Optimum Beamforming: Basic Concepts

Deterministic techniques for beamforming -> good when the signal and interference are known completely (example: null steering to cancel the inference).

What to do when interference or/and signal parameters are not known? -> Use statistical or adaptive techniques!

We further design beamforming algorithms, which are optimum in a statistical sense.

Objectives of the array processing (beamforming):

Estimate the plane-wave signal in the presence of noise and interference; statistics of noise and interference is known or must be measured; some criteria of optimality must be employed; we discuss several such criteria below.

Estimate the direction of arrival (AOA) of the signal(s) in the presence of noise and interference. This is a parameter estimation problem.

Estimate the spatial spectrum of the signal (space-time process).

A general goal is to pass the required signal without distortion and to suppress noise and interference as much as possible.

Beamforming algorithm may be optimal under one criterion and far away from optimum under another one.

Beamforming = space-domain signal processing.

Basic models of signal and interference/noise

Possible Scenarios

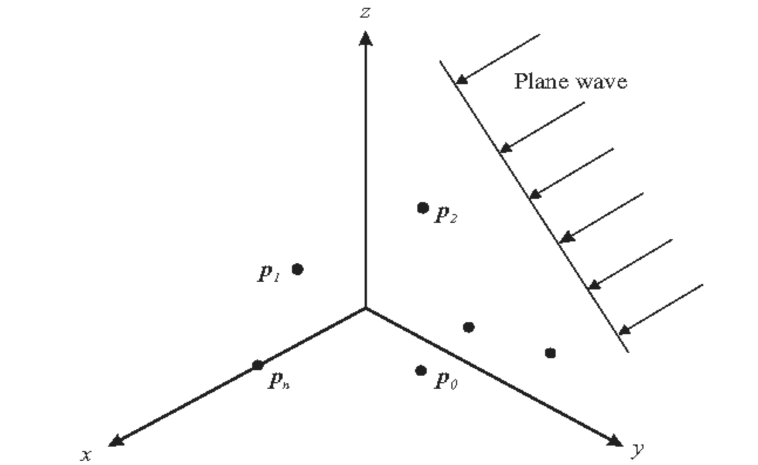


Figure 6.1 Sensor array with single plane-wave signal.

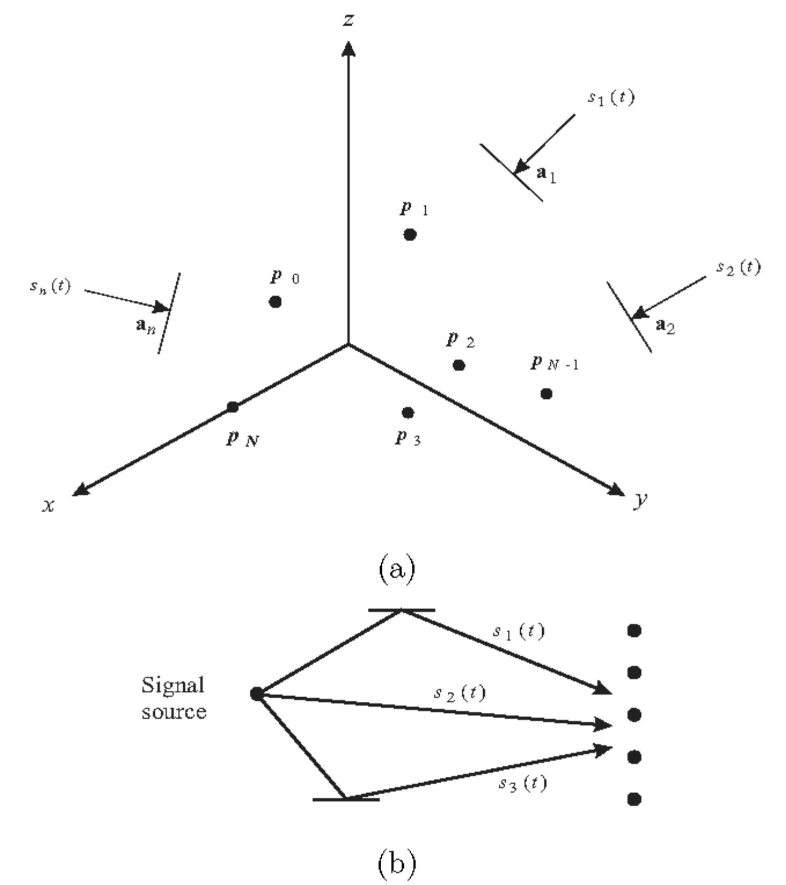


Figure 6.2 (a) Multiple plane-wave signals; (b) multipath environment.

H.L. Van Trees, Optimum Array Processing, Wiley, 2002.

Possible Scenarios

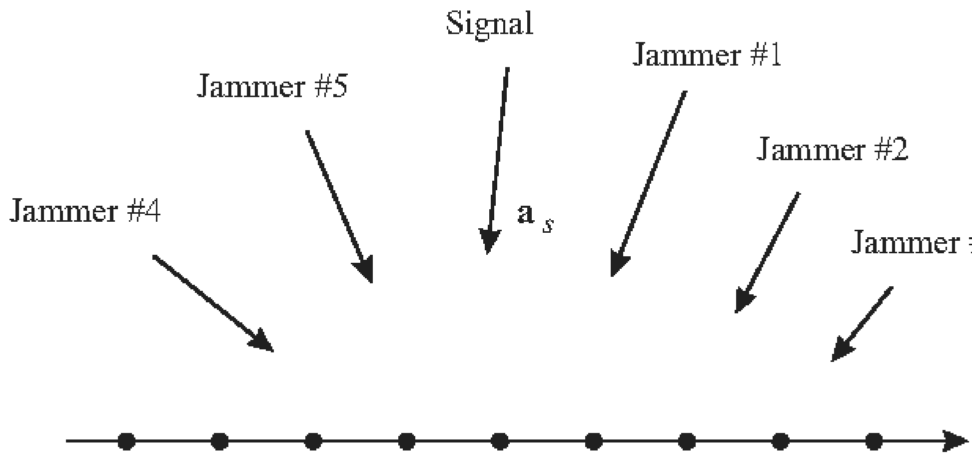


Figure 6.4 Discrete interference model.

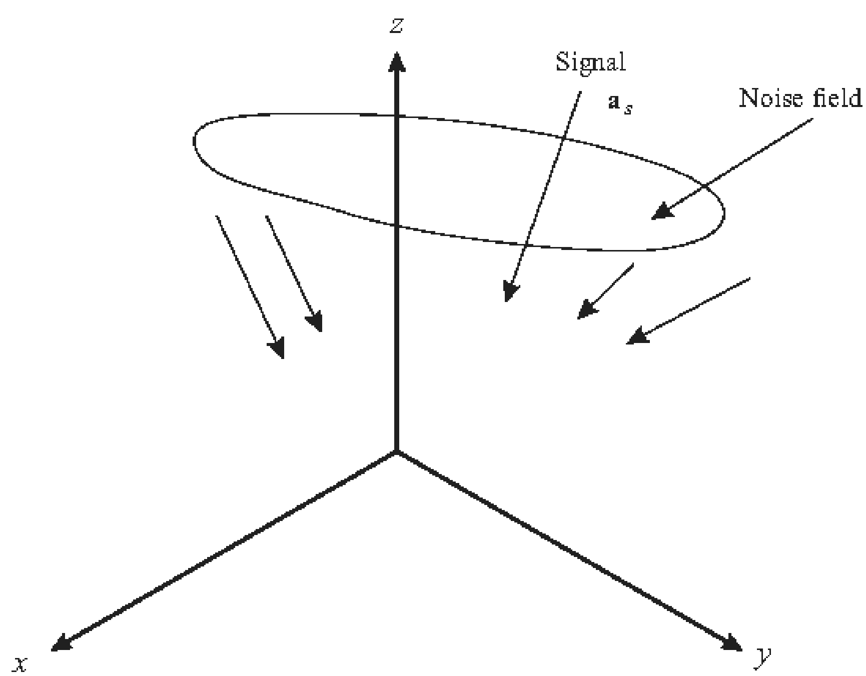


Figure 6.5 Continuous noise field.

H..L. Van Trees, Optimum Array Processing, Wiley, 2002

Optimum Beamformers

Consider the incoming signal and noise:

$$\mathbf{x} = \mathbf{x}_s + \xi \tag{7.1}$$

i.e. we use the narrowband assumption. The array output is

$$y = \mathbf{w}^+ \mathbf{x} = \sum_{i=1}^{N} w_i^* x_i \tag{7.2}$$

Assume the required signal $\mathbf{x}_s$ is a plane wave,

$$\mathbf{x}_s = x_s \cdot \mathbf{v}(\mathbf{k}_s) = x_s \mathbf{v}_s \tag{7.3}$$

where $\mathbf{k}_s$ is its wave vector, x_s is the waveform (complex envelope; may be considered as the plane-wave signal at the reference point $\mathbf{p} = 0$).

Note: for a broadband signal one has to consider

$$\mathbf{x}_s(\omega) = x_s(\omega) \cdot \mathbf{v}(\omega, \mathbf{k}_s) \tag{7.4}$$

The noise + interference correlation matrix is (in general),

$$\mathbf{S}_\xi = \mathbf{S}_I + \sigma_0^2 \mathbf{I} \tag{7.5}$$

where $\sigma_0^2 \mathbf{I}$ is the AWGN (spatially white) correlation matrix, and $\mathbf{S}_I$ includes any interference (in a form, for example, of plane waves).

Minimum Variance Distortionless Response (MVDR) Beamformer

In absence of noise,

$$y = y_s \tag{7.6}$$

In the presence of noise,

$$y = y_s + y_\xi \tag{7.7}$$

and we wish to minimize the output noise power.

The distortionless criterion is

$$\mathbf{w}^+ \mathbf{v}_s = 1 \tag{7.8}$$

The noise + interference variance (power) at the output is

$$\sigma^2_{\xi,out} = \left\langle \left| y_\xi \right|^2 \right\rangle = \mathbf{w}^+ \mathbf{S}_\xi \mathbf{w} \tag{7.9}$$

Q.: prove it!

We minimize (7.9) subject to the distortionless criterion,

$$\min_{\mathbf{w}} \mathbf{w}^+ \mathbf{S}_\xi \mathbf{w}, \text{ s.t. } \mathbf{w}^+ \mathbf{v}_s = 1 \quad .$$

by using Lagrange multipliers. The Lagrangian is

$$F = \mathbf{w}^+ \mathbf{S}_\xi \mathbf{w} + \lambda(\mathbf{w}^+ \mathbf{v}_s - 1) + \lambda^*(\mathbf{v}_s^+ \mathbf{w} - 1) \tag{7.10}$$

MVDR Beamformer: cont.

Take the derivative of F w.r.t. $\mathbf{w}^+$:

$$\nabla_{\mathbf{w}^+} F = \frac{\partial F}{\partial \mathbf{w}^+} = \mathbf{S}_\xi \mathbf{w} + \lambda \mathbf{v}_s = 0 \qquad (7.11)$$

The optimum weight vector is

$$\mathbf{w}_0^+ = -\lambda^* \mathbf{v}_s^+ \mathbf{S}_\xi^{-1} \qquad (7.12)$$

Using the constraint,

$$\lambda^* = -\left[\mathbf{v}_s^+ \mathbf{S}_\xi^{-1} \mathbf{v}_s \right]^{-1} \qquad (7.13)$$

$$\boxed{\mathbf{w}_0^+ = \frac{\mathbf{v}_s^+ \mathbf{S}_\xi^{-1}}{\mathbf{v}_s^+ \mathbf{S}_\xi^{-1} \mathbf{v}_s}} \qquad (7.14)$$

This is the optimum MVDR beamformer, which was derived first by Capon and is referred to as a Capon beamformer.

Note that this is a <u>very general result</u> - $\mathbf{S}_\xi$ can be any.

Q.: consider the special case of $\mathbf{S}_\xi = \sigma_0^2 \mathbf{I}$. Find $\mathbf{w}_0$. Does it look similar to something?

Array SNR Gain

The noise + interference power at the output with the optimal weights is

$$\sigma^2_{\xi,out} = \gamma^2 \mathbf{v}_s{}^+ \mathbf{S}_\xi^{-1} \mathbf{S}_\xi \mathbf{S}_\xi^{-1} \mathbf{v}_s = \gamma \tag{7.15}$$

$$\gamma = \left[\mathbf{v}_s{}^+ \mathbf{S}_\xi^{-1} \mathbf{v}_s \right]^{-1} \tag{7.16}$$

The output signal power is $\sigma_s^2 = \left\langle |x_s|^2 \right\rangle$ since $\mathbf{w}^+ \mathbf{v}_s = 1$.

Assume that the noise + interference power at each array element is the same, σ_ξ^2, then

$$SNIR_{in} = \frac{\sigma_s^2}{\sigma_\xi^2} \tag{7.17}$$

For simplicity, we further use *SNR* to denote *SNIR* whenever interference is present, unless indicated otherwise.

<u>The array SNR gain</u> is

$$G_0 = \frac{SNR_{out}}{SNR_{in}} = \frac{\sigma_\xi^2}{\gamma} = \sigma_\xi^2 \mathbf{v}_s^+ \mathbf{S}_\xi^{-1} \mathbf{v}_s \tag{7.18}$$

Introduce a normalized noise correlation matrix:

$$\mathbf{S}_{n,\xi} = \sigma_\xi^{-2} \mathbf{S}_\xi \tag{7.19}$$

The SNR gain is

$$G_0 = \mathbf{v}_s^+ \mathbf{S}_{n\xi}^{-1} \mathbf{v}_s \tag{7.20}$$

Comparison to Conventional Beamformer

For a conventional array (beamformer),

$$\mathbf{w}_c^+ = \frac{1}{N}\mathbf{v}_s^+ \tag{7.21}$$

The output noise + interference power is

$$\sigma_{c\xi,out}^2 = \frac{1}{N^2}\mathbf{v}_s^+\mathbf{S}_\xi\mathbf{v}_s \tag{7.22}$$

The gain is

$$G_c = N^2\left[\mathbf{v}_s^+\mathbf{S}_{n\xi}\mathbf{v}_s\right]^{-1} \tag{7.23}$$

This assumes equal-magnitude weights.

For spatially white noise:

$$\mathbf{S}_{n\xi} = \mathbf{I} \;\rightarrow\; G_0 = G_c = N$$

In other cases $G_0 \geq G_c$

Q: prove it! Hint: Use the Schwartz inequality.

Minimum Mean-Square Error (MMSE) Beamformer

We employ the same model as before:

$$\mathbf{x} = x_s \mathbf{v}_s + \xi,$$

where x_s is the required (plane wave) signal, which is assumed to be a random (unknown), $\mathbf{v}_s$ is array manifold at $\mathbf{k}_s$, and ξ is the noise + interference.

The noise correlation matrix is

$$\mathbf{S}_\xi = \mathbf{S}_I + \sigma_0^2 \mathbf{I} \qquad (7.24)$$

where σ_0^2 is the thermal noise variance (power density), $\mathbf{S}_I$ is the interference correlation matrix. $\mathbf{S}_\xi$ and $\mathbf{v}_s$ are assumed to be known (in practice, they must be estimated from measured data).

The total (signal+noise+interference) correlation matrix is

$$\mathbf{S}_x = \sigma_s^2 \mathbf{v}_s \mathbf{v}_s^+ + \mathbf{S}_\xi \qquad (7.25)$$

We assumed that the noise statistics are the same at all the array elements.

MMSE Beamformer

The mean square error is

$$\varepsilon = \left\langle \left| x_s - \mathbf{w}^+ \mathbf{x} \right|^2 \right\rangle \tag{7.26}$$

Our goal is to minimize it by proper choice of $\mathbf{w}$.

Taking the derivative $\nabla_{\mathbf{w}} \varepsilon = 0$ and assuming the signal and noise are uncorrelated, we obtain

$$\boxed{\mathbf{w}_0^+ = \frac{\sigma_s^2 \gamma}{\sigma_s^2 + \gamma} \mathbf{v}_s^+ \mathbf{S}_\xi^{-1}} \tag{7.27}$$

where $\gamma = \left[\mathbf{v}_s^+ \mathbf{S}_\xi^{-1} \mathbf{v}_s \right]^{-1}$, and σ_s^2 is the power of x_s.

Since we employ the narrowband assumption, σ_s^2 and γ are frequency-independent (flat). Generalization to non-frequency-flat case is straightforward.

The minimum error (MMSE) is

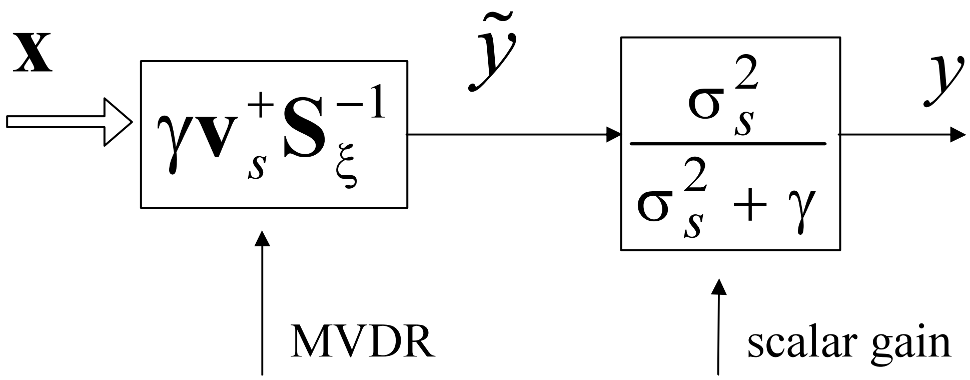
$$\varepsilon_0 = \frac{\sigma_s^2 \gamma}{\sigma_s^2 + \gamma} = \frac{\sigma_s^2 \dfrac{\sigma_\xi^2}{G_0}}{\sigma_s^2 + \dfrac{\sigma_\xi^2}{G_0}} = \frac{\sigma_s^2 \sigma_\xi^2}{\sigma_s^2 G_0 + \sigma_\xi^2} \tag{7.28}$$

where $G_0 = \sigma_\xi^2 \gamma^{-1}$ is the array gain.

For large SNR, $\sigma_s^2 \gg \sigma_\xi^2 / G_0$ and hence

$$\varepsilon_0 \approx \frac{\sigma_\xi^2}{G_0} \ll \sigma_s^2 \tag{7.29}$$

<u>Block diagram</u>



Note: MMSE beamformer is a scaled version of MVDR one.

Derivation of MMSE (homework)

The estimate of x_s is

$$\hat{x}_s = x_s + \Delta x_s \qquad (7.30)$$

where Δx_s is random error due to noise, and $\langle \hat{x}_s \rangle = x_s$, when there is no noise (i.e. unbiased estimator).

The estimation mean square error (MSE):

1) $$\varepsilon = \left\langle \left| x_s - \mathbf{w}^+ \mathbf{x} \right|^2 \right\rangle$$

$$= \left\langle |x_s|^2 \right\rangle + \left\langle \mathbf{w}^+ \mathbf{x}\mathbf{x}^+ \mathbf{w} \right\rangle - \left\langle x_s \mathbf{x}^+ \mathbf{w} \right\rangle - \left\langle x_s^* \mathbf{w}^+ \mathbf{x} \right\rangle$$

$$= \sigma_s^2 + \mathbf{w}^+ \mathbf{S}_x \mathbf{w} - S_{x_s \mathbf{x}^+} \mathbf{w} - \mathbf{w}^+ S_{x_s^* \mathbf{x}}$$

where

$$\mathbf{S}_x = \left\langle \mathbf{x}\mathbf{x}^+ \right\rangle = \sigma_s^2 \mathbf{v}_s \mathbf{v}_s^+ + \mathbf{S}_\xi, \; S_{x_s \mathbf{x}^+} = \left\langle x_s \mathbf{x}^+ \right\rangle, \; S_{x_s^* \mathbf{x}} = \left\langle x_s^* \mathbf{x} \right\rangle.$$

2) $$\frac{\partial \varepsilon}{\partial \mathbf{w}^+} = \mathbf{S}_x \mathbf{w} - S_{x_s^* \mathbf{x}} = 0 \;\rightarrow\; \mathbf{w}_0^+ = S_{x_s^* \mathbf{x}}^+ \mathbf{S}_x^{-1}$$

3) $$S_{x_s^* x} = \left\langle x_s^* x \right\rangle = \left\langle x_s^* (x_s \mathbf{v}_s + \xi) \right\rangle = \left\langle x_s^* x_s \right\rangle \mathbf{v}_s = \sigma_s^2 \mathbf{v}_s$$

4) M.I.L. : $\left(\mathbf{A}+\mathbf{x}\mathbf{x}^{+}\right)^{-1}=\mathbf{A}^{-1}-\dfrac{\mathbf{A}^{-1}\mathbf{x}\mathbf{x}^{+}\mathbf{A}^{-1}}{1+\mathbf{x}^{+}\mathbf{A}^{-1}\mathbf{x}} \rightarrow$

$$\rightarrow \mathbf{S}_{x}^{-1}=\mathbf{S}_{\xi}^{-1}-\frac{\mathbf{S}_{\xi}^{-1}\mathbf{v}_{s}\mathbf{v}_{s}^{+}\mathbf{S}_{\xi}^{-1}\sigma_{s}^{2}}{1+\mathbf{v}_{s}^{+}\mathbf{S}_{\xi}^{-1}\mathbf{v}_{s}\sigma_{s}^{2}}$$

5) $\mathbf{w}_{0}^{+}=\sigma_{s}^{2}\mathbf{v}_{s}^{+}\mathbf{S}_{\xi}^{-1}\left(1-\dfrac{\gamma^{-1}\sigma_{s}^{2}}{1+\gamma^{-1}\sigma_{s}^{2}}\right)=\dfrac{\sigma_{s}^{2}\gamma}{\sigma_{s}^{2}+\gamma}\mathbf{v}_{s}^{+}\mathbf{S}_{\xi}^{-1},$

where $\gamma=\left[\mathbf{v}_{s}^{+}\mathbf{S}_{\xi}^{-1}\mathbf{v}_{s}\right]^{-1}.$

Note : there is no distortionless constraint in MMSE: $\mathbf{w}_{0}^{+}\mathbf{v}_{s}\neq 1$

Q1: evaluate $\mathbf{w}_{0}^{+}\mathbf{v}_{s}$, is it much different from 1?

Q2: find $\mathbf{w}_{0}^{+}$ for a spatially-white noise.

Q3: find the resulting MSE.

Q4: when the desired signal is not a plane-wave one, what are the optimum weights?

Summary

Optimum beamforming. Objectives and criteria. Possible scenarios.

MVDR beamformer. Optimum weights. Array gain. Comparison to the conventional (equal-weights) beamformer.

MMSE beamformer. Optimum weights. Comparison to MVDR.

The techniques we develop also apply to other problems, i.e. optimum (adaptive) equalizers, MIMO systems, resource allocation in networks, etc.

References

1. Van Trees, Chapter 6 and Appendix 7.
2. R.A. Monzingo, T.W. Miller, Introduction to Adaptive Arrays, Wiley, 1980 (also 2nd edition, 2011), Ch. 3.1-3.3, Appendix C-E, G.
3. D.H. Brandwood, A complex gradient operator and its application in adaptive array theory, Proc. IEE, vol. 130, part F, pp.11-17, Feb. 1983.
4. L.C. Godara, Application of Antenna Arrays to Mobile Communications, Part II: Beamforming and Direction-of-Arrival Considerations, Proceedings of IEEE, v.85, N.8, pp. 1195-1245.
5. Any other adaptive antenna or adaptive filter book.

Note that some books are available in pdf files: check the library catalogue!

Homework

Fill in the details in the derivations above. Do the examples yourself.