Signal Space

Geometrical interpretation of signals via orthogonal basis function expansion:

\[ s(t) = \sum_{i=1}^{N} s_i \psi_i(t), \quad 0 \leq t \leq T, \quad s(t) \leftrightarrow \{s_i\} \Rightarrow \mathbf{s} \quad (9.1) \]

i.e. each signal \( s(t) \) is represented by vector \( \mathbf{s} \) of expansion coefficients \( \{s_i\} \).

\( \{\psi_i(t)\} \) are a set of orthonormal basis functions.

\( \int_{T} \psi_i \psi_j^*(t) dt = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} = \delta_{ij} \quad (9.2) \]

\[ E_\psi = \int_{T} \psi^2(t) dt = 1 \]

Linear independence of \( \{\psi_i(t)\} \):

\[ \sum_{i=1}^{N} \alpha_i \psi_i(t) = 0 \quad \forall t \in [0,T] \rightarrow \alpha_i = 0 \quad \forall i \quad (9.3) \]

Any complete set of LI functions can be used in (9.1), but orthonormal is more convenient.

Any LI set \( \Rightarrow \) orthonormal set via Gram-Schmidt orthogonalization process.

(9.3) \( \Rightarrow \) same as for linear independence of vectors.
Example: Fourier series

Consider periodic signal $s(t + T) = s(t) \ \forall t,$

$$s(t) = \sum_{k=-\infty}^{+\infty} c_k e^{j\frac{2\pi}{T}kt}, \quad c_k = \frac{1}{T} \int s(t)e^{-j\frac{2\pi}{T}kt} \, dt \quad (9.4)$$

Can take $\psi_k(t) = \frac{1}{\sqrt{T}} e^{j\omega t},$ $\omega = \frac{2\pi}{T} =$ fundamental frequency,

$$\int_T \psi_k(t)\psi_n^*(t) \, dt = \delta_{kn} \quad (9.5)$$

Complex vs. real form of FS: $c_k \leftrightarrow \{a_k, b_k\}$

$$s(t) = \sum_{k=0}^{\infty} a_k \cos(k\omega t) + b_k \sin(k\omega t) \quad (9.6)$$

Can take $\psi_{1,k}(t) = \sqrt{\frac{2}{T}} \cos(k\omega t),$ $\psi_{2,k}(t) = \sqrt{\frac{2}{T}} \sin(k\omega t).$

Example: M-PAM

$$s_i(t) = A_i p(t), \quad i = 1...M, \quad 0 \leq t \leq T \quad (9.7)$$

Take $\psi(t) = \frac{1}{\sqrt{E_p}} p(t).$
Example: QAM

I – Q representation of bandpass signals:

\[ x(t) = m_I(t) \cos \omega t + m_Q(t) \sin \omega t \quad (9.8) \]

Take \( \psi_1(t) = \sqrt{\frac{2}{T}} \cos \omega t \), \( \psi_2(t) = \sqrt{\frac{2}{T}} \sin \omega t \).

These basis functions are very important and are used often in communications.

How to find \( \{s_i\} \)?

\[ s_i = \int_{T} s(t) \psi_i^*(t) dt \leftrightarrow s(t) = \sum_{i} s_i \psi_i(t) \quad (9.9) \]

Signal energy: ("distance" from 0)

\[ E_s = \int_{T} |s(t)|^2 dt = |s|^2 = \sum_{i} |s_i|^2 \quad (9.10) \]

Scalar product of \( x(t) \) and \( y(t) \):

\[ x^+ y = \sum_{i} x_i^* y_i = \int_{T} x^*(t) y(t) dt \quad (9.11) \]
Distance between $x(t)$ and $y(t)$:

$$
|x - y|^2 = \sum_{i} |x_i - y_i|^2 = \int_{T} |x(t) - y(t)|^2 \, dt \tag{9.12}
$$

$\{\psi_i(t)\}$ = basis “vectors” in the signal space.

**Optimal Rx structure (Lec. 7):**

* Optimality was not proved.

* Will be proved below via probabilistic analysis in the signal space.
**Optimal Rx in Signal Space (AWGN)**

Key idea: replace \( r(t) \) by \( r \) and detect it. No loss of optimality (can be proved).

\[
r(t) = s(t) + \xi(t) \iff r = s + \xi
\]

(9.13)

\( \hat{m} \) = estimated message (Rx)
\( m \) = selected message (S)
\( s \) = transmitted signal
\( r \) = received signal

Assume \( \xi(t) = \text{AWGN} \), then

\[
R(\tau) = \xi(t)\xi^*(t+\tau) = \frac{N_0}{2} \delta(\tau), \quad \bar{\xi}(t) = 0,
\]

(9.14)

\( S_{\xi}(f) = N_0 = \text{PSD} \)

and

\[
\xi \sim N(0, \sigma^2_0 I), \quad \sigma^2_0 = \frac{N_0}{2} = \text{var}(\xi_i)
\]

(9.15)

\[
p_{\xi}(\xi) = \frac{1}{(\pi N_0)^{N/2}} \exp \left( -\frac{|\xi|^2}{N_0} \right)
\]
Also \( \mathbf{r} \sim N(\mathbf{s}, \sigma_0^2 \mathbf{I}) \) for a given \( \mathbf{s} \).

**Transmitter:** \( m \rightarrow \mathbf{s}; \quad m_i \in \{m_1 ... m_M\} = \{m_i\} \)

\[ \mathbf{s} \in \{\mathbf{s}_1 ... \mathbf{s}_M\} = \{\mathbf{s}_i\} \]

**Receiver:** \( \mathbf{r} \rightarrow \hat{\mathbf{s}} \rightarrow \hat{m} \)

**Optimal Rx:** \( P_e \rightarrow \min \),

\[ P_e = \sum_{i=1}^{M} \Pr\{m_i\} P_{e_i}, \quad P_{e_i} = \{\mathbf{s} \neq \mathbf{s}_i\} \quad (9.16) \]

Consider the equiprobable messages (*why important?):

\[ \Pr\{m_k\} = \frac{1}{M} \Rightarrow \hat{m} = m_k \text{ if } |\mathbf{r} - \mathbf{s}_k| \leq |\mathbf{r} - \mathbf{s}_i| \forall i \neq k \quad (9.17) \]

i.e. the maximum-likelihood (ML) decision rule. Can also be used when \( \Pr\{m_k\} \) are not known.

\[ \text{ML = min. distance} \quad \text{in AWGN} \quad (9.18) \]
Decision region:

\[
\Omega_k = \{ \mathbf{r} : |\mathbf{r} - \mathbf{s}_k| \leq |\mathbf{r} - \mathbf{s}_i| \quad \forall i \neq k \} \tag{9.19}
\]

so that the ML rule is

\[
\hat{m} = m_k \text{ if } \mathbf{r} \in \Omega_k \tag{9.20}
\]

\(\{\Omega_i\}_{i=1}^{M}\): split all space into a set of disjoint sets/decisions regions.

Example: BPSK, \(m_i = \pm 1\), \(\psi(t) = \alpha p(t)\).

\(\hat{m} = 1\) if \(r \in \Omega_1 \Rightarrow r > 0\)

\(\hat{m} = -1\) if \(r \in \Omega_{-1} \Rightarrow r < 0\)
Example: QPSK

\[
\psi_1(t) = \sqrt{\frac{2}{T}} \cos \omega t
\]

\[
\psi_2(t) = \sqrt{\frac{2}{T}} \sin \omega t
\]

\[
s_1 = [1, 1]^T \quad s_3 = [-1, -1]^T
\]

\[
s_2 = [-1, 1]^T \quad s_4 = [1, -1]^T
\]

\[
\Omega_1 = ? \quad \Omega_{2,3,4} = ?
\]

Signal constellation = \( \{s_i(t)\} \) in the signal space, i.e. \( \{s_i\} \).

Signals: Points (vectors) in the signal space.
Receiver Implementation

The rule (9.17) can be expressed in a different form using

$$|r - s_k|^2 = |r|^2 - 2rs_k + |s_k|^2 \quad (9.21)$$

Since \( |r|^2 \) is independent of \( s \), it can be dropped and (9.17) becomes

$$r^+s_k + c_k \geq r^+s_i + c_i, \forall i \neq k \quad (9.22)$$

where \( r^+s_k = \sum_{i=1}^{N} r_is_{ki} \) is a scalar product and \( c_k = -|s_k|^2 / 2 \). It can be expressed as

$$r^+s_k = \int_{T} r(t)s_k(t)dt \quad (9.23)$$

Q. Prove it.

(9.23) can be implemented using a correlation receiver or a matched filter receiver.

**Matched filter** impulse response

$$h_k(t) = s_k(T-t) \quad (9.24)$$

so the output sampled at time \( T \) is

$$\rho(t) * h_k(t) = \int_{T} r(t)s_k(t)dt \quad (9.25)$$
Implementation

Correlation Receiver

\[ r(t) \]

\[ s_1(t) \]

\[ s_2(t) \]

\[ s_M(t) \]

\[ \int_0^T ( ) dt \]

\[ + \]

\[ \text{Select Largest at } t=T \]

\[ \hat{m} \]

Matched Filter Receiver

\[ r(t) \]

\[ s_1(T-t) \]

\[ s_2(T-t) \]

\[ s_M(T-t) \]

\[ + \]

\[ + \]

\[ + \]

\[ \text{Select Largest at } t=T \]

\[ \hat{m} \]
Another Form of Implementation

Correlation receiver in the signal space

Q.: detector =?
An Example: BPSK

Time-domain expression of the signal is

\[ s_i(t) = (-1)^i A \cdot \psi(t), \quad i = 0, 1 \]  \hspace{1cm} (9.26)

where \( \psi(t) \) is a basis pulse shape (may be \( \cos(\omega t) \)); \( A \) is the amplitude.

Constellation:

The ML receive computes the following decision variable,

\[ z = \int_{T} r(t) \psi(t) dt \]  \hspace{1cm} (9.27)

and sets

\[ \hat{m} = \begin{cases} 0, & \text{if } z < 0 \\ 1, & \text{if } z > 0 \end{cases} \]

Q.: block diagram of the receiver?
Comparison of M-ary Modulation Schemes

**M-PSK:** the phase can assume $M$ different values

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos \left( \omega_c t + \frac{2\pi}{M} i \right), \quad i = 0, 1, \ldots, M - 1$$  \hspace{1cm} (9.28)

$$0 \leq t \leq T$$

$\log_2 M$ bits are transmitted by each symbol.

The symbol error probability (SER):

$$P_{se} \approx 2Q \left( \sqrt{\frac{2E}{N_0}} \sin \left( \frac{\pi}{M} \right) \right) = \alpha Q \left( \sqrt{\frac{\beta E}{N_0}} \right)$$  \hspace{1cm} (9.29)

**Q:** constellation example? Minimum distance $d_{\text{min}}$?

<table>
<thead>
<tr>
<th>Table 6.4</th>
<th>Bandwidth and Power Efficiency of M-ary PSK Signals</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>2</td>
</tr>
<tr>
<td>$\eta_B = \frac{R_b}{B^*}$</td>
<td>\makecell{0.5}</td>
</tr>
<tr>
<td>$E_b/N_0$ for BER=10^{-6}</td>
<td>\makecell{10.5}</td>
</tr>
</tbody>
</table>

* $B$: First null bandwidth of M-ary PSK signals

T.S. Rappaport, Wireless Communications, Prentice Hall, 2002

see (8.25): power/bandwidth efficiency tradeoff!
**M-ary Quadrature AM (M-QAM)**

M levels with different phases and amplitudes

\[ s_i(t) = a_i \cos \omega_c t + b_i \sin \omega_c t \quad (9.30) \]

\(a_i\) and \(b_i\) - I and Q components.

The BER of M-QAM (\(M = 2^k\) and \(k\) is even):

\[ P_e \approx \frac{4\left(1 - 1/\sqrt{M}\right)}{\log_2 M} Q\left[\sqrt{\frac{3 \log_2 M}{M-1}} \cdot \frac{E_b}{N_0}\right] \quad (9.31) \]

<table>
<thead>
<tr>
<th>(M)</th>
<th>4</th>
<th>16</th>
<th>64</th>
<th>256</th>
<th>1024</th>
<th>4096</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\eta_B)</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

\(E_b/N_0\), for BER = 10^{-6}:

<table>
<thead>
<tr>
<th>(M)</th>
<th>(E_b/N_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>10.5</td>
</tr>
<tr>
<td>16</td>
<td>15</td>
</tr>
<tr>
<td>64</td>
<td>18.5</td>
</tr>
<tr>
<td>256</td>
<td>24</td>
</tr>
<tr>
<td>1024</td>
<td>28</td>
</tr>
<tr>
<td>4096</td>
<td>33.5</td>
</tr>
</tbody>
</table>

The threshold SNR: \(\gamma_{th} \sim 10 \log_2 M\)

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**Frequency Shift Keying (FSK)**

M-ary FSK:

\[ s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t + \varphi), \quad i = 1, \ldots, M, \quad 0 \leq t \leq T \]  

(9.32)

Note: signal orthogonality imposes a limit on \( \Delta \omega = \omega_{i+1} - \omega_i \).

Q: find min \( \Delta \omega \) such that the signals are orthogonal.

For orthogonal BFSK (coherently detected),

\[ P_e = Q\left(\sqrt{\gamma}\right) \]  

(9.33)

Note: non-coherent detection results in different BER,

\[ P_{e,N} = e^{-\gamma/2}/2 \]  

(9.34)

Performance loss is a few dBs.

For \( M > 2 \), the tight upper bound on SER is

\[ P_{es} \leq (M - 1)Q\left(\sqrt{\gamma}\right) \]  

(9.35)

for orthogonal signals and coherent demodulation.

<table>
<thead>
<tr>
<th>Table 6.6</th>
<th>Bandwidth and Power Efficiency of Coherent M-ary FSK [Zie92]</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>2</td>
</tr>
<tr>
<td>( \eta_B )</td>
<td>0.4</td>
</tr>
<tr>
<td>( E_b/N_o ) for BER = 10(^{-6} )</td>
<td>13.5</td>
</tr>
</tbody>
</table>

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**SNR for Analog and Digital Systems**

SNR in an analog systems is

\[ SNR_a = \frac{S}{N}, \]  

(9.36)

where \( S \) is the signal power, \( N \) is the noise power.

Note that \( S = E / T_s \) and \( R_s = 1 / T_s \). Furthermore, \( N = N_0 \cdot \Delta f \). Hence

\[ \gamma = \frac{E_s}{N_0} = \frac{S \cdot T_s}{N / \Delta f} = \frac{S}{N} \cdot \frac{\Delta f}{R_s} = SNR_a \frac{\Delta f}{R_s} \]  

(9.37)

For \( R_s = \Delta f \), they are the same. \( \frac{\Delta f}{R_s} \) is inverse of the bandwidth efficiency, \( R_s / \Delta f \) \( (= \eta) \).

Similarly,

\[ \gamma_b = \frac{E_b}{N_0} = SNR_a \frac{\Delta f}{R} \]  

(9.38)

where \( R = \) bit rate (bit/s).
SER and Signal Constellation

SER can be expressed through minimum distance between points of the signal constellation.

Generic bound for the SER is (coherent detection),

\[
P_e \leq (M - 1) Q \left( \frac{d_{\text{min}}^2}{2 N_0} \right)
\]

(9.39)

where \(d_{\text{min}}\) is determined from signal constellation (minimum distance), see the next slide.

An approximation at high SNR:

\[
P_e \approx N_e Q \left( \frac{d_{\text{min}}^2}{2 N_0} \right)
\]

(9.40)

where \(N_e\) = # of nearest neighbours.

The BER can be approximated at high SNR as

\[
P_b \approx \frac{1}{\log_2 M} P_e
\]

(9.41)

Q.: what is an interpretation of (9.41)?
Note: if bit energy is used, then $d_{\text{min}} = 2\sqrt{E_b}$ for QPSK since $E = 2E_b$ (2 bits per symbol!). For the other (binary) modulations, $E = E_b$. 
Comparison of Various Modulation Formats

Fundamental limit is provided by the Shannon’s channel capacity theorem (AWGN channel):

\[
C = \Delta f \log_2 (1 + \gamma) \tag{9.42}
\]

- \( C \) = channel capacity [bit/s]
- \( \Delta f \) = bandwidth [Hz]
- \( \gamma \) = SNR, \( \gamma = \frac{P}{N} \), where \( P \) - signal power, \( N \) - noise power.

**Error free transmission** is possible if \( R < C \), and is not possible for \( R > C \); \( R \) = bit rate [b/s].

Using \( E_b = P T_b, \gamma_b = \frac{E_b}{N_0} = \gamma \frac{\Delta f}{R} \),

\[
\frac{C}{\Delta f} = \log_2 \left( 1 + \gamma_b \frac{R}{\Delta f} \right) \tag{9.43}
\]

The maximum possible data rate \( R = C \), then

\[
\frac{R}{\Delta f} \leq \log_2 \left( 1 + \gamma_b \frac{R}{\Delta f} \right) \tag{9.44}
\]

\( \frac{R}{\Delta f} \) (bit/s/Hz) is a spectral efficiency. Required SNR is

\[
\gamma_b \geq \frac{2^{\frac{R}{\Delta f}} - 1}{R/\Delta f} \geq \ln 2 = -1.6dB \tag{9.45}
\]

LB is monotonically increasing in \( \frac{R}{\Delta f} \) - power/bandwidth efficiency tradeoff (see the tables).
Fundamental Limit: Spectral Efficiency [bit/s/Hz] vs. SNR/bit [dB]

\[ R_b/W = \frac{2^{R/\Delta f} - 1}{R / \Delta f} \geq -1.6 dB \]

SNR: Eb/N0 [dB]

Spectral efficiency [bit/s/Hz] vs. SNR [dB]

achievable
**Important Conclusions:**

- **Power-limited region:** For large bandwidth available. For $f \Delta \rightarrow \infty$, $\ln 2 b = -1.6 b$. No error-free transmission is possible for $\gamma < -1.6 b$.

- **Bandwidth-limited region:** For large power available. Trade-off: increasing power efficiency decreases spectrum efficiency and vice-versa.

**Spectral Efficiency of Digital Modulation:**

- Null-to-null bandwidth (or absolute bandwidth assuming a raised cosine pulse with $\alpha = 1$) of a modulated signal, $M$-PSK and $M$-QAM:
  \[ R = \frac{2}{\log_2 M} \delta = \frac{2}{\log_2 M} \delta \]

- $M$-FSK (non-coherent):
  \[ R = \frac{2}{\log_2 M} \delta = \frac{2}{\log_2 M} \delta \]

- $M$-PSK (QAM) is much better than $M$-FSK in terms of spectral efficiency.

---

**Figure 9.6** Bandwidth-efficiency plane.
Most high-rate systems use M-PSK (up to $M=8$) and M-QAM ($M\geq 8$).
Note: for Nyquist (sinc) pulse, remove 2.
BER: Comparison of Various Modulation Formats

BER in AWGN channels

SNR: $\frac{E_b}{N_0}$ [dB]

- OOK
- BPSK
- DPSK
- QPSK
- 8PSK
- 8QAM
- Coherent BFSK
- Non-coherent BFSK
Maximum achievable rate with coding:

\[ R_{\text{max}} = \frac{\log_2 (1 + SNR)}{1 - h(P_e)} \quad \text{[bit/symbol]} \]  

(9.49)

\[ h(P_e) = -P_e \log_2 P_e - (1 - P_e) \log_2 (1 - P_e) = \text{binary entropy} \]
Summary

- Geometric representation of signals via signal space.
- Optimum receiver (MAP, ML) in the signal space.
- BPSK, QPSK, QAM.
- M-ary modulation formats. Comparison.
- Power and bandwidth efficiency.
- BER and SER.
- Fundamental limits. Channel capacity.

Reading:

- Rappaport, Ch. 6 (expect 6.11, 6.12).
- Other books (see the reference list).

Note: Do not forget to do end-of-chapter problems. Remember the learning efficiency pyramid!
Appendix: Optimal Rx

Optimal Rx selects $\hat{m} = m_k$ with largest a posteriori probability:

$$\hat{m} = m_k \quad \text{if} \quad Pr\{m_k | r\} \geq Pr\{m_i | r\} \quad \forall i \neq k \quad (9.50)$$

$$Pr\{m_k | r\} = \frac{Pr\{r | m_k\} Pr\{m_k\}}{Pr\{r\}} \quad (9.51)$$

To prove its optimality, observe that the probability of correct decision $Pr\{c | m_k\}$ given that the Rx selects $\hat{m} = m_k$ is $Pr\{c | m_k\} = Pr\{m_k | r\}$, which is maximized by (9.50).

This rule also maximizes unconditional probability of correct decision ($P_c = 1 - P_e$):

$$P_c = \int Pr\{r\} Pr\{c | r\}dr, \quad Pr\{c | r\} = \sum_{k=1}^{M} Pr\{m_k | r\} Pr\{m_k\} \quad (9.52)$$

From (9.50), (9.51), the maximum a posteriori probability (MAP) decision rule follows:

$$\hat{m} = m_k \quad \text{if} \quad Pr\{r | s_k\} Pr\{s_k\} \geq Pr\{r | s_i\} Pr\{s_i\} \quad \forall i \neq k \quad (9.53)$$

since $Pr\{s_i\} = Pr\{m_i\}, \quad Pr\{r | s_i\} = Pr\{r | m_i\}$, i.e. decide in favor of such $m_k$ that maximizes a posteriori probability of observed $r$. 
From (9.15),

$$\Pr\{r|s_k\} = P_{\xi}(r - s_k) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{|r - s_k|^2}{N_0}\right)$$

(9.54)

So that the decision rule in (9.53) becomes

$$\hat{m} = m_k \text{ if } |r - s_k|^2 + c_k \leq |r - s_i|^2 + c_i, \forall i \neq k$$

(9.55)

where $c_k = -N_0 \ln \Pr\{m_k\}$. 