**Max. SNR/MMSE V-BLAST**

Consider regular (ZF) V-BLAST first.

Basic channel model:

\[ y = Hx + \xi = \sum_{i=1}^{m} h_i x_i + \xi \]  

(1)

Detection step \( i \): assume the \( T \) symbols \( [x_1...x_{i-1}] \) has been correctly detected,

\[ \hat{x}_j = x_j, \quad j = 1...i-1 \]  

(2)

Subtract the contribution of already detected symbols from \( y \),

\[ y' = y - \sum_{j=1}^{i-1} x_j h_j = \sum_{k=i}^{m} h_k x_k + \xi = H_{(i-1)} x_{(i-1)} + \xi \]  

(3)

where \( H_{i-1} = [h_i, h_{i+1},.., h_m], \quad x_{(i-1)} = [x_i,x_{i+1},..,x_m]^{T} \). This is interference cancellation stage.

Next, project out interference from yet to detected symbols \( x_{(i)} \):

\[ y'' = P_i y' = P_i h_i x_i + P_i \xi \]  

(4)

where \( P_i = I - H_i \left( H_i^* H_i \right)^{-1} H_i^* \) is the projection matrix (orthogonal to span\{\( h_{i+1}...h_m \})

Finally, do MRC using \( y'' \) to maximize output SNR:

\[ \hat{y}_i = a_i^* y'' = h_i^+ P_i h_i x_i + h_i^+ P_i \xi \]  

(5)

where \( a_i = h_i \) are MRC weights. (5) can be compactly expressed as:

\[ \hat{y}_i = w_i^+ y', \quad w_i = P_i h_i \]  

(6)

where we used the fact that \( P_i^+ = P_i \).

The output SNR is:

\[ \gamma_i = \frac{\left| \left[ h_i^+ P_i h_i x_i \right] \right|^2}{\sigma_0^2} = \frac{h_i^+ P_i h_i}{\sigma_0^2} \]  

(7)

assuming \( \left| x_i \right|^2 = 1 \) (unit power constellation). \( \hat{y}_i \) is the decision variable to find \( x_i \).

This algorithm is sometimes called ZF (zero-forcing) V-BLAST as \( P_i \) cancels completely ISI (inter-stream interference) from \{\( x_{i+1}...x_m \}). It does not minimize BER, however.
Max. SNR V-BLAST

Consider step $i$ and find such weights $w_i$ that the output SNR is maximized,

$$\hat{y}_i = w_i^H y' = r_{si} + r_{\xi i}$$

$$r_{si} = w_i^H h_i x_i, \quad r_{\xi i} = \sum_{k=i+1}^{m} w_i^H h_k x_k + w_i^H \xi$$

Output signal and noise/interference powers:

$$P_s = \left( |r_{si}|^2 \right) = |w_i^H h_i|^2$$

$$P_\zeta = \left( |r_{\xi i}|^2 \right) = \left( w_i^H H_i x x^H H_i^+ w_i \right) + \sigma_0^2 |w_i|^2 =$$

$$= w_i^H \left( \sigma_0^2 I + H_i H_i^+ \right) w_i$$

where we have used $\left( x_{(i+1)} x_{(i+1)}^+ \right) = I$, $\left( \xi_\zeta \xi_\zeta^+ \right) = \sigma_0^2 I$ (i.e. i.i.d. signals and noise). Finally, the output SNR is

$$\gamma_i = \frac{P_s}{P_\zeta} = \frac{w_i^H h_i h_i^+ w_i}{w_i^H R_\zeta w_i}$$

where $R_\zeta = \sigma_0^2 I + H_i H_i^+$ is noise and ISI correlation matrix.

Optimization problem:

$$\max_{w_i} \gamma_i$$

The solution is

$$w_i = R_\zeta^{-1} h_i$$

and the max SNR is

$$\gamma_{i,\text{max}} = h_i^H R_\zeta^{-1} h_i$$

Compare to ZF solution (7); for large average SNR, $\sigma_0^2 \rightarrow 0$,

$$R_\zeta^{-1} = \left( \sigma_0^2 I + H_i H_i^+ \right)^{-1} \approx \frac{1}{\sigma_0^2} P_i$$

and max SNR solution is very close to ZF solution (7),

$$\gamma_{i,\text{max}} \approx \gamma_{i,ZF}$$
Max SNR solution (12)-(13) has very important property.

**Theorem:** Max SNR V-BLAST achieves MIMO capacity.

**Proof:**

\[
C = \log \left| I + \frac{\rho}{m} HH^+ \right| = \log \left| I + \frac{\rho}{m} H_i H_i^+ + \frac{\rho}{n} h_i h_i^+ \right|
\]

\[
= \log \left| I + \frac{\rho}{m} H_i H_i^+ \right| + \Delta_i
\]

\[
\Delta_i = \log \left( 1 + \frac{\rho}{m} h_i^+ \left( I + \frac{\rho}{m} H_i H_i^+ \right)^{-1} h_i \right)
\]

Note that with our normalization, \( \langle |x|^2 \rangle = 1 \),

\[
\frac{\rho}{m} = \frac{1}{\sigma_0^2}
\]

and

\[
\Delta_i = \log (1 + \gamma_i), \quad \gamma_i = h_i^+ \left( \sigma_0^2 I + H_i H_i^+ \right)^{-1} h_i
\]

Comparing (19) to (13), we conclude that \( \gamma_i \) is the output SNR of the max SNR processing at step 1 (considering \( T_x = 2 \ldots m \) as sources of interference, ISI). Hence, \( \Delta_i \) is the capacity at step 1.

Applying the same expansion to \( \log \left| I + \frac{\rho}{m} H_i H_i^+ \right| \), one obtains:

\[
C = \sum_{i=1}^{m} \Delta_i; \quad \Delta_i = \log (1 + \gamma_i)
\]

where \( \Delta_i \) is the capacity of \( i \)-th stream, and \( \gamma_i \) is the SNR with max SNR processing. Q.E.D.

Note: from (14), one may conclude that asymptotically, \( \sigma_0^2 \ll 1 \), ZF V-BLAST also achieves MIMO capacity.

**MMSE BLAST**

In a similar way, one may consider MMSE solution to the stream separation problem in (1)

\[
\min_{w_i} \varepsilon_i^2, \quad \varepsilon_i^2 = \langle |x_i - w_i y|^2 \rangle
\]

Using

\[
\frac{d\varepsilon_i^2}{dw_i} = 0
\]

one finds MMSE weights as

\[
w_i = \left( \sigma_0^2 I + HH^+ \right)^{-1} h_i
\]
After some manipulations, it can be shown that max SNR and MMSE weights are related as

\[ \varepsilon_{i,\min}^2 = 1 - \mathbf{h}_i^+ \left( \sigma_0^2 \mathbf{I} + \mathbf{H}_i^+ \right)^{-1} \mathbf{h}_i \]  \hspace{1cm} (24)

and, hence, MMSE solution also provides max SNR.

Important relationship between min MMSE and max SNR:

\[ \frac{1}{\varepsilon_{\min,i}^2} = 1 + \gamma_i \] \hspace{1cm} (26)

**Exercise:** prove (14), (23), (25), (26).