The Concept of Beamforming

Generic representation of the array output signal,[1]

$$y = \mathbf{w}^{+}\mathbf{x} = \sum_{i=1}^{N-1} w_i^{*} x_i \tag{4.1}$$

where w_i – complex weights, control the array pattern; y and x_i - narrowband signals;

"+" denotes Hermitian conjugate: $\mathbf{w}^{+} = \left(\mathbf{w}^{T}\right)^{*}$

The array pattern is

$$F(\mathbf{k}) = \left|\mathbf{w}^{+}\mathbf{v}(\mathbf{k})\right| = \left|\sum_{i=0}^{N-1} w_i^{*} v_i(\mathbf{k})\right| \tag{4.2}$$

For narrowband signals, the frequency response of array elements is flat and can be presented by a complex number (w_i); x_i are complex amplitudes,

$$x_i = a_i \exp(j\varphi_i) \tag{4.3}$$

Beamforming is to choose $\mathbf{w}$ in such a way as to obtain a desired array pattern $F(\mathbf{k})$, ($F(\theta)$ in 2-D case), i.e. find such w_i that provide desired $F(\mathbf{k})$.

[1] Notations: bold capital (**K**) – matrices; bold lower case (**k**) – vectors; lower case regular (k) – scalars; $\mathbf{k}_i$ - i-th column of **K**.

Beamforming Using Pattern Sampling

For simplicity, consider 2-D case: $F(\theta)$ *and* $\mathbf{v}(\theta)$

Sample $F(\theta)$ at M=N distinct points,

$$F_j = F(\theta_j)\ ,\ j = 1,2...,M$$

$$F_j = \mathbf{w}^+ \mathbf{v}(\theta_j) \tag{4.4}$$

Define array manifold matrix

$$v_{ij} = v_i(\theta_j) \quad \Rightarrow \mathbf{f}^T = \mathbf{w}^+ \mathbf{V}$$

The solution is

$$\mathbf{w}^+ = \mathbf{f}^T \mathbf{V}^{-1} \tag{4.5}$$

$\mathbf{w}$ provides a pattern, which is the same as the desired pattern at M points θ_j (but not necessarily in between).

<u>Important example</u>: forming (N-1) zeros in the pattern (in given directions). Assume that the main beam is at broadside,

$\theta_1 = \frac{\pi}{2}$ and $F(\theta_1) = 1$, F_j becomes

$$\mathbf{f}^T = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \end{bmatrix} = \mathbf{e}_1^T \tag{4.6}$$

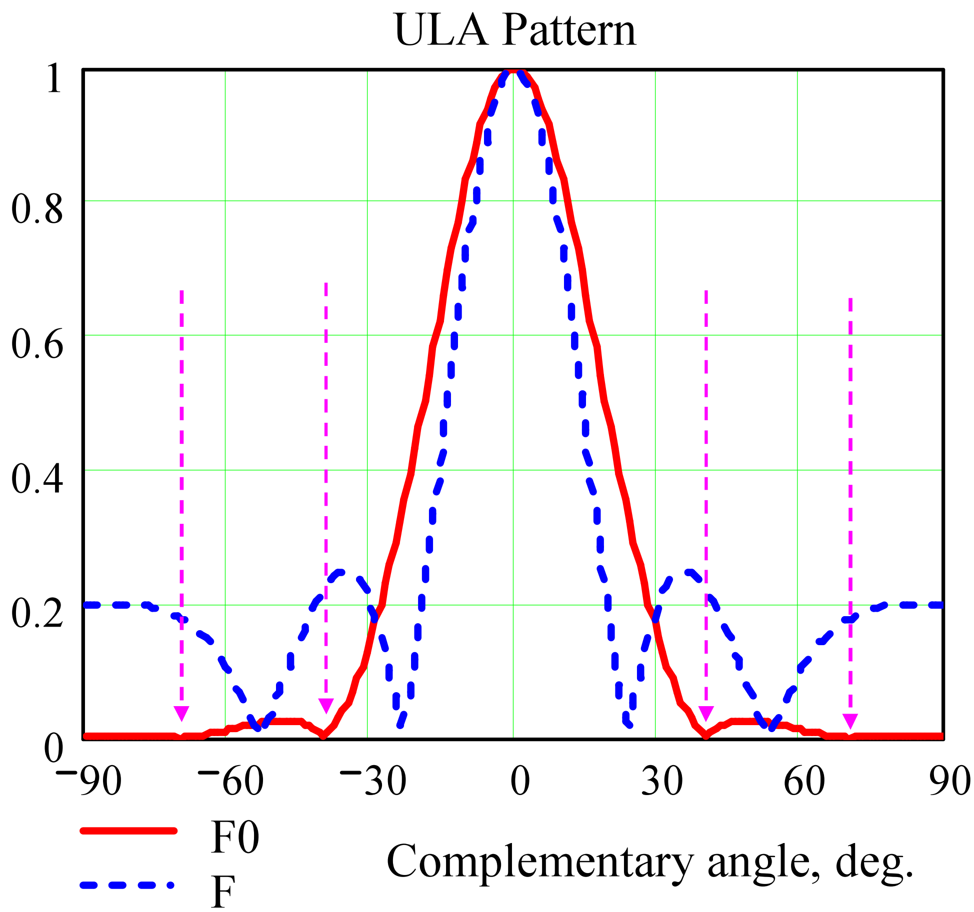
Using (4.5), we find

$$\mathbf{w}^+ = \mathbf{e}_1^T \mathbf{V}^{-1} \tag{4.7}$$

provides the desired pattern.

<u>Conclusion</u>: N-element array is able to form (N-1) nulls.

Null Steering: Example 1



$N = 5;\ d = \lambda / 2;$

$\boldsymbol{\theta} = [0, 40, -40, 70, -70]^T$, $\mathbf{w} = [0.09, 0.25, 0.33, 0.25, 0.09]^T$

$\mathbf{f} = [1, 0, 0, 0, 0]^T$

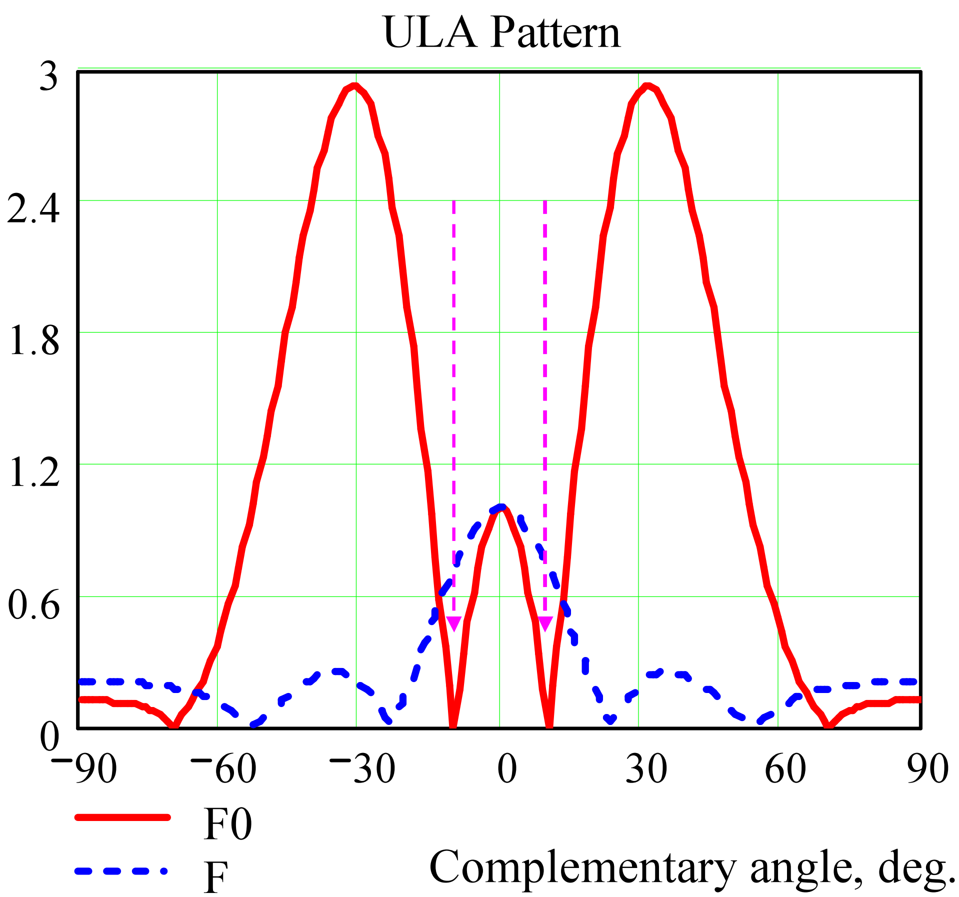
Q1: explain why only 4 zeros can be formed (and not more).

Q2: do this example yourself.

Q3: do this example for N = 10; plot the array patterns as in the figure above and discuss the impact of N.

Null Steering: Example 2

Be careful when positioning zeros!



$$N = 5;\ d = \lambda/2;$$

$$\boldsymbol{\theta} = [0, 10, -10, 70, -70]^T,\ \mathbf{w} = [0.9, 0.2, -1.2, 0.2, 0.9]^T$$

$$\mathbf{f} = [1, 0, 0, 0, 0]^T$$

Q1: explain why there is a problem in this case and suggest a way to avoid it.

Q2: do this example yourself.

Q3: do this example for N = 15; plot the array patterns as in the figure above and discuss the impact of N.

Another example: beam steering in a given direction

Basic idea - compensate the phase shifts due to the wave propagation.

If the beam is to be steered at $\mathbf{k}_0$ then

$$\mathbf{w} = \frac{1}{N}\mathbf{v}(\mathbf{k}_0) = \frac{1}{N}\left[e^{-j\mathbf{k}_0\mathbf{p}_0} \; e^{-j\mathbf{k}_0\mathbf{p}_1} \ldots e^{-j\mathbf{k}_0\mathbf{p}_{N-1}}\right]^T \tag{4.8}$$

The normalized pattern is

$$F(\mathbf{k}) = \mathbf{w}^+\mathbf{v}(\mathbf{k}) = \frac{1}{N} Tr\{\mathbf{v}(\mathbf{k}-\mathbf{k}_0)\} = \frac{1}{N}\sum_{i=0}^{N-1} v_i(\mathbf{k}-\mathbf{k}_0) \tag{4.9}$$

and $F(\mathbf{k}_0) = 1$.

Another form of the pattern,

$$F(\theta) = \frac{\sin\frac{N(\psi-\psi_0)}{2}}{N\sin\frac{\psi-\psi_0}{2}} \tag{4.10}$$

where $\psi = \frac{2\pi}{\lambda} d\cos\theta,\ \psi_0 = \frac{2\pi}{\lambda} d\cos\theta_0$.

Grating Lobes

Grating lobe: there may be additional "main" lobe(s) steered in a different (other than θ_0) direction. This is a parasitic lobe and is called "grating lobe".

Introduce new angle (measured from broadside)

$$\bar{\theta} = \frac{\pi}{2} - \theta \quad \Rightarrow \quad \psi = \frac{2\pi}{\lambda} d \sin\bar{\theta} \tag{4.11}$$

Grating lobe condition:

$$\frac{\psi - \psi_0}{2} = \pm\pi \Rightarrow \sin\bar{\theta} = \sin\bar{\theta}_0 \pm \lambda / d \tag{4.12}$$

Doesn't exist when

$$d < \frac{\lambda}{1 + \left|\sin\bar{\theta}_0\right|} \tag{4.13}$$

It is very important; if the condition is not respected, there is a spurious response that can not be distinguished from the main beam of the array.

Two special cases.

No steering:

$$\overline{\theta}_0 = 0 \Rightarrow d < \lambda \tag{4.14}$$

Steering in the entire half-plane:

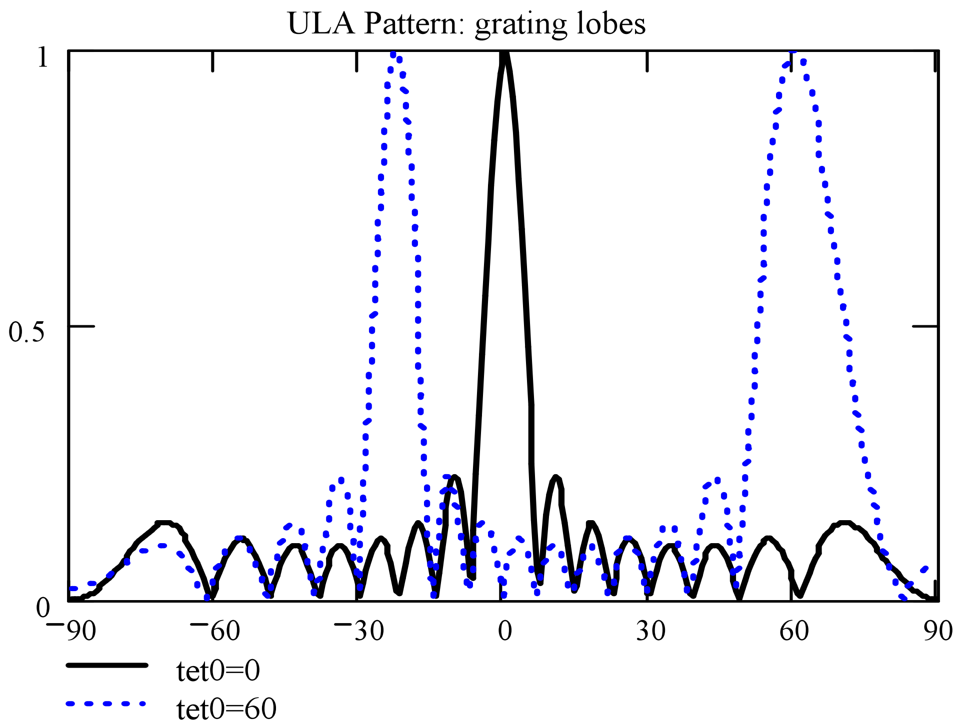
$$\overline{\theta}_0 = \pm\frac{\pi}{2} \Rightarrow d < \frac{\lambda}{2} \tag{4.15}$$

If this not feasible to implement, some other measures must be taken:

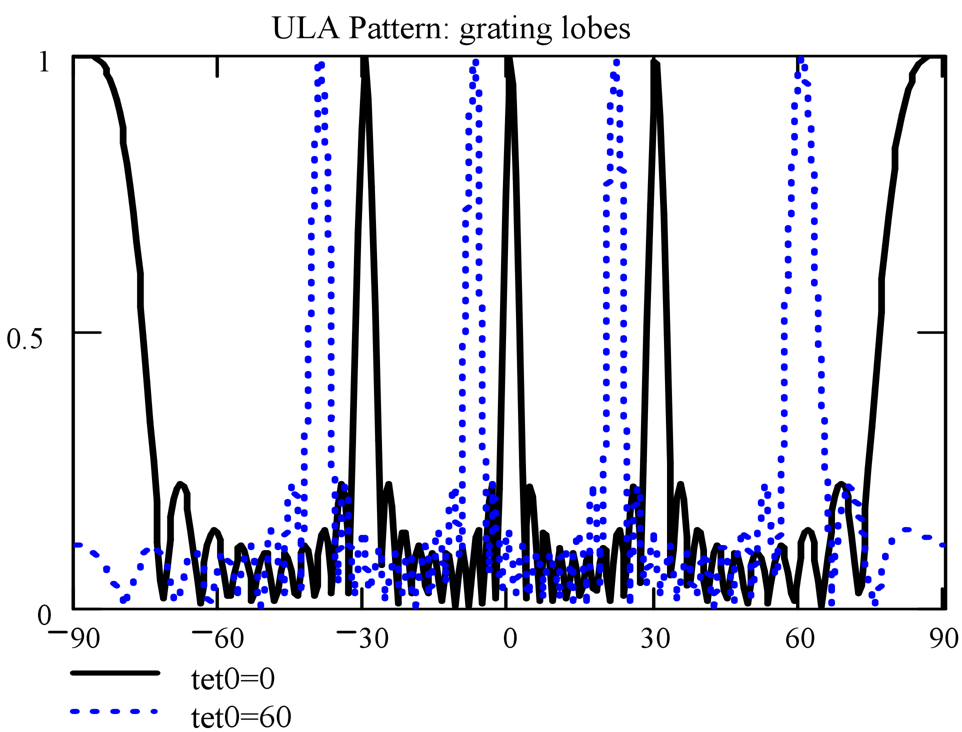
Element pattern: zero at the grating lobe directions.

Irregular element location (random arrays).

Grating Lobes: Example



$d = 0.8\lambda;$
$N = 10$



$d = 2\lambda;$
$N = 10$

Q.: do the examples above with $N = 5$ and $N = 15$, plot array patterns and discuss the impact of N and d on the array pattern and on its grating lobes.

Array Performance Parameters

Uniform linear array (ULA) gain:

$$G = \left[\sum_{n=0}^{N-1} \sum_{m=0}^{N-1} w_m w_n^* \mathrm{sinc}\left(\frac{2\pi d(n-m)}{\lambda}\right) \right]^{-1} \qquad (4.16)$$

where $\mathrm{sinc}(x)=\sin(x)/x$ (see Van Trees for a derivation), and the weights are normalized:

$$\max F(\theta) = 1 \rightarrow \sum_{i=0}^{N-1} |w_i| = |\mathbf{w}|_1 = l_1(\mathbf{w}) = 1$$

A more compact form of (4.16):

$$G = \left(\mathbf{w}^+ \mathbf{S} \mathbf{w} \right)^{-1} \qquad (4.17)$$

where

$$S_{nm} = \mathrm{sinc}(2\pi d(n-m)/\lambda) \qquad (4.18)$$

Note that $\mathbf{w}$ includes steering as well.

Consider the special case of $d = n\lambda/2$ and an ULA with non-uniform weighting:

$$S_{nm} = \delta_{nm}, \ G = \left\{ \sum_{i=1}^{N-1} |w_i|^2 \right\}^{-1} = \left(\mathbf{w}^+ \mathbf{w} \right)^{-1} = |\mathbf{w}|^{-2} \qquad (4.19)$$

where $|\mathbf{w}| = \sqrt{\mathbf{w}^+ \mathbf{w}}$ is the norm (L_2) of a vector (length).

If the weights are of the same magnitude,

$$|w_i| = 1/N \Rightarrow G = N \qquad (4.19a)$$

Note that beam steering <u>does not</u> change the gain (isotropic elements).

The array gain (in this case) is reciprocal of the magnitude squared of the weight vector, $G = |\mathbf{w}|^{-2}$.

If $d \neq n\lambda/2$, the gain will depend on the steering direction.

Uniform weighting maximizes the gain of the ULA with $d = n\lambda/2$.

Q1: prove it!

Hint : use Lagrange multipliers or, better, the Schwarz inequality.

Q2: verify (4.19a) by computing the gain G numerically using its integral expression in (3.6) or (3.8) of Lecture 3, for N=5, 10, 15 and for $d = n\lambda/2$ with $n = 1,2,3$. For N =10, plot $G(d)$ for d in the interval $[0.1...4]\lambda$. Use any suitable math software, e.g. Matlab.

SNR Gain

Important function of a receive array is to increase signal to noise ratio (SNR), which is characterized the SNR gain assuming that the noise is spatially white.

The SNR gain is also known as "noise gain".

Consider the incoming wave consisting of the required signal and noise.

The array element outputs are

$$x_i = s_i + \xi_i \tag{4.20}$$

where $s_i = a_i e^{j\varphi_i}$ and ξ_i are the signal and noise at i-th element.

The array output is

$$y = a\sum_i w_i^+ e^{j\varphi_i} + \sum_i w_i^+ \xi_i = a\sum_i \left| w_i \right| + \sum_i w_i^+ \xi_i \tag{4.21}$$

where a is the required signal amplitude.

SNR Gain

The total (signal+noise) output power

$$P = \left\langle |y|^2 \right\rangle = \underbrace{|a|^2 \sum_{i,j} |w_i||w_j|}_{\text{signal power}} + \underbrace{\sigma_0{}^2 \sum_i |w_i|^2}_{\text{noise power}}; \qquad (4.22)$$

$$\sigma_0{}^2 = \left\langle |\xi_i|^2 \right\rangle \qquad (4.23)$$

$$R_\xi = \left\langle \xi\xi^+ \right\rangle = \sigma_0{}^2 \mathbf{I} \qquad \text{(spatially white noise)} \qquad (4.24)$$

Normalize the weights, $\sum_i |w_i| = 1$. ($F_{\max} = ?$).

The output SNR,

$$SNR_{out} = \frac{P_s}{P_n} = \frac{|a|^2}{\sigma_0{}^2 \sum_i |w_i|^2} = \frac{SNR_{in}}{|\mathbf{w}|^2} \qquad (4.25)$$

$$SNR_{in} = |a|^2 / \sigma_0{}^2 \qquad (4.26)$$

The SNR gain

$$A = \frac{SNR_{out}}{SNR_{in}} = |\mathbf{w}|^{-2} \qquad (4.27)$$

SNR Gain

For ULA with $d = n\lambda/2 \rightarrow A = G$

In general, $d \neq n\lambda/2 \rightarrow A \neq G$

For a uniformly weighted array, $A = N$.

Q1: prove the statements above.

Q2: can you make a more definite statement about the relationship between A and G, e.g. $A \geq G$ or $A \leq G$, when $d \neq \lambda/2$? Justify your answer.

Summary

Concept of beamforming.

Array pattern synthesis for given levels. Null forming.

Grating lobes. Criteria of no grating lobes.

Array performance parameters: array gain and SNR gain.

Effect of random perturbations. Sensitivity function.

References:

H.L. Van Trees, Optimum Array Processing, Wiley, New York, 2002.

Homework: fill in the details in the derivations above. Do the examples yourself.

Appendix: Sensitivity and Tolerance Factor

Assume there are random errors (perturbations) in the array weights and the element locations,

$$\begin{cases} w_i = g_i e^{j\varphi_i};\ g_i = g_{i0}(1+\Delta_i),\ \varphi_i = \varphi_{i0} + \Delta\varphi_i \\ \mathbf{p}_i = \mathbf{p}_{i0} + \Delta\mathbf{p}_i \end{cases} \qquad (4.28)$$

$\Delta_i, \Delta\varphi_i$ - amplitude and phase errors, $\Delta\mathbf{p}_i$ - location errors

For each array element, g_{i0}, φ_{i0}, $\mathbf{p}_{i0}$ are the nominal values.

The expected power pattern (conventional pattern is random),

$$\left\langle \left|F(\mathbf{k})\right|^2 \right\rangle = \left\langle \sum_{i=0}^{N-1} \sum_{l=1}^{N-1} g_i e^{j\left(\varphi_i - \mathbf{k}\mathbf{p}_i\right)} g_l e^{-j\left(\varphi_l - \mathbf{k}\mathbf{p}_l\right)} \right\rangle \qquad (4.29)$$

Assume that all variations are independent of each other, and that Δ_i *and* $\Delta\varphi_i$ are i.i.d Gaussian.

After some manipulations,

$$\overline{F^2} = \left\langle \left|F(\mathbf{k})\right|^2 \right\rangle = \sum_{i \neq l}^{N-1} g_{i0} g_{l0} e^{j(\varphi_{i0} - \varphi_{l0})} e^{-j\mathbf{k}(\mathbf{p}_{i0} - \mathbf{p}_{l0})} e^{-(\sigma_\varphi{}^2 + \sigma_\lambda{}^2)}$$
$$+ \sum_{i=0}^{N-1} (1 + \sigma_g{}^2) g_{i0}^2 \qquad (4.30)$$

$\sigma_\varphi{}^2$ - variance of φ , $\sigma_g{}^2$ - variance of g, $\sigma_\lambda{}^2$ - variance of $2\pi\mathbf{p}/\lambda$,

$$\sigma_\varphi{}^2 = \left\langle (\varphi - \varphi_0)^2 \right\rangle \tag{4.31}$$

$$\sigma_g{}^2 = \left\langle (g - g_0)^2 \right\rangle \tag{4.32}$$

$$\sigma_\lambda{}^2 = \left\langle \left| \mathbf{p} - \mathbf{p}_0 \right|^2 \right\rangle \cdot \left(\frac{2\pi}{\lambda} \right)^2 \tag{4.33}$$

$\overline{F^2}$ can be presented as

$$\overline{F^2} = \left| F(\mathbf{k}) \right|^2 e^{-(\sigma_\varphi{}^2 + \sigma_\lambda{}^2)} + \sum_{i=0}^{N-1} g_{i0}^2 (1 + \sigma_g^2 - e^{-(\sigma_\varphi{}^2 + \sigma_\lambda{}^2)}) \tag{4.34}$$

$$= \left| F(\mathbf{k}) \right|^2 e^{-(\sigma_\varphi{}^2 + \sigma_\lambda{}^2)} + \left| \mathbf{w}_0 \right|^2 (1 + \sigma_g^2 - e^{-(\sigma_\varphi{}^2 + \sigma_\lambda{}^2)}) \tag{4.35}$$

<u>Random variations has 2 effects:</u>

1st term is an attenuated pattern without errors.

2nd term raises the pattern uniformly - side lobes are not low anymore.

2nd term is critical since it limits array ability to cancel interference.

Introduce the sensitivity function

$$T_s = \sum_i |w_{i0}|^2 = |\mathbf{w}_0|^2 \tag{4.36}$$

For small variance, the 2nd term

$$\varepsilon^2 = T_s[1+\sigma_g{}^2 - e^{-(\sigma_\varphi{}^2+\sigma_\lambda{}^2)}] \approx T_s[\sigma_g{}^2 + \sigma_\varphi{}^2 + \sigma_\lambda{}^2] \tag{4.37}$$

Note that $T_s = A^{-1}$ and, hence, ε^2 depends on A.

The larger the noise gain, the smaller the sensitivity and ε^2 ("noise side lobe level").

Due to ε^2 , we cannot put a perfect null in the direction of interference —> limit on null depth.

<u>Example</u>:

$$\sigma_t{}^2 = [\sigma_g{}^2 + \sigma_\varphi{}^2 + \sigma_\lambda{}^2] = 10^{-2} \; and \quad A = 10^2$$

What is the maximum null depth?

$$L_0 = \varepsilon^2 = \frac{\sigma_t{}^2}{A} = 10^{-4} = -40dB$$

If $\sigma_t{}^2$ increases to 0.1, null depth decreases to -30 dB.

The sensitivity function above holds for any array geometry.