Signal Space

Geometrical interpretation of signals via orthogonal basis function expansion:

\[ s(t) = \sum_{i=1}^{N} s_i \psi_i(t), \quad 0 \leq t \leq T, \quad s(t) \leftrightarrow \{s_i\} \Rightarrow \mathbf{s} \quad (9.1) \]

i.e. each signal \( s(t) \) is represented by vector \( \mathbf{s} \) of expansion coefficients \( \{s_i\} \).

\( \{\psi_i(t)\} \) are a set of orthonormal basis functions,

\[
\int_{T}^{T} \psi_i \psi^*_j(t) dt = \begin{cases} 
1, & i = j \\
0, & i \neq j 
\end{cases} = \delta_{ij} 
\quad (9.2)
\]

\[ E_{\psi} = \int_{T}^{T} \psi^2(t) dt = 1 \]

Linear independence of \( \{\psi_i(t)\} \):

\[
\sum_{i=1}^{N} \alpha_i \psi_i(t) = 0 \quad \forall t \in [0,T] \rightarrow \alpha_i = 0 \quad \forall i \quad (9.3)
\]

Any complete set of LI functions can be used in (9.1), but orthonormal is more convenient.

Any LI set \( \Rightarrow \) orthonormal set via Gram-Schmidt orthogonalization process.

(9.3) \( \Rightarrow \) same as for linear independence of vectors.
**Example: Fourier series**

Consider periodic signal $s(t + T) = s(t) \ \forall t,$

$$s(t) = \sum_{k=-\infty}^{+\infty} c_k e^{\frac{j2\pi kt}{T}}, \quad c_k = \frac{1}{T} \int_{T} s(t)e^{-\frac{j2\pi kt}{T}} dt \quad (9.4)$$

Can take $\psi_k(t) = \frac{1}{\sqrt{T}} e^{j\omega t}, \quad \omega = \frac{2\pi}{T} = \text{fundamental frequency},$

$$\int_{T} \psi_k(t)\psi^*_n(t)dt = \delta_{kn} \quad (9.5)$$

Complex vs. real form of FS: $c_k \leftrightarrow \{a_k, b_k\}$

$$s(t) = \sum_{k=0}^{\infty} a_k \cos(k\omega t) + b_k \sin(k\omega t) \quad (9.6)$$

Can take $\psi_{1,k}(t) = \sqrt{\frac{2}{T}} \cos(k\omega t), \quad \psi_{2,k}(t) = \sqrt{\frac{2}{T}} \sin(k\omega t).$

**Example: M-PAM**

$$s_i(t) = A_i p(t), \quad i = 1...M, \quad 0 \leq t \leq T \quad (9.7)$$

Take $\psi(t) = \frac{1}{\sqrt{E_p}} p(t).$
Example: QAM

I – Q representation of bandpass signals:

\[ x(t) = m_I(t) \cos \omega t + m_Q(t) \sin \omega t \]  \hspace{1cm} (9.8)

Take \( \psi_1(t) = \sqrt{\frac{2}{T}} \cos \omega t \), \( \psi_2(t) = \sqrt{\frac{2}{T}} \sin \omega t \).

These basis functions are very important and are used often in communications.

How to find \( \{s_i\} \)?

\[ s_i = \int_{T} s(t) \psi_i^*(t) dt \leftrightarrow s(t) = \sum_{i} s_i \psi_i(t) \]  \hspace{1cm} (9.9)

Signal energy: (“distance” from 0)

\[ E_s = \int_{T} |s(t)|^2 dt = |s|^2 = \sum_{i} |s_i|^2 \]  \hspace{1cm} (9.10)

Scalar product of \( x(t) \) and \( y(t) \):

\[ x^+ y = \sum_{i} x_i^* y_i = \int_{T} x^*(t) y(t) dt \]  \hspace{1cm} (9.11)
Distance between $x(t)$ and $y(t)$:

$$\|x - y\|^2 = \sum_{i} |x_i - y_i|^2 = \int_{T} |x(t) - y(t)|^2 \, dt \quad (9.12)$$

$\{\psi_i(t)\} =$ basis “vectors” in the signal space.

**Optimal Rx structure (Lec. 7):**

* Optimality was not proved.

* Will be proved below via probabilistic analysis in the signal space.
Optimal Rx in Signal Space (AWGN)

Key idea: replace $r(t)$ by $\mathbf{r}$ and detect it. No loss of optimality (can be proved).

$$r(t) = s(t) + \xi(t) \iff \mathbf{r} = \mathbf{s} + \xi$$  \hspace{1cm} (9.13)

Assume $\xi(t) = \text{AWGN}$, then

$$R(\tau) = \xi(t)\xi^*(t + \tau) = \frac{N_0}{2} \delta(\tau), \quad \xi(t) = 0,$$  \hspace{1cm} (9.14)

$$S_{\xi}(f) = N_0 \quad \text{PSD}$$

and

$$\xi \sim N(0, \sigma_0^2 \mathbf{I}), \quad \sigma_0^2 = \frac{N_0}{2} = \text{var}(\xi_i)$$

$$p_{\xi}(\xi) = \frac{1}{(\pi N_0)^{N/2}} \exp \left( -\frac{\|\xi\|^2}{N_0} \right)$$  \hspace{1cm} (9.15)
Also \( \mathbf{r} \sim N(\mathbf{s}, \sigma_0^2 \mathbf{I}) \) for a given \( \mathbf{s} \).

Transmitter: \( m \rightarrow \mathbf{s}; \quad m_i \in \{m_1...m_M\} = \{m_i\} \)
\( \mathbf{s} \in \{\mathbf{s}_1...\mathbf{s}_M\} = \{\mathbf{s}_i\} \)

Receiver: \( \mathbf{r} \rightarrow \hat{s} \rightarrow m \)

Optimal Rx: \( P_e \rightarrow \min \),

\[
P_e = \sum_{i=1}^{M} \Pr\{m_i\} P_{e_i}, \quad P_{e_i} = \{\hat{s} \neq \mathbf{s}_i\} \tag{9.16}
\]

Consider the equiprobable messages (why important?):

\[
\Pr\{m_k\} = \frac{1}{M} \Rightarrow m = m_k \text{ if } |\mathbf{r} - \mathbf{s}_k| \leq |\mathbf{r} - \mathbf{s}_i| \quad \forall i \neq k \tag{9.17}
\]

i.e. the maximum-likelihood (ML) decision rule. Can also be used when \( \Pr\{m_k\} \) are not known.

\[
\text{ML = min. distance in AWGN} \tag{9.18}
\]
Decision region:

$$\Omega_k = \{ \mathbf{r} : |\mathbf{r} - \mathbf{s}_k| \leq |\mathbf{r} - \mathbf{s}_i| \ \forall i \neq k \}$$

(9.19)

so that the ML rule is

$$m = m_k \text{ if } \mathbf{r} \in \Omega_k$$

(9.20)

$$\{ \Omega_i \}_{i=1}^M$$: split all space into a set of disjoint sets/decisions regions.

Example: BPSK, $$m_i = \pm 1, \ \psi(t) = \alpha p(t)$$.

\[ m = 1 \text{ if } r \in \Omega_1 \Rightarrow r > 0 \]

\[ m = -1 \text{ if } r \in \Omega_{-1} \Rightarrow r < 0 \]
Example: QPSK

\[
\psi_1(t) = \sqrt{\frac{2}{T}} \cos \omega t
\]

\[
\psi_2(t) = \sqrt{\frac{2}{T}} \sin \omega t
\]

\[
s_1 = [1,1]^T \quad s_3 = [-1,-1]^T
\]

\[
s_2 = [-1,1]^T \quad s_4 = [1,-1]^T
\]

\[
\Omega_1 = ? \quad \Omega_{2,3,4} = ?
\]

Signal constellation = \{s_i(t)\} in the signal space, i.e. \{s_i\}.

Signals: Points (vectors) in the signal space.
**Receiver Implementation**

The rule (9.17) can be expressed in a different form using

\[ |\mathbf{r} - \mathbf{s}_k|^2 = |\mathbf{r}|^2 - 2\mathbf{r}\mathbf{s}_k + |\mathbf{s}_k|^2 \]  

(9.21)

Since \(|\mathbf{r}|^2\) is independent of \(\mathbf{s}\), it can be dropped and (9.17) becomes

\[ \mathbf{r}^+\mathbf{s}_k + c_k \geq \mathbf{r}^+\mathbf{s}_i + c_i, \forall i \neq k \]  

(9.22)

where \(\mathbf{r}^+\mathbf{s}_k = \sum_{i=1}^{N} r_i s_{ki}\) is a scalar product and \(c_k = -|\mathbf{s}_k|^2 / 2\). It can be expressed as

\[ \mathbf{r}^+\mathbf{s}_k = \int r(t)s_k(t)dt \]  

(9.23)

Q. Prove it.

(9.23) can be implemented using a correlation receiver or a matched filter receiver.

**Matched filter** impulse response

\[ h_k(t) = s_k(T - t) \]  

(9.24)

so the output sampled at time \(T\) is

\[ \rho(t)^*h_k(t) = \int r(t)s_k(t)dt \]  

(9.25)
**Implementation**

**Correlation Receiver**

\[ s_1(t) \rightarrow X \rightarrow \int_0^T c_1 dt \rightarrow + \]

\[ s_2(t) \rightarrow X \rightarrow \int_0^T c_2 dt \rightarrow + \]

\[ s_M(t) \rightarrow X \rightarrow \int_0^T c_M dt \rightarrow + \]

Select Largest at \( t = T \)

\[ \hat{m} \]

**Matched Filter Receiver**

\[ r(t) \]

\[ s_1(T-t) \rightarrow + \]

\[ s_2(T-t) \rightarrow + \]

\[ s_M(T-t) \rightarrow + \]

Select Largest at \( t = T \)

\[ \hat{m} \]
Another Form of Implementation

Correlation receiver in the signal space

Q.: detector =?
An Example: BPSK

Time-domain expression of the signal is

\[ s_i(t) = (-1)^i A \cdot \psi(t), \quad i = 0, 1 \]  \hspace{1cm} (9.26)

where \( \psi(t) \) is a basis pulse shape (may be \( \cos(\omega t) \)); \( A \) is the amplitude.

Constellation:

The ML receive computes the following decision variable,

\[ z = \int_{T} r(t)\psi(t) dt \]  \hspace{1cm} (9.27)

and sets

\[ \hat{m} = \begin{cases} 
0, & \text{if } z < 0 \\
1, & \text{if } z > 0
\end{cases} \]

Q.: block diagram of the receiver?
Comparison of M-ary Modulation Schemes

**M-PSK:** the phase can assume $M$ different values

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left(\omega_c t + \frac{2\pi}{M} i\right), \quad i = 0, 1, ..., M - 1$$

$$0 \leq t \leq T$$

$log_2 M$ bits are transmitted by each symbol.

The symbol error probability (SER):

$$P_{se} \approx 2Q\left(\sqrt{\frac{2E}{N_0}} \sin\left(\frac{\pi}{M}\right)\right) = \alpha Q\left(\sqrt{\frac{B E}{N_0}}\right)$$

**Q:** constellation example? Minimum distance $d_{\text{min}}$?

<table>
<thead>
<tr>
<th>$M$</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_B = R_b/B^*$</td>
<td>0.5</td>
<td>1</td>
<td>1.5</td>
<td>2</td>
<td>2.5</td>
<td>3</td>
</tr>
<tr>
<td>$E_b/N_0$ for BER=10^-6</td>
<td>10.5</td>
<td>10.5</td>
<td>14</td>
<td>18.5</td>
<td>23.4</td>
<td>28.5</td>
</tr>
</tbody>
</table>

*T.S. Rappaport, Wireless Communications, Prentice Hall, 2002*

*rectangular pulse or RC pulse with $\alpha = 1$*

see (9.45) for the power/bandwidth efficiency tradeoff.
**M-ary Quadrature AM (M-QAM)**

M levels with different phases and amplitudes

\[ s_i(t) = a_i \cos \omega_c t + b_i \sin \omega_c t \]  \hspace{1cm} (9.30)

\( a_i \) and \( b_i \) - I and Q components.

\[ \begin{align*}
Q(b_i) & \quad \text{Q(b_i)} \\
\circ & \quad \circ \\
\circ & \quad \circ \\
\circ & \quad \circ
\end{align*} \]

\[ \begin{align*}
\text{I(a_i)} & \quad \text{I(a_i)} \\
\circ & \quad \circ \\
\circ & \quad \circ \\
\circ & \quad \circ
\end{align*} \]

M=4

\[ \begin{align*}
Q & \quad \text{Q} \\
\circ & \quad \circ \\
\circ & \quad \circ \\
\circ & \quad \circ \\
\circ & \quad \circ \\
\circ & \quad \circ \\
\circ & \quad \circ \\
\circ & \quad \circ
\end{align*} \]

\[ \begin{align*}
\text{I} & \quad \text{I} \\
\circ & \quad \circ \\
\circ & \quad \circ \\
\circ & \quad \circ \\
\circ & \quad \circ \\
\circ & \quad \circ \\
\circ & \quad \circ \\
\circ & \quad \circ
\end{align*} \]

M=16

The BER of M-QAM (\( M = 2^k \) and \( k \) is even):

\[ P_e \approx \frac{4(1-1/\sqrt{M})}{\log_2 M} Q\left[ \sqrt{\frac{3\log_2 M}{M-1} \cdot \frac{E_b}{N_0}} \right] \]  \hspace{1cm} (9.31)

| Table 6.5 Bandwidth and Power Efficiency of QAM [Zie92] |
|-----------------|-------|-------|-------|-------|-------|-------|-------|
| \( M \)         | 4     | 16    | 64    | 256   | 1024  | 4096  |
| \( \eta_B \)    | 1     | 2     | 3     | 4     | 5     | 6     |
| \( E_b/N_0 \) for BER = 10^{-6} \) | 10.5  | 15    | 18.5  | 24    | 28    | 33.5  |

T.S. Rappaport, Wireless Communications, Prentice Hall, 2002

The threshold SNR: \( \gamma_{th} \sim 10 \log M \)
**Frequency Shift Keying (FSK)**

M-ary FSK:

\[ s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t + \varphi), \quad i = 1, \ldots, M \quad 0 \leq t \leq T \] (9.32)

Note: signal orthogonality imposes a limit on \( \Delta \omega = \omega_{i+1} - \omega_i \).

Q: find min \( \Delta \omega \) such that the signals are orthogonal.

For orthogonal BFSK (coherently detected),

\[ P_e = Q\left(\sqrt{\gamma}\right) \] (9.33)

Note: non-coherent detection results in different BER,

\[ P_{e,N} = e^{-\gamma/2}/2 \] (9.34)

Performance loss is a few dBs.

For \( M > 2 \), the tight upper bound on SER is

\[ P_{e,s} \leq (M - 1)Q\left(\sqrt{\gamma}\right) \] (9.35)

for orthogonal signals and coherent demodulation.

**Table 6.6** Bandwidth and Power Efficiency of Coherent M-ary FSK [Zie92]

<table>
<thead>
<tr>
<th>M</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta_B )</td>
<td>0.4</td>
<td>0.57</td>
<td>0.55</td>
<td>0.42</td>
<td>0.29</td>
<td>0.18</td>
</tr>
<tr>
<td>( E_{b}/N_o ) for BER = 10^{-6}</td>
<td>13.5</td>
<td>10.8</td>
<td>9.3</td>
<td>8.2</td>
<td>7.5</td>
<td>6.9</td>
</tr>
</tbody>
</table>

T.S. Rappaport, Wireless Communications, Prentice Hall, 2002
**SNR for Analog and Digital Systems**

SNR in an analog systems is

\[
SNR_a = \frac{S}{N},
\]

(9.36)

where \( S \) is the signal power, \( N \) is the noise power.

Note that \( S = E / T_s \) and \( R_s = 1 / T_s \). Furthermore, \( N = N_0 \cdot \Delta f \). Hence

\[
\gamma = \frac{E_s}{N_0} = \frac{S \cdot T_s}{N / \Delta f} = \frac{S}{N} \cdot \frac{\Delta f}{R_s} = SNR_a \frac{\Delta f}{R_s}
\]

(9.37)

For \( R_s = \Delta f \), they are the same. \( \frac{\Delta f}{R_s} \) is inverse of the bandwidth efficiency, \( R_s / \Delta f \) (= \( \eta \)).

Similarly,

\[
\gamma_b = \frac{E_b}{N_0} = SNR_a \frac{\Delta f}{R}
\]

(9.38)

where \( R \) = bit rate (bit/s).
**SER and Signal Constellation**

SER can be expressed through minimum distance between points of the signal constellation.

Generic bound for the SER is (coherent detection),

\[
P_e \leq (M - 1) Q \left( \frac{d_{\text{min}}^2}{2N_0} \right)
\]  

(9.39)

where \(d_{\text{min}}\) is determined from signal constellation (minimum distance), see the next slide.

An approximation at high SNR:

\[
P_e \approx N_e Q \left( \frac{d_{\text{min}}^2}{2N_0} \right)
\]  

(9.40)

where \(N_e\) = # of nearest neighbours.

The BER can be approximated at high SNR as

\[
P_b \approx \frac{1}{\log_2 M} P_e
\]  

(9.41)

Q.: what is an interpretation of (9.41)?
Comparison of Various Modulation Formats

Fundamental limit is provided by the Shannon’s channel capacity theorem (AWGN channel):

\[ C = \Delta f \log_2 (1 + \gamma) \]  

(9.42)

\( C = \) channel capacity [bit/s]
\( \Delta f = \) bandwidth [Hz]
\( \gamma = \) SNR, \( \gamma = P / N \), where \( P \) - signal power, \( N \) - noise power.

Almost error-free transmission is possible if \( R < C \), and is not possible for \( R > C \); \( R = \) bit rate [b/s].

Using \( E_b = P T_b \), \( \gamma_b = \frac{E_b}{N_0} = \gamma \frac{\Delta f}{R} \),

\[ \frac{C}{\Delta f} = \log_2 \left( 1 + \gamma_b \frac{R}{\Delta f} \right) \]  

(9.43)

Since \( R < C \) for reliable communications, then

\[ \frac{R}{\Delta f} < \log_2 \left( 1 + \gamma_b \frac{R}{\Delta f} \right) \]  

(9.44)

\( R / \Delta f \) (bit/s/Hz) is the spectral efficiency. Required SNR is

\[ \gamma_b > \frac{2^{R/\Delta f} - 1}{R / \Delta f} \geq \ln 2 = -1.6dB \]  

(9.45)

LB is monotonically increasing in \( R / \Delta f \) - power/bandwidth efficiency tradeoff (see the tables).
Fundamental Limit: Spectral Efficiency [bit/s/Hz] vs. SNR/bit [dB]

![Graph showing spectral efficiency vs. SNR]

\[ \gamma_b = \frac{2^{R/\Delta f} - 1}{R/\Delta f} \geq -1.6 \text{dB} \]
Important Conclusions:

Power-limited region: for large bandwidth available. For \( f \Delta \to \infty \),

\[
\ln 2 \cdot 1.6 \text{dB} = \gamma \Rightarrow \gamma < -1.6 \text{dB}.
\]

No error-free transmission is possible for \( \gamma < -1.6 \text{dB} \)!

Bandwidth-limited region: for large power available. Trade-off: increasing power efficiency decreases spectrum efficiency and vice versa.

Note: for \( f \Delta \to \infty \),

\[
0 \leq \frac{C}{P}N_0 = 0.5 \ln 2.
\]

Spectral Efficiency of Digital Modulation:

Null-to-null bandwidth (or absolute bandwidth assuming a raised cosine pulse with \( \alpha = 1 \)) of a modulated signal, \( M \)-PSK and \( M \)-QAM:

\[
2 \log_2 R f M = \Delta
\]

or

\[
2 \log_2 R f M = \Delta
\]

(9.46)

\( M \)-FSK (non-coherent):

\[
2 \log_2 R f M = \Delta
\]

(9.47)

\( M \)-PSK(QAM) is much better than \( M \)-FSK in terms of spectral efficiency.

Most high-rate systems use \( M \)-QAM, \( M \geq 4 \).

Figure 9.6 Bandwidth-efficiency plane.
Spectrum of Digital Modulation (RF)

![Graph of digital modulation spectrum](image)

- BPSK
- QPSK
- 8PSK
- MSK

(f - fc) * Tb dB

0 0.5 1 1.5 2

-60 -40 -20 0 20 40 60 dB

0 0.5 1 1.5 2 (f - fc) * Tb

Lecture 9
BER: Comparison of Various Modulation Formats

BER in AWGN channels

SNR: Eb/N0 [dB]

Perror

OOK
BPSK
DPSK
QPSK
8PSK
8QAM
Coherent BFSK
Non-coherent BFSK

+ + + OOK

BPSK

DPSK

QPSK

8PSK

8QAM

Coherent BFSK

Non-coherent BFSK

Maximum achievable rate with coding:

\[ R_{\text{max}} = \frac{\log_2(1 + \text{SNR})}{1 - h(P_e)} \quad \text{[bit/ symbol]} \quad (9.49) \]

\[ h(P_e) = -P_e \log_2 P_e - (1 - P_e) \log_2 (1 - P_e) = \text{binary entropy} \]
SOME REFERENCES ON CHANNEL CODING/MODULATION:

Books:


Review Papers:

STATE-OF-THE-ART (IN OPTICAL COMMUNICATIONS)

Record-High 17.3-bit/s/Hz Spectral Efficiency Transmission over 50 km Using Probabilistically Shaped PDM 4096-QAM

Samuel L.I. Olsson¹, Junho Cho¹, Sethumadhavan Chandrasekhar¹, Xi Chen¹, Ellsworth C. Burrows¹, and Peter J. Winzer¹

¹Nokia Bell Labs, Holmdel, New Jersey, 07733, United States

16384-QAM TRANSMISSION AT 10 GBD OVER 25-KM SSMF USING POLARIZATION-MULTIPLEXED PROBABILISTIC CONSTELLATION SHAPING

Xi Chen, Junho Cho, Andrew Adaniecki, and Peter Winzer

Nokia Bell Labs, Holmdel, New Jersey, United States

10.66 Peta-Bit/s Transmission over a 38-Core-Three-Mode Fiber

Georg Rademacher¹, Benjamin J. Puttnam¹, Ruben S. Luís¹, Jun Sakaguchi¹, Werner Klaus¹, Tobias A. Eriksson¹,², Yoshinari Awaži¹, Tetsuya Hayashi³, Takuji Nagashima³, Tetsuya Nakanishi³, Toshiki Taru³, Taketoshi Takahata⁴, Tetsuya Kobayashi⁴, Hideaki Furukawa¹, and Naoya Wada¹

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²Royal Institute of Technology (KTH), AlbaNova University Center, 106 91 Stockholm, Sweden
³Sumitomo Electric Industries, Ltd., 1 Toya-cho, Nakae-ku, Yokohama 244-8588, Japan
⁴Optoquest Co. Ltd., 1335 Harachi, Ageo, Saitama 362-0021, Japan

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Summary

- Geometric representation of signals via signal space.
- Optimum receiver (MAP, ML) in the signal space.
- BPSK, QPSK, QAM.
- M-ary modulation formats. Comparison.
- Power and bandwidth efficiency.
- BER and SER.
- Fundamental limits. Channel capacity.

Reading:

- Rappaport, Ch. 6 (expect 6.11, 6.12).
- Other books (see the reference list).

Note: Do not forget to do end-of-chapter problems. Remember the learning efficiency pyramid!
Appendix: Optimal Rx

Optimal Rx selects \( m = m_k \) with largest a posteriori probability:

\[
m = m_k \quad \text{if} \quad \Pr\{m_k|r\} \geq \Pr\{m_i|r\} \quad \forall i \neq k \tag{9.50}
\]

\[
\Pr\{m_k|r\} = \frac{\Pr\{r|m_k\}\Pr\{m_k\}}{\Pr\{r\}} \tag{9.51}
\]

To prove its optimality, observe that the probability of correct decision \( \Pr\{c|m_k\} \) given that the Rx selects \( m = m_k \) is \( \Pr\{c|m_k\} = \Pr\{m_k|r\} \), which is maximized by (9.50).

This rule also maximizes unconditional probability of correct decision \( P_c = 1 - P_e \):

\[
P_c = \int \Pr\{r\}\Pr\{c|r\}dr, \quad \Pr\{c|r\} = \sum_{k=1}^{M} \Pr\{m_k|r\}\Pr\{m_k\} \tag{9.52}
\]

From (9.50), (9.51), the maximum a posteriori probability (MAP) decision rule follows:

\[
m = m_k \quad \text{if} \quad \Pr\{r|s_k\}\Pr\{s_k\} \geq \Pr\{r|s_i\}\Pr\{s_i\} \quad \forall i \neq k \tag{9.53}
\]

since \( \Pr\{s_i\} = \Pr\{m_i\} \), \( \Pr\{r|s_i\} = \Pr\{r|m_i\} \), i.e. decide in favor of such \( m_k \) that maximizes a posteriori probability of observed \( r \).
From (9.15),

$$\Pr\{r|s_k\} = P_{\xi}(r - s_k) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{|r - s_k|^2}{N_0}\right)$$

(9.54)

So that the decision rule in (9.53) becomes

$$m = m_k \text{ if } |r - s_k|^2 + c_k \leq |r - s_i|^2 + c_i, \forall i \neq k$$

(9.55)

where $c_k = -N_0 \ln \Pr\{m_k\}$.  