Question 1: A square-wave (digital) test signal of 50\% duty cycle phase modulates a transmitter where \( s(t) = 10 \cos(\omega_c t + \theta(t)) \).

The carrier frequency is 60 MHz and the peak phase deviation is 90\(^\circ\). Assume that the test signal is of the unipolar NRZ type with a period of 1 ms and that it is symmetrical about \( t=0 \). Find the exact spectrum of \( s(t) \).

Solution:

First, if we want to sketch the \( m(t) \) based on the information provided in the question, we would have the figure below.
Question 1: A square-wave (digital) test signal of 50% duty cycle phase modulates a transmitter where \( s(t) = 10 \cos (\omega_t + \theta(t)) \). The carrier frequency is 60 MHz, and the peak phase deviation is 90°. Assume that the test signal is of the unipolar NRZ type, with a period of 1 ms and that it is symmetrical about \( t=0 \). Find the exact spectrum of \( s(t) \).

Solution:

First, if we want to sketch the \( m(t) \) based on the information provided in the question, we would have the figure below.
From the reference book we know that a BPSK signal is represented by
\[ s(t) = A_c \cos \left( \omega_c t + D_p m(t) \right) \]
where \( D_p = R \Delta \theta \) (value of the peak deviation) \[ \Delta \theta = \frac{\pi}{2} \]
If we want to obtain the spectrum of \( s(t) \) we first need to obtain the Fourier transform.
\[ S(f) = 10 \cos \left( \omega_c t + \Delta \theta m(t) \right) \]
\[
\left( \cos \left( \frac{\theta - \phi}{2} \right) \right) = \frac{10}{2} \left( e^{j(\omega_c t + \Delta \theta m(t))} + e^{-j(\omega_c t + \Delta \theta m(t))} \right) \\
= 5 \left( e^{j\omega_c t} e^{-j \Delta \theta m(t)} + e^{-j \omega_c t} e^{j \Delta \theta m(t)} \right)
\]
Let's consider \( y(t) = e^{j \Delta \theta m(t)} \) then \( y^*(t) = e^{-j \Delta \theta m(t)} \).
We also know that \( m(t) \) is a periodic function then \( y(t) \) would be a periodic function too. Therefore we need to obtain the Fourier series for it.
\[
y(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn \frac{2 \pi}{T_m} t}, \quad T_m = 1 \text{ ms}
\]
\[ C_n = \frac{1}{T_m} \int_T^T y(t) e^{-jn \frac{2\pi}{T_m} t} \, dt \]

\[ = \frac{1}{T_m} \int_{\frac{T_m}{4}}^{\frac{3T_m}{4}} y(t) e^{-jn \frac{2\pi}{T_m} t} \, dt \]

\[ = \frac{1}{T_m} \left[ \int_{\frac{T_m}{4}}^{\frac{\frac{T_m}{4}}{4}} e^{j\alpha t} e^{-jn \frac{2\pi}{T_m} t} \, dt + \int_{\frac{\frac{T_m}{4}}{4}}^{\frac{\frac{T_m}{4}}{4}} e^{-j\alpha t} e^{-jn \frac{2\pi}{T_m} t} \, dt \right] \]

\[ = \frac{1}{T_m} \left[ e^{j\Delta \theta} \frac{e^{-jn \frac{2\pi}{T_m} t}}{-jn \frac{2\pi}{T_m}} \bigg|_{\frac{T_m}{4}}^{\frac{\frac{T_m}{4}}{4}} + e^{-j\Delta \theta} \frac{e^{jn \frac{2\pi}{T_m} t}}{-jn \frac{2\pi}{T_m}} \bigg|_{\frac{T_m}{4}}^{\frac{\frac{T_m}{4}}{4}} \right] \]

\[ = \frac{1}{T_m} \cdot \frac{1}{n\pi} \left[ e^{j\Delta \theta} \sin \left( n\pi \frac{\frac{T_m}{4}}{2} \right) + e^{-j\Delta \theta} \left( \frac{e^{jn\pi \frac{\frac{T_m}{4}}{2}}}{-2j} - \frac{e^{-jn\pi \frac{\frac{T_m}{4}}{2}}}{-2j} \right) \right] \]

\[ = \frac{1}{n\pi} \left[ e^{j\Delta \theta} \sin \left( n\frac{\frac{T_m}{4}}{2} \right) + e^{-j\Delta \theta} \left( \frac{e^{jn\pi \frac{\frac{T_m}{4}}{2}}}{-2j} - \frac{e^{-jn\pi \frac{\frac{T_m}{4}}{2}}}{-2j} \right) \right] \]

\[ = \frac{1}{n\pi} \left[ e^{j\Delta \theta} \sin \left( n\frac{\frac{T_m}{4}}{2} \right) + \gamma^n \sin \left( n\frac{\frac{T_m}{4}}{2} \right) \right] \]

\[ = \frac{\sin \left( n\frac{\frac{T_m}{4}}{2} \right)}{n\pi} \left| \gamma^n + e^{j\Delta \theta} \right| = \frac{1}{\pi} \sin^{\gamma} \left( \frac{n\pi}{2} \right) \left| (\gamma^n) + e^{j\Delta \theta} \right|
Now applying $\Delta \theta = \frac{\pi}{2}$

$$C_n = \frac{1}{2} \sin \left( \frac{\pi}{2} (e^{i\frac{\theta}{2}} + (-1)^n) \right) = \frac{1}{2} \sin \left( \frac{\pi}{2} (e^{i\theta} + (-1)^n) \right)$$

$$S_Y(f) = \sum_{n=-\infty}^{\infty} C_n S(f - nf_0), f_0 = \frac{1}{T_m}$$

Back to $s(t) = 5 (e^{j2\pi f_c t} y(t) + e^{-j2\pi f_c t} y^*(t))$

Considering that we know $e^{j2\pi f_c t} y(t) \xrightarrow{F.T.} S_Y(f-f_c)$

$$F.T. \{ s(t) \} = 5 \left( S_Y(f-f_c) + S_Y^*(f+f_c) \right)$$

$$= 5 \left( \sum_{n=-\infty}^{\infty} C_n S(f-nf_0-f_c) + \sum_{n=-\infty}^{\infty} C_n^* S(f-nf_0+f_c) \right)$$

$$= 5 \left( \sum_{n=-\infty}^{\infty} C_n S(f-nf_0-f_c) + \sum_{n=-\infty}^{\infty} C_n^* S(f+nf_0+f_c) \right)$$

$$f_0 = \frac{1}{T_m}, \quad T_m = 1 \text{ ms} \quad \Rightarrow \quad f_0 = 1 \text{ kHz}$$

$$f_c = \frac{\omega_c}{2\pi} = 60 \text{ MHz}$$
In order to obtain the bandwidth we should consider $C_n$ values.

$$C_n = 0 \implies \frac{1}{2} \left( e^{i \theta} + e^{-i \theta} \right) \text{sinc} \left( \frac{n \pi}{2} \right) = 0$$

$$\text{sinc} \left( \frac{n \pi}{2} \right) = 0 \implies \sin \left( \frac{n \pi}{2} \right) = 0$$

$$\frac{n \pi}{2} = k \pi \text{ for } k = 1, 2, \ldots \implies 1\text{st null } \implies k = 1$$

$$\frac{n \pi}{2} = \pi \implies n = 2$$

$$\Delta f = 4f_0 = 4 \times 1 \text{kHz} = 4 \text{kHz}$$
Question 2

In a binary PCM system, the output signal-to-quantization-noise is to be held to a minimum of 40 dB. Determine the number of required levels and find the corresponding output signal-to-quantization-noise ratio.

Solution:

First case: Sinusoid amplitude is equal to the level of the maximum allowed level.

Based on lecture notes we have \( SQNR = -\beta + 4.8 + 6n \)

\[ \beta = \frac{\eta_{\text{max}}^2}{P_n} \quad \text{and} \quad \eta(t) = A \sin(wt) \Rightarrow \begin{cases} \eta_{\text{max}} = A \\ P_n = \frac{A^2}{2} \end{cases} \Rightarrow \beta = \frac{\frac{A^2}{2}}{\frac{A^2}{2}} = 2 = 3\text{dB} \]

\[ SQNR = -3 + 4.8 + 6n = 1.8 + 6n \]

We want \( SQNR \gg 40 \Rightarrow 1.8 + 6n > 40 \Rightarrow n > 6.36 \Rightarrow n = 7 \]
Second case: The sinusoid amplitude is half of the maximum allowed level of the quantizer.

\[ \beta = \frac{\hat{x}_{\text{max}}}{P_x} \]
\[ x(t) = A \sin(\omega t) \Rightarrow \text{Sinusoid amplitude} = A \]
\[ \hat{x}_{\text{max}} = 2x \text{ sinusoid amplitude} = 2A \]

\[ \beta = \frac{\hat{x}_{\text{max}}}{P_x} = \frac{4A^2}{A^2/2} = 8 = 9 \text{dB} \]

\[ \text{SNR} = -9 + 4.8 + 6n = -4.2 + 6n \]

\[ \text{SNR} = -4.2 + 6n \Rightarrow n = \frac{1}{6} (\text{SNR} + 4.2) = \frac{1}{6} (40 + 4.2) \]
\[ = \frac{44.2}{6} \Rightarrow n = 8 \]

One more bit is required because of the 6-dB rule.