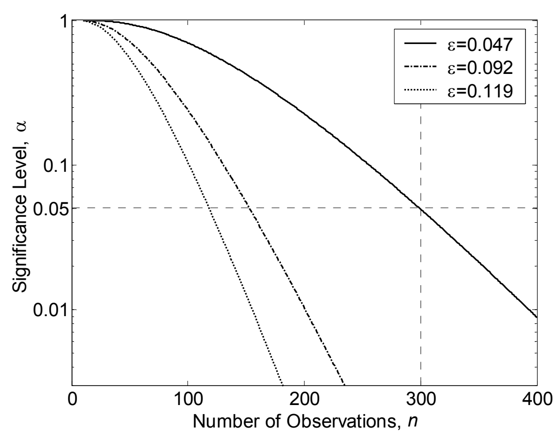

Fig. 1. α vs. n in the χ^2 test ($K = 10$, $m = 2$).

In the following statistical analysis we use (4), (5) and (6) to choose appropriate statistical test parameters.

IV. MIMO Channel Statistical Analysis

In this section we analyze the experimental data based on the measurements of the 8x8 5.2 GHz indoor MIMO channel reported in [6]. However, the procedure is general enough to be applied to any channel. The MIMO channel was measured at $F = 193$ frequency bins equally spread over

120MHz frequency band at the central frequency of 5.2GHz. At each frequency bin, $n = 130$ spatial realizations of the 8x8 MIMO complex channel matrix were taken at 8 different locations (Rx1, Rx2,…, Rx7, and Rx9) and 3 different directions (D1, D2, and D3) in each location. As a result we have (3x8x130x193x8x8) 6-dimensional complex channel transfer matrix (for details see [6]). In further discussion, we will denote by $\mathbf{H}(n, f)$ the n^{th} spatial realization of the 8x8 complex channel matrix measured at f^{th} frequency bin in a considered location and direction.

Below we compare the outage capacity distribution of the measured channel to the Gaussian model in [5] in a statistically-rigorous way. For this purpose, we initially test whether the channel is Rayleigh distributed, uncorrelated, frequency flat and non-degenerated. At the end, we study how fast the outage capacity distribution of the measured channel converges to the Gaussian model with respect to different MIMO system orders $(r \times t)$ and *SNR*.

1. Channel Gain Distribution

In order to analyze the gain distribution of the measured channel, we considered $\mathbf{H}_{ij}(n, f)$, $n = 1..130$, for different, i, j, f, and for different locations and directions. Each $\mathbf{H}_{ij}$ was normalized giving a set of normalized matrices $\mathbf{W}$ whose i, j element was defined as:

$$\mathbf{W}_{ij}(n, f) = \left(\mathbf{H}_{ij}(n, f) - \overline{\mathbf{H}}_{ij}(f)\right) / \sigma_{\mathbf{H}_{ij}}(f) \qquad (7)$$

where $\overline{\mathbf{H}}_{ij}(f)$ is the sample mean, and $\sigma^2_{\mathbf{H}_{ij}}(f)$ is the sample variance of $\mathbf{H}_{ij}$ with respect to n.

The χ^2 test was applied on the set of $\mathbf{W}_{ij}(n, f)$, $n = 1..130$. As a null hypothesis, it was assumed that each $\left|\mathbf{W}_{ij}\right|$ is Rayleigh distributed. All $\left|\mathbf{W}_{ij}\right|$ were arranged in $K = 10$ intervals to provide at least five expected $\left|\mathbf{W}_{ij}\right|$ falling into each interval [8]. The significance level we chose is $\alpha = 0.05$, which based on (4) corresponds to $\varepsilon = 0.119$ for $n = 130$. This is a compromise between low α and not very big ε. For every considered configuration, the assumption that $\left|\mathbf{W}_{ij}\right|$ is Rayleigh distributed was accepted. We also noticed that the measured channel does not have line-of-sight (LOS) component, since the estimated LOS factors $\left|\overline{\mathbf{H}}_{ij}(f)\right| / \sigma_{\mathbf{H}_{ij}}(f)$ are very low (around -30dB) in each considered configuration.

2. *Tx and Rx correlations*

In order to test TX and RX correlations, we estimated sample correlations between different antenna elements in the transmitter and the receiver at different frequencies bins, locations and directions. Then, we applied the generalized t-test of correlation, where H_0 means that the measured MIMO channel is Rayleigh distributed and *uncorrelated* both at the transmitter and the receiver. We chose $\alpha = 0.05$, which based on (5) corresponds to $\varepsilon = 0.054$ given $n \cdot t = n \cdot r = 130 \cdot 8$ observations. Only in few cases the null hypothesis was accepted, in all the others the test shows that there is a statistically significant correlation (in some cases > 0.75) between different antenna elements at the transmitter and the receiver. Therefore, unlike some MIMO channel models, the measured MIMO channel cannot be considered uncorrelated. We also observed much more severe correlation at the receiver than at the transmitter. We explain this by the fact that the angular spread is smaller in the transmitter rather than in the receiver.

3. Channel frequency response

To test the channel frequency response, we considered the following ratio in each location and direction:

$$F_{ij}(f_1, f_2) = \sigma^2_{\mathbf{H}_{ij}}(f_1) \Big/ \sigma^2_{\mathbf{H}_{ij}}(f_2) \qquad (8)$$

for every i, j, f_1 and f_2. H_0 is that the measured indoor MIMO channel is Rayleigh distributed and it has identical power gain over all measured frequency bins, i.e. it is flat within the considered frequency band of 120MHz at 5.2GHz. We chose $\alpha = 0.1$, which following (6) corresponds to $\varepsilon = 0.205$ for $n = 130$.

For all considered configurations, H_0 was rejected, i.e. the channel has different power gain at different frequency bins. Therefore, the channel is statistically *frequency selective* within the considered frequency band.

4. Outage Capacity Distribution

Using (2), the sample outage capacity $C(n)$ of the measured MIMO channel was calculated as:

$$C(n) = \frac{1}{F} \cdot \sum_{j=1}^{F} \log_2 \det\left(\mathbf{I} + \rho/\mathrm{t} \cdot \mathbf{H}(n, j) \cdot \mathbf{H}^{+}(n, j)\right) \qquad (9)$$

At each location and direction the channel matrix was normalized over all measured frequencies and spatial realizations so that:

$$\frac{1}{n \cdot F} \sum_{i=1}^{n} \sum_{j=1}^{F} trace\left\{\mathbf{H}^{+}(i, j) \cdot \mathbf{H}(i, j)\right\} = r \cdot t \qquad (10)$$

We also noticed that $Rank\left\{\sum_{i=1}^{n} \sum_{j=1}^{F} \mathbf{H}(i, j)\right\} = 8$ (the full rank of $\mathbf{H}$) in all locations and directions. Therefore, the measured channel is *non-degenerated* or it has no "keyholes".

In order to analyze the distribution of the outage capacity, we considered the adjusted sample outage capacity:

$$\hat{C}(n) = \left(C(n) - \overline{C}\right) / \sigma_C \qquad (11)$$

where $\overline{C}$ is a sample mean, and σ^2_C is a sample variance of C with respect to n. The χ^2 test was then applied on $\hat{C}$ in different locations, directions, ρ and different channel orders. To test different orders ($r \times t$) the right-upper corners with appropriate size were picked up from the 8x8 MIMO channel matrices and substituted into (9). Following the Gaussian model of the outage capacity distribution given in [5], H_0 is that $\hat{C}$ is Gaussian with zero mean and unit variance. All $\hat{C}$ were arranged in $K = 10$ intervals to provide at least five expected $\hat{C}$ falling into each interval [8]. Since the number of observation we had in each tested location and direction is $n = 130$, we chose $\alpha = 0.1$, which corresponds to $\varepsilon = 0.092$ in (4), see Fig. 1.

Some of the results are presented below. The measured test statistics $|T_n|$ of the χ^2 test in location Rx7D3 for different orders and $\rho = 20dB$ are given in Tab. 1. If $|T_n| \le \varepsilon$, H_0 was accepted and the corresponding cell is shadowed.

$r \setminus t$	1.	2.	3.	4.	5.	6.	7.	8.
1.	*0.12*	*0.03*	*0.03*	*0.05*	*0.09*	*0.06*	*0.11*	*0.11*
2.	*0.12*	*0.10*	*0.07*	*0.01*	*0.04*	*0.05*	*0.09*	*0.09*
3.	*0.10*	*0.06*	*0.09*	*0.08*	*0.09*	*0.03*	*0.06*	*0.11*
4.	*0.03*	*0.09*	*0.14*	*0.12*	*0.07*	*0.07*	*0.05*	*0.11*
5.	*0.09*	*0.08*	*0.10*	*0.06*	*0.06*	*0.07*	*0.06*	*0.06*
6.	*0.07*	*0.11*	*0.11*	*0.05*	*0.06*	*0.04*	*0.06*	*0.07*
7.	*0.08*	*0.15*	*0.17*	*0.04*	*0.07*	*0.08*	*0.11*	*0.11*
8.	*0.05*	*0.07*	*0.20*	*0.04*	*0.07*	*0.11*	*0.11*	*0.16*

Tab. 1 Results of χ^2 test in Rx7D3 for different MIMO orders ($r \times t$), $\rho = 20dB$ and $\varepsilon = 0.092$.

Following the Gaussian model [5], H_0 is supposed to be more frequently accepted for higher orders as well as for lower ρ, as suggested in [4]. However, as the order of the MIMO channel increases the χ^2 test does not give systematically the expected results (see Tab. 1). The same was observed when ρ decreases.

As known, the χ^2 test gives an integral evaluation of a measured distribution. However, from the practical point of view, it is also important to know the outage capacity distribution on the distribution tales where P_{out} is low, i.e. in the region of the high quality of service. In order to evaluate $\hat{C}$ differentially in the region of small probabilities, we built the sample distributions of $\hat{C}$ computed for different locations, directions, orders and ρ. As an example, this distribution for the 2x2 MIMO channel is given for $\rho = 20dB$ in Fig. 2. We also give the $\pm\sigma$ error range given $n = 130$ and plot Gaussian CDF with zero mean and unit variance assumed in H_0. As can be seen, the deviation of the sample distribution of $\hat{C}(n)$ from the Gaussian CDF computed in Rx4D1 and Rx7D3 exceeds the $\pm\sigma$ error range. Especially, this deviation is large on the tales for $P_{out} < 0.1$.

Unfortunately, no conclusions can be made as to whether the outage capacity distribution of the measured channel follows the Gaussian model. In fact, $\alpha = 0.1$ and $\varepsilon = 0.092$ are too large to make adequate decisions. As stated above, the only way to reduce α and ε simultaneously is to increase n. The appropriate number of measured observations, we propose for future experiments, is about $n = 300$. Then as follows from Fig. 1, for $\alpha = 0.05$ the corresponding critical value is $\varepsilon = 0.047$. On one hand the proposed n is quite moderate; on the other hand, since both α and ε as twice as lower then their current values, the accuracy of the χ^2 test will increase.

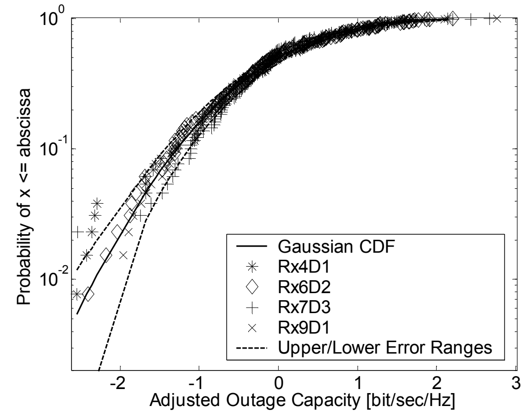

Fig. 2. 2x2 MIMO channel adjusted outage capacity sample CDF in different locations ($\rho = 20dB$), see (11).

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