Information Theory & Electromagnetism: Are They Related?

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Introduction

- Extensive progress in information & communication theory and electromagnetic science in last 50 years
- The two areas are completely disconnected
- Doesn’t fit the internal structure: the only carrier of information is the electromagnetic field!
- Prediction: the future integration is inevitable
- Current research is stimulated by MIMO
- This seems to be the closed point between the two areas
- Purpose: not only to answer, but to ask questions!
- May be somewhat speculative

Information Theory

- Random variables/processes, entropy, mutual information, channel capacity
- Fundamental limits on communications
- Birth date: 1948, Shannon's "Communication in the presence of noise"
- **Basics:**
- Entropy –> average information content of the source per symbol:

$$H(X) = -\sum_{i=1}^{N} p_i \log p_i \ , \ p_i = \Pr\{x_i\}$$

- It is a measure of uncertainty about x (on average): the more is known about x, the less is the entropy

Information Theory: Basics

- Two or more R.V. -> joint & conditional probabilities -> joint & conditional entropies.
- Joint entropy: $$H(X,Y) = -\sum_{i,j} p\left(x_i, y_j\right) \log p\left(x_i, y_j\right)$$
- Conditional entropy: the entropy of x given y, averaged over y: $$H\left(\mathrm{X} \middle| Y\right) = -\sum_{i,j} p\left(x_i, y_j\right) \log p\left(x_i \middle| y_j\right)$$
- It is a measure of uncertainty about x provided y is known. Since y may provide some information about x,

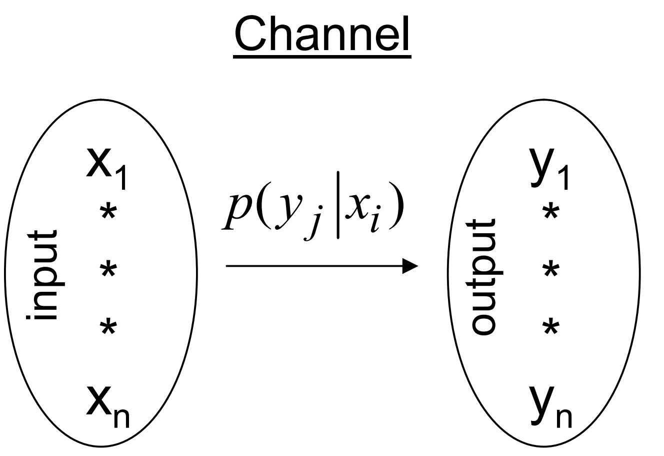
$$H\left(\mathrm{X} \middle| Y\right) \leq H(X)$$

Information Theory: Basics

- The mutual information:

$$I\left(X,Y\right)=H(X)-H\left(X\middle|Y\right)\ \text{[bit/symbol]}$$

- H(X) – measure of uncertainty about X, H(X/Y) – measure of uncertainty about X provided we know Y. The difference gives a decrease in uncertainty due to knowledge of Y.



- Channel capacity:

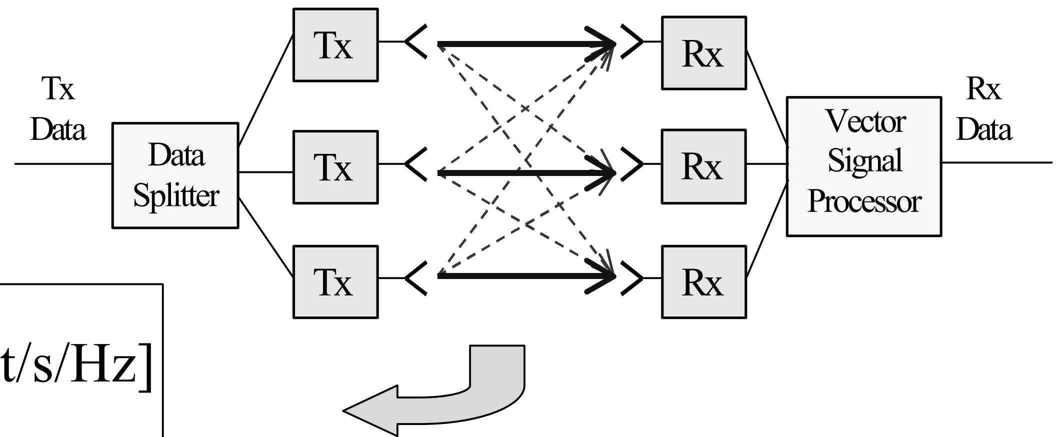
AWGN

$$C=\max_{p(x)} I\left(X,Y\right)\ \text{[bit/symbol]}=\Delta f\log(1+SNR)\ \text{[bit/s]}$$

Shannon, 1948

- This is the most fundamental notion in communication & information theory. It gives the fundamental limit on reliable communication over noisy channel.
- *Error-free transmission is possible at* $R\le C$ *only!*

Recent Development: MIMO



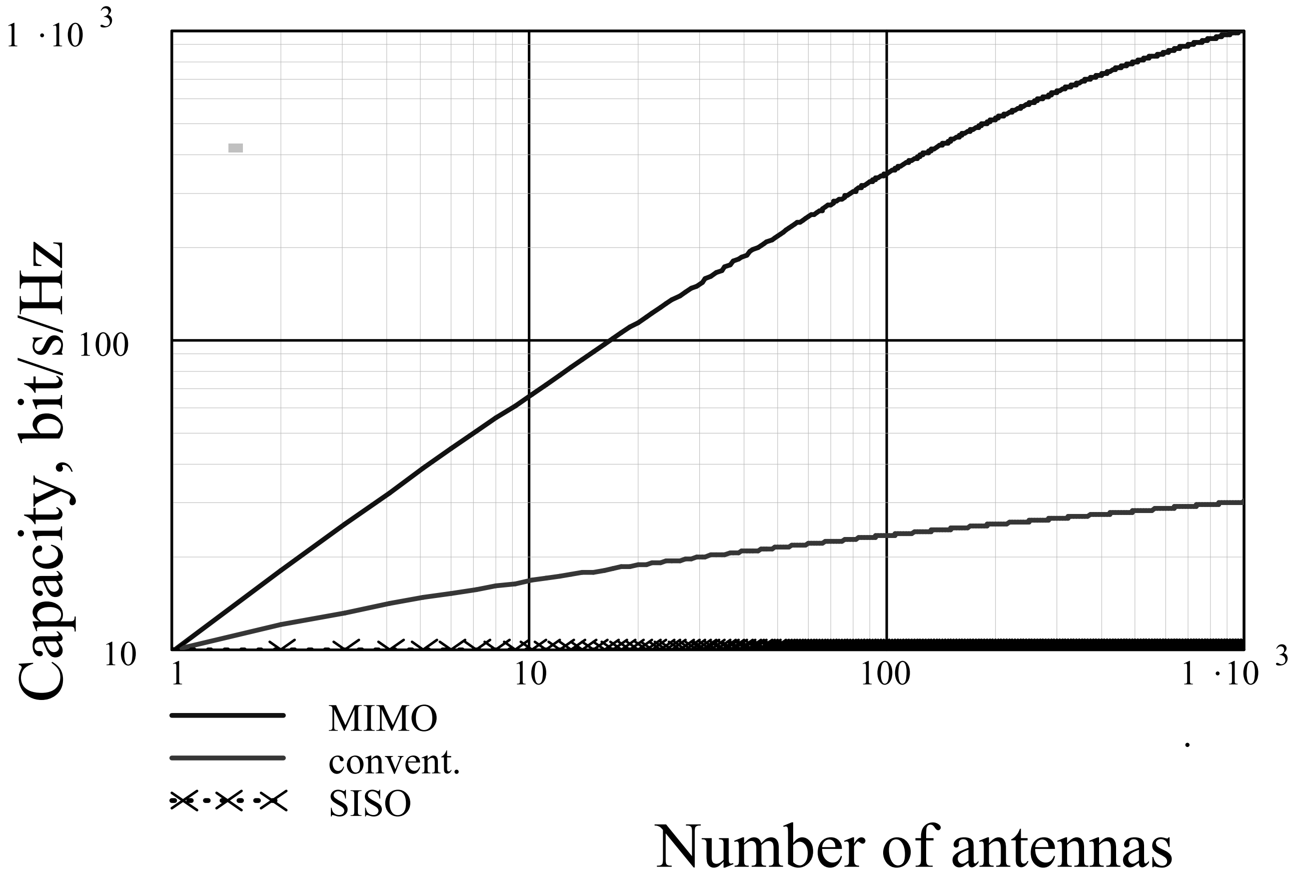
- Matrix AWGN channel -> celebrated Foschini-Telatar formula:

$$C = \log_2 \det\left(\mathbf{I} + \frac{\rho}{n}\mathbf{G}\mathbf{G}^{+}\right) \text{ [bit/s/Hz]}$$

G – normalized channel gain matrix, n – number of Tx/Rx antennas, ρ - SNR

- Enormous channel capacity -> 10 fold increase has been demonstrated
- Multipath is not enemy, but ally !
- MIMO channel capacity crucially depends the propagation channel **G**
- The impact of electromagnetism comes through **G**

MIMO Spectral Efficiency



How to Avoid Electromagnetism

- If you don’t want to learn it – avoid it!
- Are there many options? (to carry the information)
- Nature provides few of them (fundamental interactions):
  1. Electromagnetism
  2. Gravitation
  3. Strong nuclear force
  4. Weak nuclear force
- The latter two – short range only (10^{-15} & 10^{-18} m)
- Long-life particles can be used as well, but difficult to detect,
  – neutrino

How to Avoid Electromagnetism

- Summary:
  - the two fundamental forces are out forever (short range)
  - the gravity is temporarily out: very weak, don’t know whether the waves exist
  - the particles are temporarily out: difficult to produce and control
- Conclusion:
  - no many options
  - electromagnetism remains the only feasible candidate in the foreseeable future -> you have to learn it!

Electromagnetism = Maxwell

- **Maxwell equations**:

$$\nabla \cdot \mathbf{D} = \rho, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

- Fields in source-free region -> **wave equation**:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \quad \nabla^2 \mathbf{H} - \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

- There are 6 field components (“polarization degrees of freedom”). Anyone can be used for communication.
- Only two of them “survive” in free space (“poor” scattering) .

Information Theory + Electromagnetism = Spatial Capacity

- Channel model: $\mathbf{y} = \mathbf{G}\mathbf{x} + ?$
- Channel matrix **G** is controlled by Maxwell:

$$\mathbf{G} = \mathbf{G}(\mathbf{E}) \leftarrow \nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

- Definition of spatial capacity:

$$S = \max_{p(\mathbf{x}),\, \mathbf{E}} \left\{ I\left(\mathbf{x}, \{\mathbf{y}, \mathbf{G}(\mathbf{E})\}\right) \right\}$$

$$\text{const.: } \left\langle \mathbf{x}^{+}\mathbf{x} \right\rangle \le P_T,\ \nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0,\ \mathbf{E} = \mathbf{E}_0 \forall \{\mathbf{r}, t\} \in B$$

- Note 1: max is taken over both p(x) and E
- Note 2: conv. power constraint + E=harmonic function for given boundaries

Another View of Spatial Capacity

- Start with MIMO capacity: $C = \log_2 \det\left(\mathbf{I} + \frac{\rho}{n}\mathbf{G}\mathbf{G}^{+}\right)$
- Varying the channel **G** varies the capacity.
- Find the maximum!

$$S = \max_{\mathbf{G}}\left\{C\left(\mathbf{G}\right)\right\},\ \text{const.: } \mathbf{G} \in\ \mathcal{S}\left(\text{Maxwell}\right)$$

- Constraint: due to Maxwell, explicit form is unknown
- Additional constraints: limited aperture etc. (practical)

- Does this maximum exists? If so, what is it ? What are the main factors that have an impact on it?

Spatial Capacity: Correlation Approach

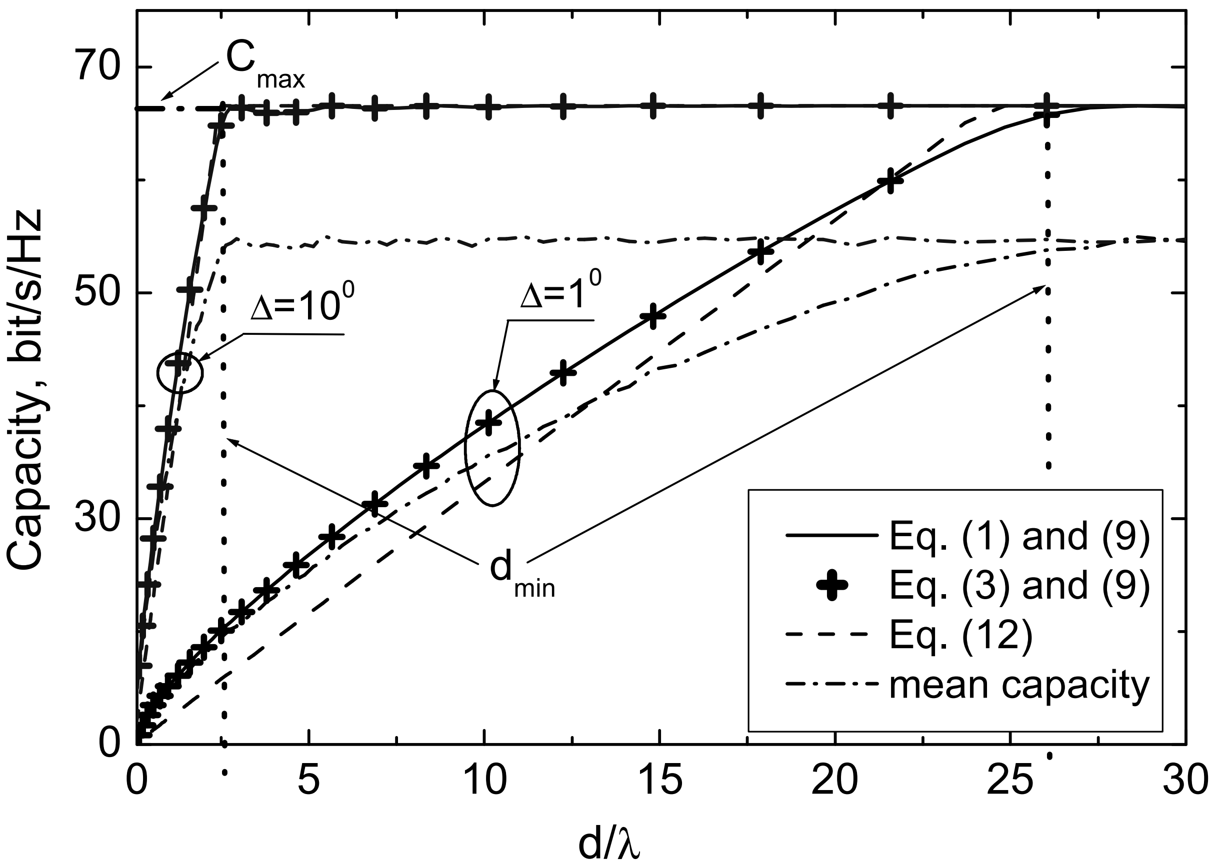
- How to find the fundamental limit, which is due to the laws of electromagnetism only?
- Get rid of all design-specific details!
- The following assumptions are adopted:
  - *limited region of space* is considered (similar to limited power)
  - the *richest scattering*: infinite number of ideal scatterers, uniformly distributed, which do not absorb the EM waves
  - Tx & Rx antenna elements are *ideal field sensors*, with no size and no mutual coupling
- Capacity is linear in the number of antennas -> use as many antennas as possible!
- Is there any limit to this?

$$C = \log_2 \det\left(\mathbf{I} + \frac{\rho}{n}\mathbf{G}\mathbf{G}^{+}\right) \underset{\mathbf{G}\to\mathbf{U}}{\longrightarrow} C = n\log_2\left(1+\frac{\rho}{n}\right) \overset{???}{\underset{n\to\infty}{\longrightarrow}} \frac{\rho}{\ln 2}$$

Spatial Capacity: Correlation Approach

- Increasing the number of antennas increases capacity at first.
- Later, one has to reduce antenna spacing to accommodate more antennas within limited space.
- This increases correlation and decreases capacity!
- Some minimum antenna spacing must be respected in order to avoid loss in capacity.
- 2-D analysis shows that this limit is about half a wavelength (Jakes): $\boxed{d_{\min} \approx \lambda / 2}$
- 3-D case – roughly the same (sinc)

Capacity vs. Antenna Spacing



$$\Delta = 360^0 \rightarrow d_{\min} = l / 2$$

Spatial Capacity: Correlation Approach

- Limited region of space -> limited number of antennas (due to the minimum spacing!)
- Use "sphere packing" argument to estimate it:

$$n_{opt} \approx \frac{6V}{V_S} = \frac{288V}{\pi\lambda^3} \quad \Rightarrow \quad C_{\max} \approx n_{opt} \log_2 \left(1 + \rho / n_{opt}\right)$$

- where *V* is the volume of the space region, ? is SNR, and factor 6 is due to 6 "polarizational" degrees of freedom.
- C_{max} is the maximum capacity the region of space of volume *V* is able to provide.

Spatial Capacity: Spatial Sampling Approach

- Antennas just sample the field at various points in space
- Sampling theorem can be used to determine the required number and positions of the antennas
- 3-D Fourier transform in the spatial domain is the key to applying the sampling theorem
- Key difference between temporal and spatial sampling:
  - temporal sampling -> 1-D, f_{max}, sampling interval/rate, no direction
  - spatial sampling -> 3-D, spectrum 2-D boundary (not f_{max}), sampling cell/density, direction is important
- EM field itself posses certain number of degrees of freedom; number of antennas should not be larger
- Information theoretical properties of EM fields

Electromagnetism in Frequency Domain

- Frequency-domain representation:

$$\phi(\mathbf{r},\omega) = \int \phi(\mathbf{r},t) e^{-j\omega t} dt \quad \Rightarrow \quad \nabla^2 \phi(\mathbf{r},\omega) + \left(\omega / c\right)^2 \phi(\mathbf{r},\omega) = 0$$

- where ϕ is any of the components of **E** or **H** .
- Plane-wave spectrum expansion:

$$\phi(\mathbf{k},\omega) = \int \phi(\mathbf{r},\omega) e^{j\mathbf{k}\cdot\mathbf{r}} d\mathbf{r}$$

$$\phi(\mathbf{r},t) = \frac{1}{(2\pi)^4} \iint \phi(\mathbf{k},\omega) e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})} d\mathbf{k} d\omega$$

$$\Rightarrow \quad \left(|\mathbf{k}|^2 - \left(\frac{\omega}{c}\right)^2 \right) \phi(\mathbf{k},\omega) = 0$$

- Key observation: the channel matrix entries must satisfy the same wave equation!

$$\left(|\mathbf{k}|^2 - \left(\frac{\omega}{c}\right)^2 \right) \mathbf{g}_j(\mathbf{k},\omega) = 0$$

$$g_{ij} = \frac{1}{(2\pi)^3} \int \mathbf{g}_j(\mathbf{k},\omega) e^{-j\mathbf{k}\cdot\mathbf{r}_i} d\mathbf{k} \quad \Leftarrow$$

Plane-Wave Spectrum and Sampling

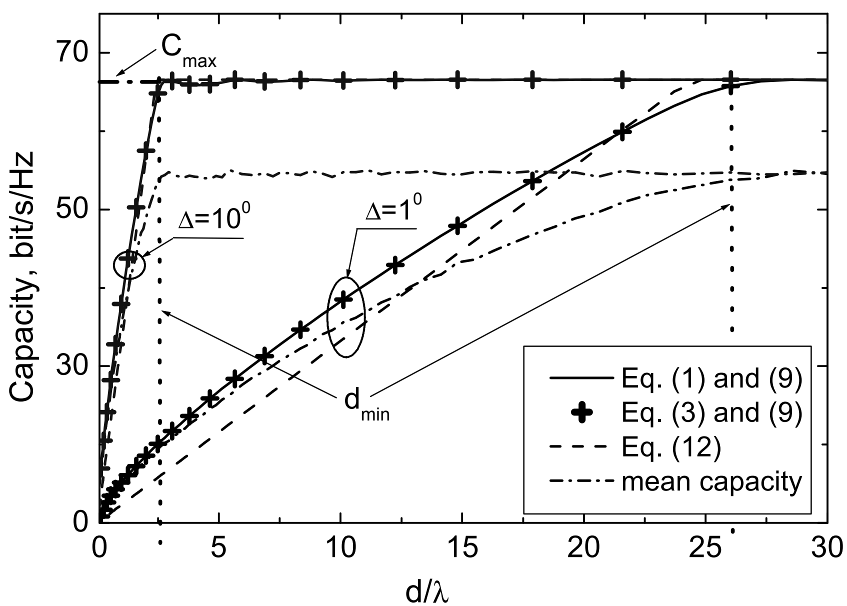
- Plane-wave spectrum is band-limited, $k_x, k_y, k_z \leq |\mathbf{k}| = \omega / c$
- This assumes no evanescent waves with imaginary wavenumber
- Apply the sampling theorem. Sampling interval (in each dimension) is

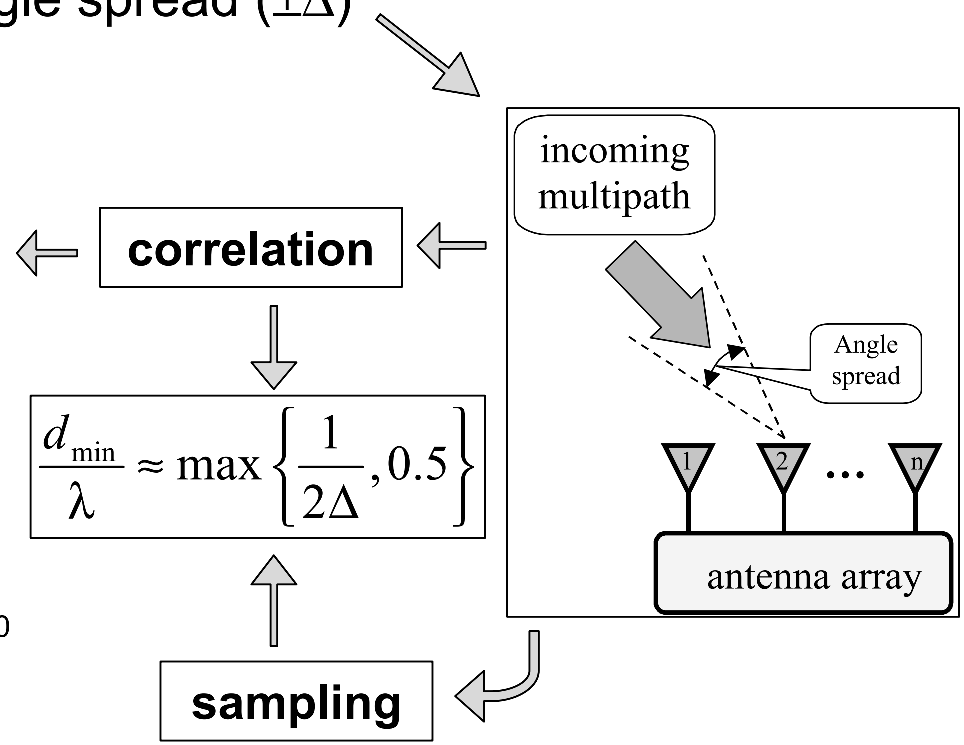
$$\Delta x = \Delta y = \Delta z = \frac{\lambda}{2}$$

- The field can be recovered completely from its samples -> no loss of information
- Conclusion: <u>minimum antenna spacing</u> is lamda/2 -> this is a fundamental limit!
- For given aperture L (1-D), $n_{\max} = 2L / \lambda$
- This limits the capacity according to $C_{\max} = n \log_2 \left(1 + \frac{\rho}{n} \right)$
- The limit is fundamental and is imposed by Maxwell !
- The same limit as for the correlation argument

Correlation and Sampling

- For given angular spread, the correlation and sampling approaches produce roughly the same results!
- Salz-Winters Model: Incoming multipath signals arrive to the linear antenna array within some angle spread (±Δ)



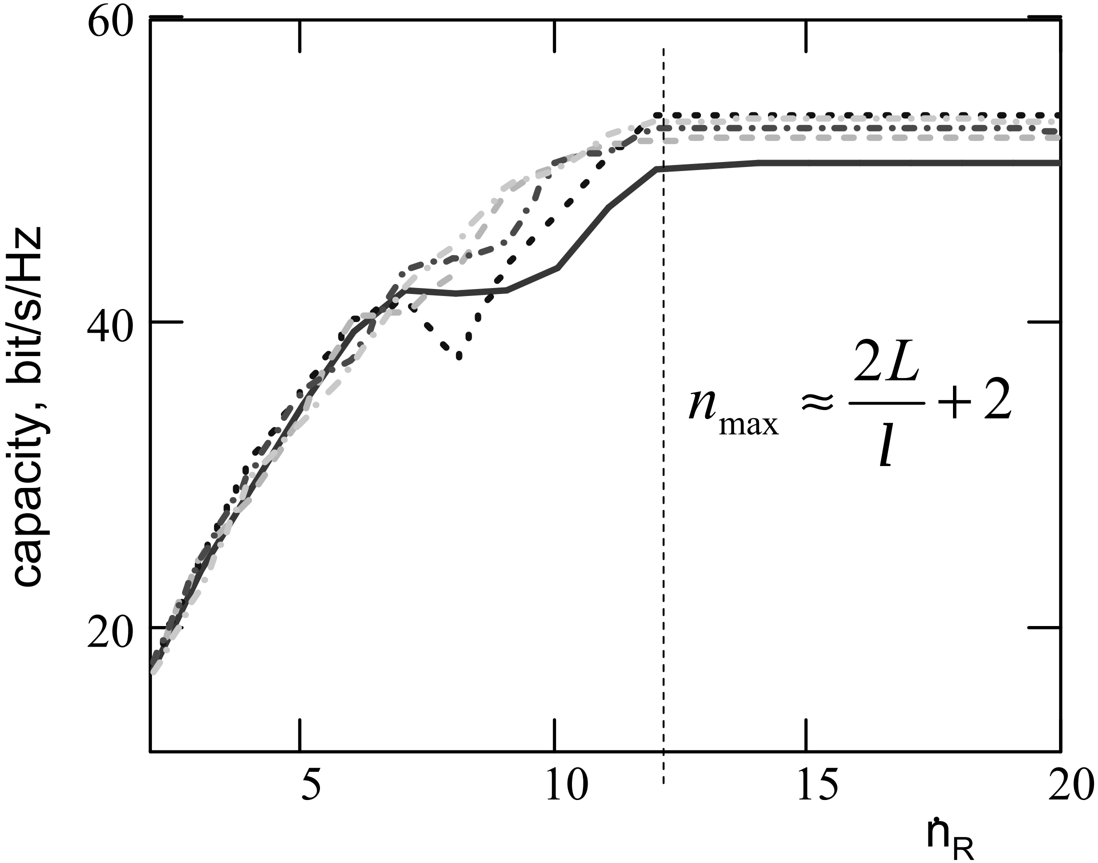


Some Flaws in The Argument

- Implicit assumption: infinite number of antennas is used (the number of samples must be infinite!)
- May be close to that in temporal domain (i.e., millions of samples) but not in the spatial!
- Truncation error must be carefully evaluated
- Some bounds
  - sampling with guard band (over sampling) → $$\left|\Delta(t)\right| \le \frac{4\max\{\left|x(t)\right|\}}{\pi^2 N(1-r)}$$
  - sampling over finite interval → $$\left|\Delta(t)\right| \le \frac{\sqrt{2}E}{\pi}\left|\sin\frac{\pi t}{\Delta t}\right|\sqrt{\frac{T\Delta t}{T^2-t^2}}$$ $$\Delta t = T/N$$
  - sampling finite energy signal
- Truncation error -> 0 as N -> infinity
- In terms of capacity?
- Tx degrees of freedom

Truncation Error and Capacity

- Fix n_T and increase n_R for fixed L=5 lambda
- Rich-multipath quasi-static channel

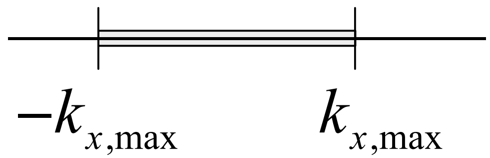


2-D and 3-D Sampling

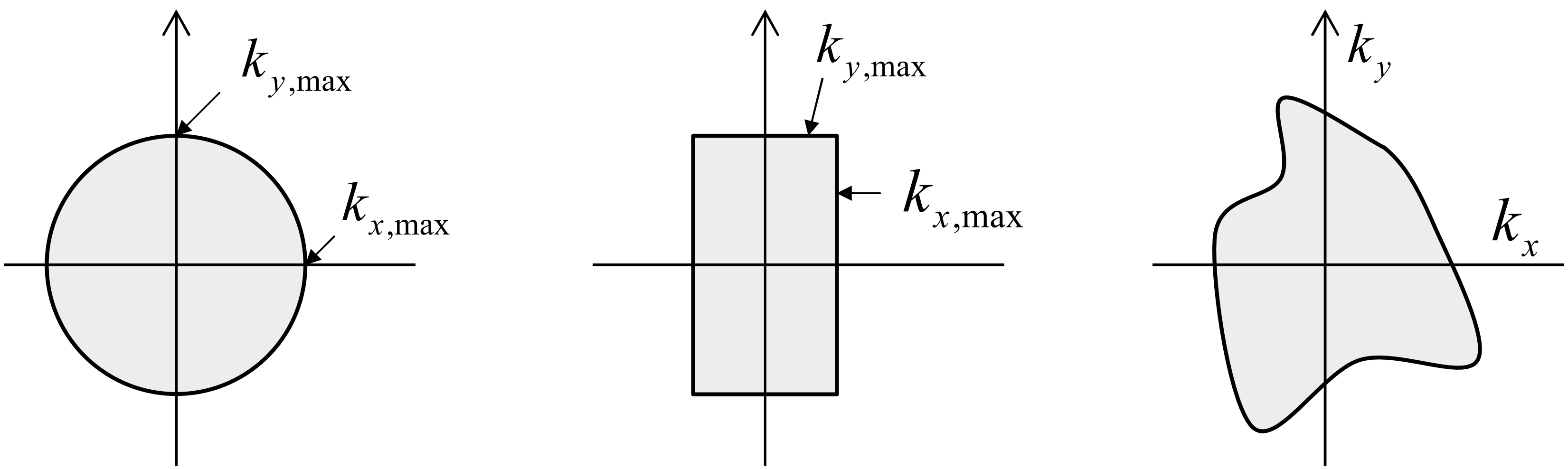
- 1-D antenna -> simple sampling (like temporal)
- 2-D and 3-D cases -> many possibilities, much richer structure
- Minimum spacing is different!
  - 2-D: $\Delta x = \Delta y = \lambda / \sqrt{3}$
  - 3-D: $\Delta x = \Delta y = \Delta z = \lambda / \sqrt{2}$
- Each additional dimension possesses less degrees of freedom than the previous one
- Rectangular lattice is not optimum!

2-D and 3-D Sampling: Spectral Support

1-D sampling:



2-D sampling

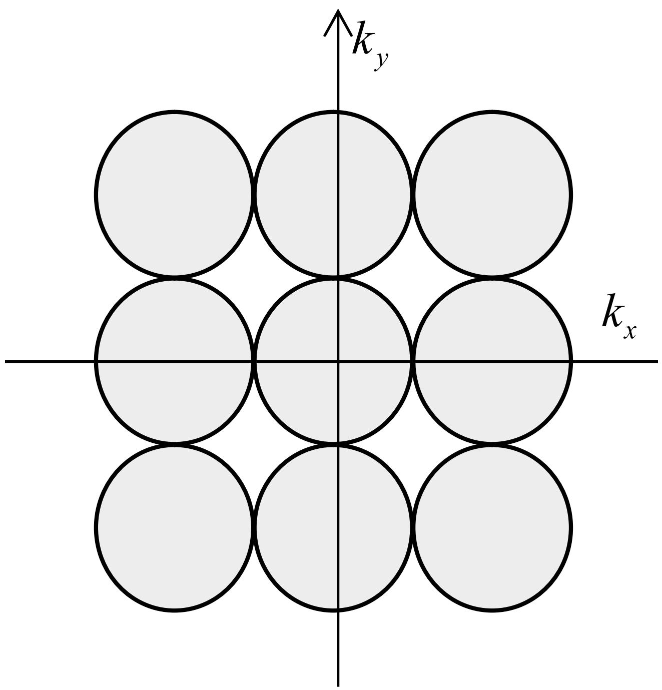


Best sampling strategy: depends on the spectral support

Spectral Support of Sampled Signal

Rectangular sampling

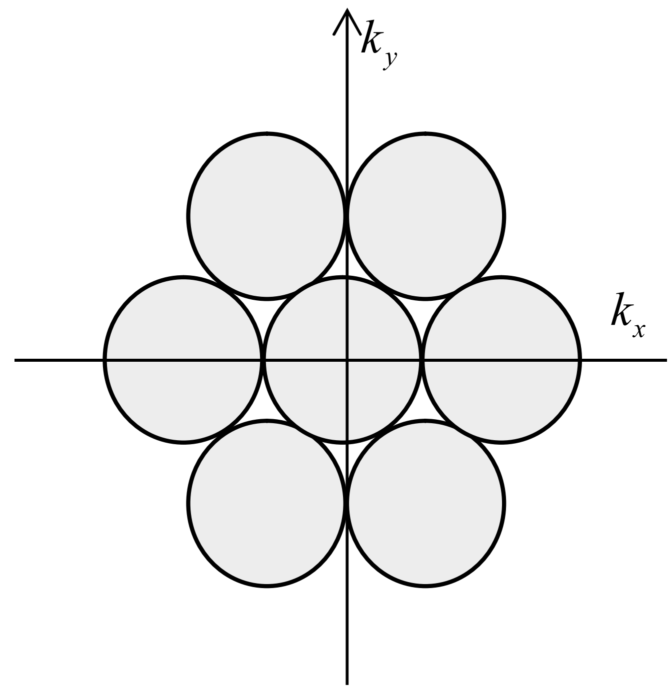
$$\Delta r_{\min} = l / 2$$



$$\mathbf{V} = \begin{bmatrix} l/2 & 0 \\ 0 & l/2 \end{bmatrix}$$

Optimum sampling

$$\Delta r_{min} = l / \sqrt{3}$$



$$\mathbf{V} = \begin{bmatrix} l/2 & -l/2 \\ l/\sqrt{3} & l/\sqrt{3} \end{bmatrix}$$

EM Degrees of Freedom and Quantum Field Theory

- From continuous to discrete variables
- EM field in rectangular volume (a,b,c): $\mathbf{E}(\mathbf{r}) = \sum_{\mathbf{k}} \mathbf{A}_{\mathbf{k}} e^{j\mathbf{kr}}$
- where $k_x = \frac{2p}{a} n_x,\ k_y = \frac{2p}{b} n_y,\ k_z = \frac{2p}{c} n_z,\ n_x, n_y, n_z$ - integer
- For $|\mathbf{k}| \leq k_{\max}$, number of degrees of freedom is finite
- Standard approach in quantum electrodynamics: expansion of the field into oscillators (eigenmodes)
- Information capacity of quantum fields?
- Link between information theory and quantum field theory?

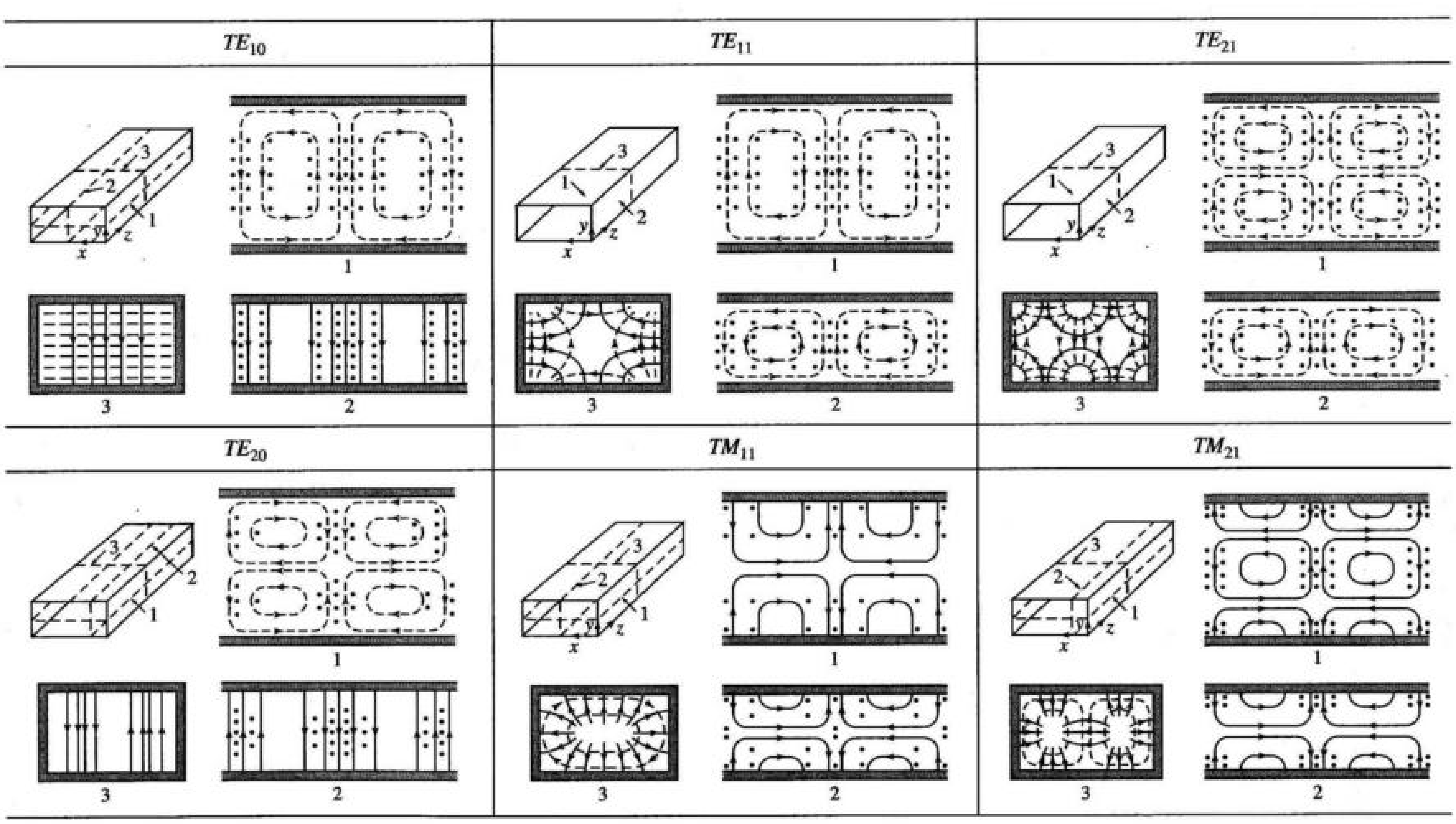
Capacities of Waveguide and Cavity Channels: Why?

- Waveguides / cavities can model corridors, tunnels and other confined space channels,
- This is a canonical problem, it allows to develop appropriate techniques, which can be further extended to more complex problems,
- It allows to shed light on the relation between information theory and electromagnetism in most clear form
- the limits imposed by Maxwell on achievable channel capacity follow immediately.

Basic Idea

- Any field inside of a waveguide is a combination of eigenmodes $$\mathbf{E}(\mathbf{r}) = \sum_{\mathbf{k}} \mathbf{E}_{\mathbf{k}}(\mathbf{r})$$
- All eigenmodes are orthogonal (lossless, homogeneous waveguide), $$\iint_S \mathbf{E}_{\mu} \mathbf{E}_{\nu} dS = c\delta_{\mu\nu} \rightarrow \mathbf{G} = \mathbf{I}$$
- Use the eigenmodes as independent sub-channels !
- MIMO capacity is maximum: $C = N \log_2 \left(1 + \rho / N\right)$
- Need to evaluate N -> electromagnetic analysis
- Lossy/inhomogeneous waveguide -> coupling of eigenmodes. Loss in capacity is low if r<0.5

Waveguide Modes



D. M. Pozar, Microwave Engineering, Wiley

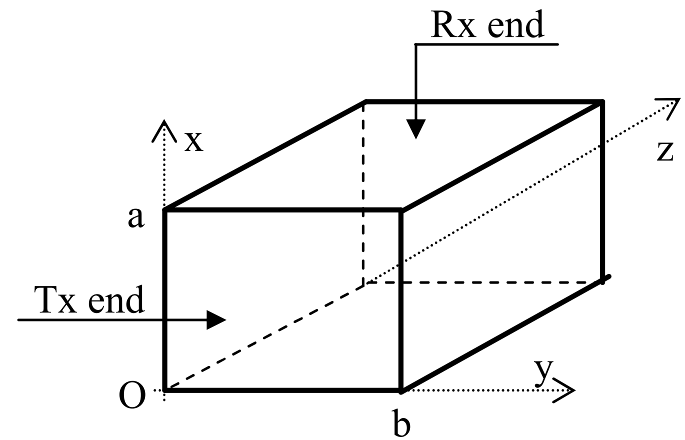
Basic System Architecture

- System architecture is based on the mode orthogonality
- Tx end: all the possible modes are excited (sounds crazy to electromagnetic experts!)
- Rx end: EM field is measured on the cross-sectional area + correlation receiver (with each eigenmode)
- Spatial sampling may be used to reduce the number of field sensors
- Equivalent channel matrix (Tx end-Rx end-correlator output): $\mathbf{G} = \mathbf{I}$

Rectangular Waveguide

- No evanescent waves:

$$\gamma_{mn}^2 = \left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2 \leq \left(\frac{\omega}{c_0}\right)^2$$



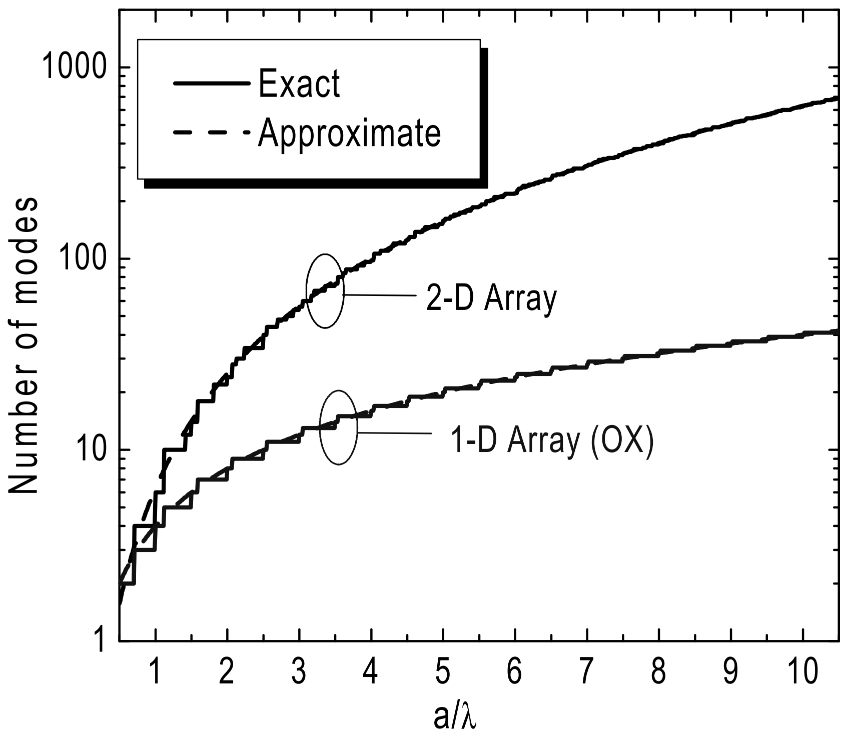
- This limits the number of modes, i.e. possible (m,n) pairs:

$$\left(\frac{m\lambda}{a}\right)^2 + \left(\frac{n\lambda}{b}\right)^2 \leq 4 \rightarrow N \approx \frac{2\pi ab}{\lambda^2}$$

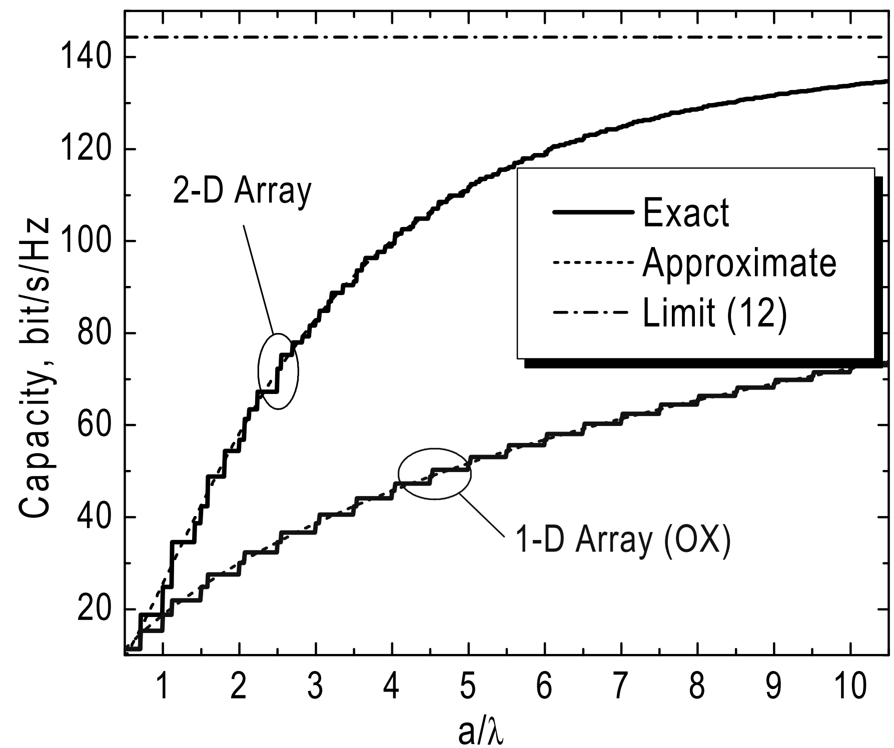
- The number of modes (i.e. channels) is determined by the waveguide cross-section
- This, in turn, limits the capacity -> the limit is fundamental !

Capacity of Rectangular Waveguide

Number of modes in a rectangular waveguide for *a*=*b*.



MIMO capacity of a rectangular waveguide for *a*=*b* and SNR=20 dB.



<u>one of applications: optics</u>

$$\lim_{N\to\infty} C = \frac{\rho}{\ln 2}$$

Capacity of Rectangular Waveguide

- What happens if a linear (1-D) array is used ?
- The number of channels decreases!
- OX array: modes are orthogonal if $m_1 \neq m_2$
- OY array: modes are orthogonal if $n_1 \neq n_2$
- Number of modes:

$$N_{xy} \approx \frac{2\pi ab}{\lambda^2} \rightarrow N_x \approx \frac{4a}{\lambda} \text{ or } N_y \approx \frac{4b}{\lambda}$$

- Maximum "reasonable" number of antennas:

$$C = \frac{\rho}{\ln 2} \sum_{i=0}^{\infty} \frac{(-1)^i}{i+1} \left(\frac{\rho}{N} \right)^i \approx \frac{\rho}{\ln 2} \left(1 - \frac{\rho}{2N} \right) \rightarrow N_{\max} \approx \rho$$

- Max. "reasonable" size:

$$\frac{a_{\max}}{\lambda} \approx \sqrt{\frac{\rho}{2\pi}} \text{ (2-D array)}, \quad \frac{a_{\max}}{\lambda} \approx \frac{\rho}{4} \text{ (1-D OX array)}$$

Circular Waveguide

- Similar approach can be used

$$\gamma_{mn} = \frac{p_{mn}}{a} \text{ (E mode)}, \gamma_{mn} = \frac{p'_{mn}}{a} \text{ (H mode)} \le \frac{\omega}{c_0}$$

- where p and p’ are the roots of Bessel functions and their derivatives, a is the radius
- The number of modes is limited by

$$p_{mn} \le 2\pi a / \lambda \text{ (E modes)}, p'_{mn} \le 2\pi a / \lambda \text{ (H modes)}$$

- and, using wavenumber space filling, approximately

$$N \approx \frac{10a^2}{\lambda^2}$$

Some Remarks

- Compare rectangular and circular waveguides:

$$N_{xy} \approx \frac{2\pi ab}{\lambda^2} \to N_x \approx \frac{4a}{\lambda} \text{ or } N_y \approx \frac{4b}{\lambda} \qquad N_{cir} \approx \frac{10a^2}{\lambda^2}$$

- The number of modes is determined by the cross-section (in terms of wavelength)
- Conjecture: this is true for arbitrary cross-section,

$$N_{arb} \sim S / \lambda^2$$

- In all cases, this corresponds to sampling at $\Delta r \sim \lambda / 2$
- Structure of EM field has a profound impact on capacity!

Rectangular Cavity

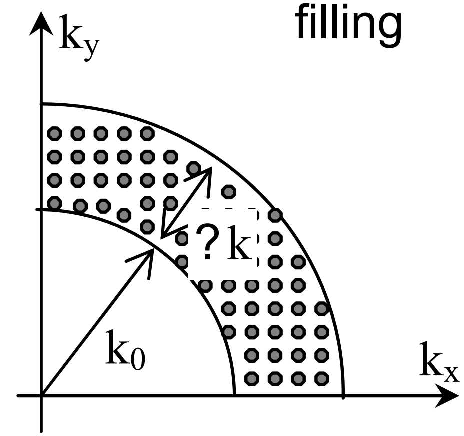
- Eigenmodes exist at certain frequencies only:

$$k^2 = \left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2 + \left(\frac{\pi p}{c}\right)^2 = \left(\frac{\omega}{c_0}\right)^2$$

- Consider a narrow band of frequencies and find the number of eigenmodes for $k \in [k_0, k_0 + \Delta k]$: $N_c \approx \frac{8\pi V_c}{\lambda^3}\frac{\Delta f}{f_0}$

- Orthogonality: $\iiint_{V_c} \mathbf{E}_\mu \mathbf{E}_\nu dV = c\delta_{\mu\nu}$

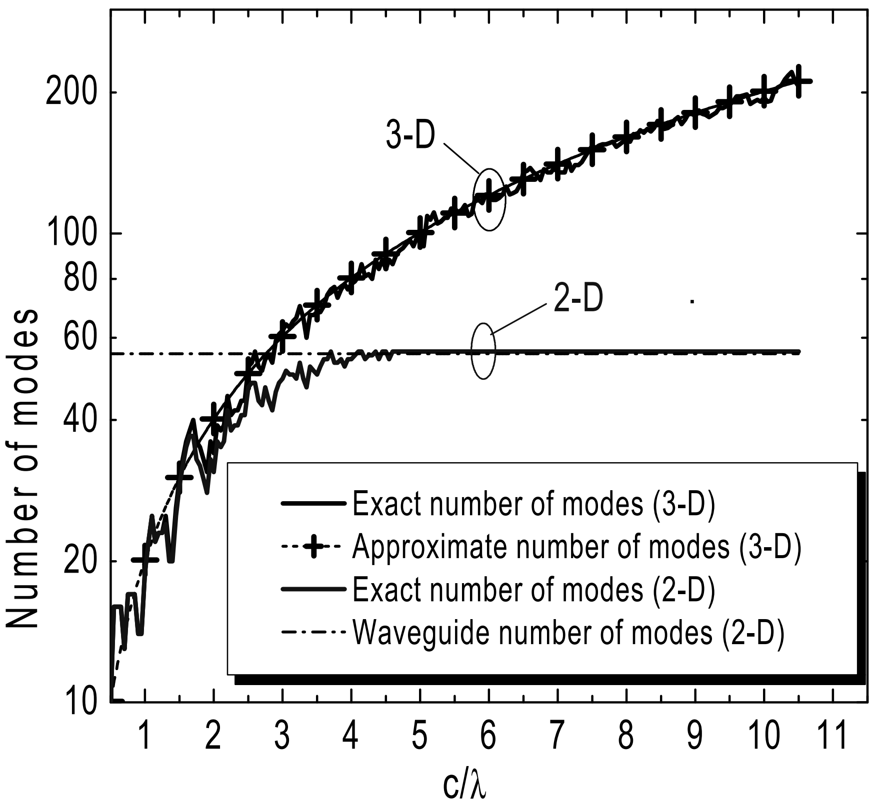
wavenumber space filling



- Reduced 2-D version: modes with different (m,n) are orthogonal
- Critical length: $c > c_t = \frac{f_0 \lambda}{4\Delta f} \rightarrow N_c \approx N_w$
- Long cavity is the same as waveguide

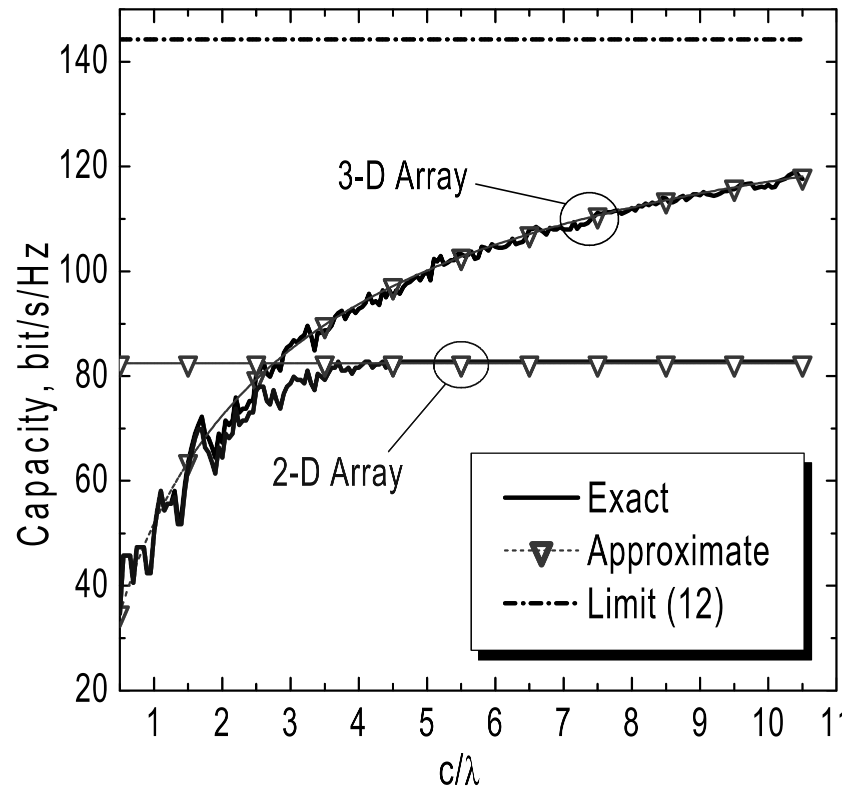
Capacity of Rectangular Cavity

Number of orthogonal modes



Capacity in a rectangular cavity

$a = 4\lambda,\ b = 2\lambda \quad \Delta f / f_0 = 0.01$



Conclusions

- Information theory and electromagnetism: a fundamental link exists
- Future unification is inevitable
- Maxwell limits MIMO capacity
- Half a wavelength is a fundamental limit
- EM field has a finite number of degrees of freedom
- MIMO capacity in confined spaces – eigenmode analysis
- MIMO capacity in open spaces – spatial sampling
- New insights into waveguide performance
- Waveguide has a limited capacity! (by both IT and EM)