MIMO Channel Capacity

When the $T_x$ signal correlation matrix $P = \mathbb{E}\{xx^+\} \neq I$, the MIMO capacity is

$$C = \log \left| 1 + \frac{1}{\sigma_0^2} PHP^+ \right|$$  \hspace{1cm} (1)

If CSI (channel state information) is available at the $T_x$, $P$ can be chosen to maximize $C$, subject to the total $T_x$ power constrain:

$$\sum_{i=1}^{m} P_{ii} = tr(P) \leq P_T$$  \hspace{1cm} (2)

where $P_{ii}$ is the $i$-th $T_x$ power.

Consider the $m \times n$ MIMO channel,

$$y = Hx + \xi$$  \hspace{1cm} (3)

Using the SVD of $H$,

$$H = U \Sigma V^+$$  \hspace{1cm} (4)

where $U, V$ are $n \times n$ and $m \times m$ unitary matrices, $U^+U = V^+V = I$, and

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$$  \hspace{1cm} (5)

and $\Sigma_1 = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ are non-zero singular values of $H$.

Using (4) and (3),

$$y = U \Sigma V^+ x + \xi, \quad \tilde{y} = \Sigma \tilde{x} + \tilde{\xi}$$  \hspace{1cm} (6)

$$\tilde{y}_i = \sigma_i \tilde{x}_i + \tilde{\xi}_i, \quad i = 1, 2, \ldots, k$$  \hspace{1cm} (7)

where $\tilde{y} = U^+y$, $\tilde{x} = V^+x$, $\tilde{\xi} = U^+\xi$. Note that multiplication by a unitary matrix does not change statistics of a random vector and, hence, does not affect the mutual information and the capacity.

Hence, the channel in (7) has the same capacity as the original channel (3). But the channel in (7) is as of $k$ independent sub-channels, with per-sub-channel SNR:

$$\gamma = \lambda_i \frac{P_i}{\sigma_0^2}$$  \hspace{1cm} (8)

and $\lambda_i = \sigma_i^2$ are eigenvalues of $HH^+$, and $P_i = P_{ii}$. Its capacity is

$$C = \sum_{i=1}^{k} \log_2 \left( 1 + \lambda_i \frac{P_i}{\sigma_0^2} \right)$$  \hspace{1cm} (9)
Optimum $P_i$ can be found using **water-filling technique** as follows:

$$P_i = \left[ \mu - \frac{\sigma_0^2}{\lambda_i} \right]_+, \quad i = 1, 2, .., k_1$$

(10)

$$\sum_{i=1}^{k_1} P_i = P_T$$

(11)

where $[x]_+ = x$ if $x > 0$ and $0$ otherwise; $k_1$ is the number of active eigenmodes (i.e. with non-zero $P_i$), and constant $\mu$ is found from (10). Note that (10) and (11) also give (implicitly) $k_1$.

Water-filling technique can be formulated as iterative algorithm as follows [1]:

1) order eigenvalues, set iteration index $p=0$

2) find $\mu$ as follows

$$\mu = \frac{1}{k - p} \left( P_T + \sigma_0^2 \sum_{i=1}^{k-p} \frac{1}{\lambda_i} \right)$$

(12)

3) set $P_i$ using (10) with $k_1 = k - p$

4) if there is zero $P_i$, set $p = p + 1$, eliminate $\lambda_i$ and go to step 2

5) finish when all $P_i$ $(i=1, 2, .., k-p)$ are non-zero.

This algorithm gives all non-zero $P_i$. All the other $P_i$ are zeros (i.e., those eigenmodes are not used).

**Proof of the water-filling technique**:

using Lagrange multipliers with the following goal function,

$$F = \sum_i \log(1 + \frac{\lambda_i P_i}{\sigma_0^2}) - \alpha \left( \sum_i P_i - P_T \right)$$

(13)

$$\frac{dF}{dP_i} = 0, \quad \frac{dF}{d\alpha} = 0$$

(14)

where $\alpha$ is a Lagrange multiplier. From (14), one obtains (10) and (11).

Finally, optimum $P$ is found using (4)

$$P = VD^+V^T$$

(15)

where $D = \text{diag} [p_1, p_2, .., p_{k_1}, 0, .., 0]$.

(16)

**Effect of $T_x$ CSI on the Capacity** [2]

Compare the MIMO channel capacity in 2 cases:

1) no $T_x$ CSI (uninformed $T_x$-UT)

2) full $T_x$ CSI (informed $T_x$-IT)
In case 1, the capacity is given by (9) with $P_i = \frac{P_T}{m}$

$$C_{UT} = \sum_{i=1}^{k} \log(1 + \frac{P_T}{m\sigma_0^2} \lambda_i)$$  \hspace{1cm} (17)

In case 2, the capacity is given by (9) with $P_i$ given by (10)

$$C_{IT} = \sum_{i=1}^{k_i} \log(1 + \frac{\lambda_i P_i}{\sigma_0^2})$$  \hspace{1cm} (18)

$$P_i = \left[ \mu - \frac{\sigma_0^2}{\lambda_i} \right]_+$$  \hspace{1cm} (18a)

$$\sum_{i=1}^{k_i} P_i = P_T$$  \hspace{1cm} (18b)

Consider the ratio

$$\beta = \frac{C_{IT}}{C_{UT}}$$  \hspace{1cm} (19)

when $P_T/\sigma_0^2 \to \infty$ i.e. high SNR mode.

Assuming that $P_T=\text{const}$ and $\sigma_0^2 \to \infty$, it is clear from (18a) and (18b) that $P_i=P_T/m$ (assuming $k=m$, i.e. full-rank channel), and

$$\frac{C_{IT}}{C_{UT}} \to 1 \quad \text{as} \quad \frac{P_T}{\sigma_0^2} \to \infty$$  \hspace{1cm} (20)

Hence, optimum power allocation does not provide advantage in high SNR mode – parallel transmission (spatial multiplexing) with equal powers is optimum.

Consider the case of low SNR, $\frac{P_T}{\sigma_0^2} \to 0$. Assume that $\sigma_0^2 \to \infty$, then from (18)-(18b) one finds that $P_{i_{\text{max}}} = P_T$ and all the other $P_i = 0^*$, where $i_{\text{max}}$ is the largest eigenmode index.

Hence,

$$C_{IT} = \log \left( 1 + \frac{\lambda_{\text{max}} P_T}{\sigma_0^2} \right) \approx \frac{\lambda_{\text{max}} P_T}{\sigma_0^2} \log e$$  \hspace{1cm} (21)

Similarly,

$$C_{UT} \approx \frac{P_T}{m\sigma_0^2} \sum_{i=1}^{m} \lambda_i \log e$$

Hence,

$$\frac{C_{IT}}{C_{UT}} \approx \frac{m\lambda_{\text{max}}}{\text{tr}(HH^+)}$$  \hspace{1cm} (22)

Important conclusion: in low SNR case, the best strategy is to use the largest eigenmode only → this is beamforming!
In high SNR mode, the best strategy is to use spatial multiplexing (parallel transmission on all eigenmodes).

References: