Rx Processing ("demodulation", "decoding"): V-BLAST Algorithm

Recall the basic idea:

\[ y = Hx + \xi \]  

(14.1)

where \( x \) is the Tx vector, \( y \) is the Rx vector, \( \xi \) is the AWGN.
Channel state information (CSI) is available at the Rx only. Given $\mathbf{y}$, how to find $\mathbf{x}$?

Simple solution,

$$\hat{\mathbf{x}} = \mathbf{H}^{-1} \mathbf{y} \quad (14.2)$$

is not efficient, as it requires $O(N^3)$ operations + possible noise enhancement: for ill-conditioned $\mathbf{H}$ ($\det(\mathbf{H})$ close to 0) it is not optimum.

Q: What is the MMSE solution?

The best solution (min. BER): maximum likelihood (ML) -> too complex (exponential in $N$).

Efficient solution -> V-BLAST detection algorithm (also known as decision-feedback (DF) or successive interference cancellation (SIC)).

Three major steps:
1) Interference cancellation (form already detected symbols)
2) Interference nulling (from yet to be detected symbols)
3) Optimal ordering (max. post-processing SNR)

The Rx (vector) signal can be expressed as:

\[ y = \sum_{i=1}^{n} h_i x_i + \xi \]  \hspace{1cm} (14.3)

where \( h_i \) is the i-th column of \( H \).

Let’s assume that (i-1) symbols, from the Tx 1 to (i-1), have already been detected.
The interference cancellation step: the contribution of these symbols (from the Tx 1 to (i-1) ) to .. can be cancelled:

\[ y' = y - \sum_{j=1}^{i-1} h_j \hat{x}_j \]  \hspace{1cm} (11.4)

where \( \hat{x}_j \) are the detected symbols, which are assumed to be error free.

If this is the case, then (11.4) becomes

\[ y' = \sum_{k=i}^{n} h_k x_k + \xi \]  \hspace{1cm} (11.5)

Our immediate goal now is to detect \( x_i \), which is mixed up with \( x_{i+1}, \ldots, x_{n_T} \). Hence, the interference nulling stage:

To null out the interference from \( \{x_{i+1}, \ldots, x_{n_T}\} \), project \( y' \) to the sub-space orthogonal to the sub-space spanned by \( \{x_{i+1}, \ldots, x_{n_T}\} \).
For this, use the projection matrix in (11.9). In fact, this is an interference cancellation problem we discussed before.

This stage of V-BLAST is also called “zero-forcing” (ZF) interference cancellation.

Alternative solution: MMSE interference cancellation.

Q.: which is better? Explain why.

Consider an example of nx2 V-BLAST. At step 1,

\[ y = h_1 x_1 + h_2 x_2 + \xi \]  \hspace{1cm} (11.6)

At step 2,

\[ y' = h_2 x_2 + \xi \]  \hspace{1cm} (11.7)
The last component of the V-BLAST is the optimal ordering procedure.

The order of symbol processing is organized according to their after-processing SNR’s in decreasing order, i.e, the symbol with highest after-processing SNR is detected first.

Practical way to accomplish this: detect first the symbol whose propagation vector has the lowest correlation with the other vectors.
V-BLAST Block Diagram

Start

Optimal ordering: \( \{ h_1', h_2', \ldots, h_{n_T}' \} \)

\( i = 1 \)

Interference cancellation

\( y_i' = y - \sum_{j=1}^{i-1} h_j' \hat{x}_j \)

\( 1 \)

Interference nulling

\( y_i'' = C_{i+1} y' \)

detection

\( \hat{x}_i = Q(y_i'') \)

\( i = i + 1 \)

No

\( i = n_T \) ?

Yes

End
V-BLAST Block Diagram

$C_{i+1}$ is the projection matrix to the sub-space orthogonal to $\{h'_{i+1}, \ldots, h'_{n_T}\}$. This can be expressed as

$$C_{i+1} = I - H_i \left( H_i H_i^+ \right)^{-1} H_i^+$$

(11.8)

where $H_i = \begin{bmatrix} h'_{i+1}, h'_{i+2}, \ldots, h'_{n_T} \end{bmatrix}$.

Q.: what is the equivalent of (11.8) for MMSE V-BLAST?
Optimal Ordering

If the Tx signals are of equal power, then the optimal ordering is equivalent to finding the largest

$$a_i = |C_{i+1}h_i|$$

(11.9)

i.e. at step $i$ we detect the symbol

$$j = \arg \max_k |C_i^k h_k|, \ k \in [i...n_T]$$

(11.10)

where $C_i^k$ is a projection matrix to the subspace orthogonal to

$$\{h_i, \ldots, h_{k-1}, h_{k+1}, \ldots, h_{n_T}\}$$

i.e. all the vectors $h_i$ to $h_{n_T}$ except for $h_k$.

***

The algorithm described above, i.e. V-BLAST, is also known as ordered ZF SIC (or “decision feedback interference cancellation”).

There are some modifications: unordered one, ordered MMSE SIC, etc.
See the Appendix (at the end) for an extended discussion of those. It can be proved that the BLAST achieves the full MIMO capacity [8].

Q.: write down explicitly all the steps for nx2 V-BLAST.

Useful references on V-BLAST:


**Diagonal BLAST (D-BLAST)** Basic idea – cycle Tx antennas periodically over transmitted sub-streams to provide equal conditions for each sub-stream.

Fixed V-BLAST architecture is not optimal because in fixed environment (or slowing varying), one of the sub-streams may be in worst conditions all the time.

Detailed description of the D-BLAST - see Foschini’s paper:
Hint: to facilitate understanding, consider first 2x2 D-BLAST.

Note: the cycling does not affect the system capacity; can be skipped if rates of each stream are properly allocated.

Q.: what is the difference in BER performance of V- and D-BLAST?
Maximum-Likelihood (ML) BLAST

A big disadvantage of V-BLAST is that 1st detected symbol doesn’t enjoy any diversity (due to nulling out (n-1) other symbols) when $n_T = n_R = n$, or has the lowest diversity order of all steps, $(n_R - n_T + 1)$, in the general case. This sub-stream will have the worst performance, which will dominate the overall performance due to the error propagation. Thus, the algorithm needs an improvement.

The key idea of the ML BLAST: first $m$ symbols, $m<n$, are jointly detected using the ML approach, and the remaining $n-m$ symbols are detected in a conventional way.

Advantage: diversity order for the first $m$ symbols is $m$.

Cannot do $m=n$ because ML is exponential in complexity, but it is very feasible for small $m$ (e.g. $m=2$).
**Maximum-Likelihood (ML) BLAST**

\[
\begin{align*}
&n-1 \\
n & \\
& \vdots \\
&n \\
&\vdots \\
&3 \\
&2 \\
&1
\end{align*}
\]

- Conventional V_BLAST
- Joint ML detection
  \[m=2\]
Summary

♦ V-BLAST, D-BLAST and ML-BLAST.
♦ Detailed description of the algorithms.
♦ Performance analysis.
♦ Comparison: advantages and disadvantages.
♦ Links to multiuser systems (MAC).
References


**Homework**

Fill in the details in the derivations above. Answer the questions. Do the examples yourself.
Appendix: Further Discussion of the V-BLAST and its properties.

Max. SNR/MMSE V-BLAST

Consider regular (ZF) V-BLAST first. The basic channel model:

\[ y = Hx + \xi = \sum_{i=1}^{m} h_i x_i + \xi \]  \hspace{1cm} (1)

Detection step i: assume the Tx symbols \([x_1...x_{i-1}]\) has been correctly detected,

\[ \hat{x}_j = x_j, \quad j = 1...i - 1 \]  \hspace{1cm} (2)
Subtract the contribution of already detected symbols from $y$,

$$y' = y - \sum_{j=1}^{i-1} x_j h_j = \sum_{k=i}^{m} h_k x_k + \xi = H_{(i-1)} x_{(i-1)} + \xi$$  \hspace{1cm} (3)$$

where $H_{i-1} = [h_i, h_{i+1}, \ldots, h_m]$, $x_{(i-1)} = [x_i, x_{i+1}, \ldots, x_m]^T$. This is the interference cancellation stage.

Next, project out interference from yet to detected symbols $x_{(i)}$:

$$y'' = P_i y' = P_i h_i x_i + P_i \xi,$$  \hspace{1cm} (4)$$

where $P_i = I - H_i \left( H_i^+ H_i \right)^{-1} H_i^+$ is the projection matrix (orthogonal to span\{h_{i+1}..h_m\}).
Finally, do MRC using $y''$ to maximize output SNR:

$$\hat{y}_i = \alpha_i^+ y'' = h_i^+ P_i h_i x_i + h_i^+ P_i \xi$$  \hspace{1cm} (5)

where $\alpha_i = h_i$ are MRC weights. (5) can be compactly expressed as:

$$\hat{y}_i = w_i^+ y', \quad w_i = P_i h_i$$  \hspace{1cm} (6)

where we used the fact that $P_i^+ = P_i$.

The output SNR is:

$$\gamma_i = \frac{\left\langle |h_i^+ P_i h_i x_i|^2 \right\rangle}{\left\langle |h_i^+ P_i \xi|^2 \right\rangle} = \frac{h_i^+ P_i h_i}{\sigma_0^2}$$  \hspace{1cm} (7)
assuming $\langle |x_i|^2 \rangle = 1$ (unit power constellation).  $\hat{y}_i$ is the decision variable to find $x_i$.

This algorithm is sometimes called ZF (zero-forcing) V-BLAST as $p_i$ cancels completely ISI (inter-stream interference) from $\{x_{i+1}...x_m\}$. It does not minimize BER, however.
Max. SNR V-BLAST

Consider step $i$ and find such weights $w_i$ that the output SNR is maximized,

$$\hat{y}_i = w_i^+ y' = r_{si} + r_{\xi i}$$

$$r_{si} = w_i^+ h_i x_i, \quad r_{\xi i} = \sum_{k=i+1}^{m} w_i^+ h_k x_k + w_i^+ \xi$$

Output signal and noise/interference powers:

$$P_s = \left\langle |r_{si}|^2 \right\rangle = \left| w_i^+ h_i \right|^2$$

$$P_\xi = \left\langle |r_{\xi i}|^2 \right\rangle = \left\langle w_i^+ H_i x x^+ H_i^+ w_i \right\rangle + \sigma_0^2 w_i^+ w_i = w_i^+ \left( \sigma_0^2 I + H_i H_i^+ \right) w_i$$
where we have used $\langle x_{(i+1)} x_{(i+1)}^+ \rangle = I$, $\langle \xi \xi^+ \rangle = \sigma_0^2 I$ (i.e. i.i.d. signals and noise). Finally, the output SNR is

$$\gamma_i = \frac{P_s}{P_\xi} = \frac{w_i^+ h_i h_i^+ w_i}{w_i^+ R_\xi w_i}$$

(10)

where $R_\xi = \sigma_0^2 I + H_i H_i^+$ is noise and ISI correlation matrix.

Optimization problem:

$$\max_{w_i} \gamma_i$$

(11)

The solution is

$$w_i = R_\xi^{-1} h_i$$

(12)

and the max SNR is

$$\gamma_{i,\text{max}} = h_i^+ R_\xi^{-1} h_i$$

(13)
Compare to ZF solution (7); for large average SNR, $\sigma_0^2 \to 0$,

$$R_{\xi}^{-1} = \left( \sigma_0^2 I + H_i H_i^+ \right)^{-1} \approx \frac{1}{\sigma_0^2} P_i$$ \hspace{1cm} (14)

and max SNR solution is very close to ZF solution (7),

$$\gamma_{i,\text{max}} \approx \gamma_{i,\text{ZF}}$$ \hspace{1cm} (15)

Max SNR solution (12)-(13) has very important property.

**Theorem:** Max SNR (MMSE) V-BLAST achieves the full MIMO capacity (no Tx CSI, isotropic signaling).
Proof:

\[ C = \log \left| I + \frac{\rho}{m} \mathbf{HH}^+ \right| = \log \left| I + \frac{\rho}{m} \mathbf{H}_1 \mathbf{H}_1^+ + \frac{\rho}{n} \mathbf{h}_1 \mathbf{h}_1^+ \right| \]

\[ = \log \left| I + \frac{\rho}{m} \mathbf{H}_1 \mathbf{H}_1^+ \right| + \log \left| I + \frac{\rho}{m} \left( I + \frac{\rho}{n} \mathbf{H}_1 \mathbf{H}_1^+ \right)^{-1} \mathbf{h}_1 \mathbf{h}_1^+ \right| \quad (16) \]

\[ = \log \left| I + \frac{\rho}{m} \mathbf{H}_1 \mathbf{H}_1^+ \right| + \Delta_1 \]

\[ \Delta_1 = \log \left( 1 + \frac{\rho}{m} \mathbf{h}_1^+ \left( I + \frac{\rho}{m} \mathbf{H}_1 \mathbf{H}_1^+ \right)^{-1} \mathbf{h}_1 \right) \quad (17) \]

Note that with our normalization, \[ \left\langle |x_i|^2 \right\rangle = 1, \]
\[
\frac{\rho}{m} = \frac{1}{\sigma_0^2} \quad (18)
\]

and
\[
\Delta_1 = \log(1 + \gamma_1), \quad \gamma_1 = h_1^+ \left( \sigma_0^2 I + H_1 H_1^+ \right)^{-1} h_1 \quad (19)
\]

Comparing (19) to (13), we conclude that \(\gamma_1\) is the output SNR of the max SNR processing at step 1 (considering \(T_x 2 \ldots m\) as sources of interference, ISI). Hence, \(\Delta_1\) is the capacity at step 1.

Applying the same expansion to \(\log \left| I + \frac{\rho}{m} H_1 H_1^+ \right|\), one obtains:
\[
C = \sum_{i=1}^{m} \Delta_i; \quad \Delta_i = \log(1 + \gamma_i) \quad (20)
\]

\[
\gamma_i = h_i^+ \left( \sigma_0^2 I + H_i H_i^+ \right) h_i
\]
where $\Delta_i$ is the capacity of i-th stream, and $\gamma_i$ is the SNR with max SNR processing. Q.E.D.

Note: from (14), one may conclude that asymptotically, $\sigma_0^2 \ll 1$, ZF V-BLAST also achieves MIMO capacity.

**MMSE BLAST**

In a similar way, one may consider MMSE solution to the stream separation problem in (1),

$$\min_{\mathbf{w}_i} \mathbb{E}_i^2, \quad \mathbb{E}_i^2 = \left\langle \left| x_i - \mathbf{w}_i^+ \mathbf{y} \right|^2 \right\rangle$$  \hspace{1cm} (21)

Using

$$\frac{d\mathbb{E}_i^2}{d\mathbf{w}_i} = 0$$  \hspace{1cm} (22)

one finds MMSE weights as
\[ w_i = \left( \sigma_0^2 I + H(i-1)H^+(i-1) \right)^{-1} h_i \]  \hspace{1cm} (23)

and

\[ \varepsilon_{i, \text{min}}^2 = 1 - h_i^+ \left( \sigma_0^2 I + H(i-1)H^+(i-1) \right)^{-1} h_i \]  \hspace{1cm} (24)

After some manipulations, it can be shown that max SNR and MMSE weights are related as

\[ w_{\text{MMSE}} = \frac{w_{\text{SNR}}}{1 + \gamma_i}, \quad \gamma_i = h_i^+ R_{\xi}^{-1} h_i \]  \hspace{1cm} (25)

and, hence, MMSE solution also provides max SNR.

Important relationship between min MMSE and max SNR:

\[ \frac{1}{\varepsilon_{\text{min}, i}^2} = 1 + \gamma_i \]  \hspace{1cm} (26)
Exercise: prove (14), (23), (25), (26).
Ref. [1] has an especially good chapter on MIMO systems. Highly recommended, as well as [2].