Angle Modulation

- Angle modulation: frequency modulation (FM) or phase modulation (PM).
- Basic idea: vary frequency (FM) or phase (PM) according to the message signal.
- While AM is (almost) linear, FM or PM is highly nonlinear.
- FM/PM provide many advantages (main – noise immunity) over AM, at a cost of larger bandwidth.
- Demodulation may be complex, but modern ICs allow cost-effective implementation.
- Example: FM radio (high quality, not expensive receivers).
Angle Modulation: Basic Definitions

- Angle-modulated signal (PM or FM) can be expressed as:
  \[ x(t) = A_c \cos(\psi(t)) \]

- Phase modulation:
  \[ \psi(t) = \omega_c t + \varphi(t), \quad \varphi(t) = \Delta \varphi \cdot m(t) \]

- Frequency modulation:
  \[ \psi(t) = \omega_c t + \int_0^t \Omega(\tau)d\tau, \quad \Omega(t) = \Delta \Omega \cdot m(t) \]

  for a short period of time (small \( t \)): \( \psi(t) \approx [\omega_c + \Omega(0)]t + \varphi_0 \)

- **Max phase deviation**: \( \Delta \varphi = \text{Max} \{|\varphi(t)|\} = \text{Max} \{|\psi(t) - \omega_c t|\} \)

- **Max frequency deviation**: \( \Delta \Omega = \text{Max} \{|\Omega(t)|\} = \text{Max} \{|\omega(t) - \omega_c|\} \)

- **Normalized message signal**: \( |m(t)| \leq 1 \)

Note: deviation is w.r.t. unmodulated value.
Angle Modulation: Parameters

- Instantaneous frequency:

\[
\omega(t) = \frac{d\psi(t)}{dt} = \begin{cases} 
\omega_c + \frac{d\phi(t)}{dt} &= \omega_c + \Delta\phi \frac{dm(t)}{dt}, \\
\omega_c + \Omega(t) &= \omega_c + \Delta\Omega \cdot m(t), 
\end{cases} 
\]

- Instantaneous phase:

\[
\psi(t) = \int_0^t \omega(\tau)d\tau = \begin{cases} 
\omega_c t + \phi(t) &= \omega_c t + \Delta\phi \cdot m(t), \\
\omega_c t + \int_0^t \Omega(\tau)d\tau &= \omega_c t + \Delta\Omega \int_0^t m(\tau)d\tau, 
\end{cases} 
\]

- Effect of mod. signal amplitude:

\[M(t) = A \cdot m(t), \quad \max\left[|m(t)|\right] = 1\]

\[
\begin{cases} 
\Delta\phi = k_p A, \\
\Delta\Omega = k_f A 
\end{cases} 
\]

\[k_f, k_p - \text{modulation constants}, \quad \text{Hz/V & rad./V} \]

measured in lab 3.
Angle Modulation: Examples

rectangular pulse

sawtooth pulse
Example: Sinusoidal Modulating Signal

- Assume that \( m(t) = \cos(\omega_m t) \)

- Instantaneous phase:

\[
\psi(t) = \begin{cases} 
\omega_c t + \Delta \varphi \cdot \cos(\omega_m t), & PM \\
\omega_c t + \frac{\Delta \Omega}{\omega_m} \sin(\omega_m t), & FM 
\end{cases}
\]

- Modulated signal:

\[
x(t) = \begin{cases} 
A_c \cos\left[\omega_c t + \Delta \varphi \cdot \cos(\omega_m t)\right], & PM \\
A_c \cos\left[\omega_c t + \frac{\Delta \Omega}{\omega_m} \sin(\omega_m t)\right], & FM 
\end{cases}
\]

- Modulation indices:

\[
\begin{cases} 
\beta_p = \Delta \varphi, & PM \\
\beta_f = \frac{\Delta \Omega}{\omega_m}, & FM 
\end{cases}
\]

Valid in general case as well, with \( \omega_m \to \text{max. modulating frequency} \)
Spectrum of Angle-Modulated Signal

- Consider sinusoidal modulating signal:
  \[ x(t) = A_c \cos[\omega_c t + \beta \cdot \sin(\omega_m t)] = \text{Re} \left[ A_c e^{j\beta \cdot \sin(\omega_m t)} e^{j\omega_c t} \right] \]

- Complex envelope is expanded in Fourier series:
  \[ C(t) = A_c e^{j\beta \cdot \sin(\omega_m t)} = A_c \sum_{n=-\infty}^{\infty} c_n e^{j n\omega_m t} \]

- Expansion coefficients are
  \[ c_n = \frac{1}{T_m} \int_{0}^{T_m} e^{j\beta \cdot \sin \omega_m t} e^{-j n\omega_m t} dt = \frac{1}{2\pi} \int_{0}^{2\pi} e^{j(\beta \sin u - nu)} du = J_n(\beta) \]

- Finally,
  \[ x(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos \left[ (\omega_c + n\omega_m) t \right] \]

\( J_n(\beta) \) - Bessel function of 1st kind & n-th order, \( J_{-n}(\beta) = (-1)^n J_n(\beta) \)
Spectrum of Angle Modulation: $J_n(\beta)$

*Plots of Bessel Functions $J_n(\beta)$*

Increasing $n$ moves the peak to the right!
## Spectrum of Angle Modulation: $J_n(\beta)$

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<th>$n$</th>
<th>$\beta = 0.1$</th>
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<th>$\beta = 0.5$</th>
<th>$\beta = 1$</th>
<th>$\beta = 2$</th>
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</table>
Spectrum: Examples

\[ \beta = 0.3 \]

\[ \beta = 1 \]

\[ \beta = 10 \]

\[ \beta = 5 \]
Bandwidth of Angle-Modulated Signal

• Power bandwidth (98% of the power) of angle-modulated signal (Carson’s rule):
  \[ \Delta \omega \approx 2(\beta + 1) \omega_m \]

• Power bandwidth of PM and FM signals:
  \[ \Delta \omega \approx 2(\beta + 1) \omega_m = \begin{cases} 2(\Delta \phi + 1) \omega_m, & PM \\ 2(\Delta \Omega + \omega_m), & FM \end{cases} \]

• These expressions hold for a general modulating signal as well, \( \omega_m \) - the max. modulating frequency.
• Angle modulation with large index expands spectrum!
Narrowband Angle Modulation

- Modulation index is low, $\beta << 1$
- Modulated signal can be expressed as:

\[
x(t) = A_c \cos\left[\omega_c t + \beta \cdot \sin(\omega_m t)\right] = \\
= A_c \cos(\omega_c t) + \frac{A_c \beta}{2} \cos(\omega_c + \omega_m) t - \frac{A_c \beta}{2} \cos(\omega_c - \omega_m) t
\]

- Similar to AM signal, the bandwidth is (both, PM & FM)

$$2|S_x(f)|$$

\[
\frac{A_c \beta}{2}, 180^0, \frac{A_c \beta}{2}, 0^0
\]

$$\Delta \omega \approx 2\omega_m$$
Wideband Angle Modulation

• Modulation index is high, $\beta >> 1$

• The signal bandwidth is:

\[
\Delta \omega \approx 2\beta \omega_m = \begin{cases} 
2\Delta \phi \cdot \omega_m, & \text{PM} \\
2\Delta \Omega, & \text{FM}
\end{cases}
\]

• Different for PM and FM!

• **Wideband FM**: the bandwidth is twice the frequency deviation. Does not depend on the modulating frequency.

• **Wideband PM**: the bandwidth depends on modulating frequency.

• Modulation index $\beta =$ bandwidth expansion factor.
PM Modulator

General principle:

Practical implementation:

\[ \Delta \phi \approx k \Delta f, \quad \Delta f = f_c - f_0 \]

\( k = \text{modulation constant} \)
\( f_0 = \text{resonant frequency} \)

Q.: sketch \( \Delta \phi(\Delta f) \)

FM Modulator

General principle:

variable resonant circuit

RF oscillator

\[ x(t) = A_c \cos \left( \omega_c t + \int_0^t \Omega(\tau) d\tau + \phi_0 \right) \]

Practical implementation:

Difficulty: frequency stability.

Suitable for narrowband FM only.

Narrowband Angle Modulator

Small modulation index: $\beta \ll 1$

Message signal

$x(t) = A_c \cos[\omega_c t + \beta \cdot m(t)] \approx A_c \cos \omega_c t - A_c \sin \omega_c t \cdot \beta \cdot m(t)$

Q.: can an AM modulator be used instead?

Indirect Wideband Angle Modulator

Frequency multiplier:

\[ \cos^2 \psi(t) = \frac{1}{2} \left[ 1 + \cos(2\psi(t)) \right] \]

BPF

\[ \frac{1}{2} \cos(2\psi(t)) \]
Indirect Wideband FM Transmitter
(Amstrong)

Figure 5.10  Armstrong indirect FM transmitter.

B.P. Lathi, Modern Digital and Analog Communications Systems, Oxford University Press
Direct Wideband Angle Modulator

Explain how it operates
- Hint: consider it without feedback first
- Explain why feedback is required
- Explain why frequency divider is required

FM Demodulators

- FM-to-AM conversion:

\[ x(t) = A_c \cos \left[ \omega_c t + \int_0^t \Omega(\tau) d\tau + \varphi_0 \right] \]

\[ y(t) = A_c (H_0 + \alpha \Omega(t)) \cos[*] \]

- Possible candidate: \( H(f) = H_0 + \alpha (f - f_c) \) for \( |f - f_c| < \frac{B_c}{2} \)

- Possible candidate: \( |H(f)| = 2\pi f \) (differentiator)
FM Slope Detector

- Circuit diagram:

- Magnitude frequency response:
Balanced Discriminator: Block Diagram

Balanced Discriminator: Circuit Diagram

Phased Locked Loop (PLL) Detector

\[ v_{in}(t) = A_{in} \sin[\omega_c t + \varphi_{in}(t)] \]

\[ v_0(t) = A_0 \cos[\omega_c t + \varphi_0(t)] \]

\[ \omega_{VCO}(t) = \frac{d}{dt}(\omega_c t + \varphi_0(t)) = \omega_c + \alpha v_2(t) \]

\[ v_1(t) = \frac{A_1 A_2}{2} \sin[\varphi_{in}(t) - \varphi_0(t)] + (2\omega_c)\text{term} \]

Informally,

\[ \varphi_{in}(t) \approx \varphi_0(t) \]

\[ v_2(t) \approx \frac{1}{\alpha} \frac{d}{dt}\varphi_{in}(t) \]
PLL Detector: Linear Model

\[ v_1(t) = \frac{A_1A_2}{2} \sin[\varphi_{in}(t) - \varphi_0(t)] + (2\omega_c) \text{term} \]

\[ v_2(t) = \frac{A_1A_2}{2} \sin[\varphi_{in}(t) - \varphi_0(t)] \approx \frac{A_1A_2}{2} (\varphi_{in}(t) - \varphi_0(t)) = K_d \Delta \varphi(t) \]

\[ \frac{d}{dt} \varphi_0(t) = \alpha v_2(t) \]

\[ v_2(t) = \frac{1}{\alpha} \frac{d}{dt} [\varphi_{in}(t) - \Delta \varphi(t)] \approx \frac{1}{\alpha} \frac{d}{dt} \varphi_{in}(t) \]
FM Demodulator: Lab 3

- explain its operation!
Comparison of AM and FM/PM

- AM is simple (envelope detector) but no noise/interference immunity (low quality).
- AM bandwidth is twice or the same as the modulating signal (no bandwidth expansion).
- Power efficiency is low for conventional AM.
- DSB-SC & SSB – good power efficiency, but complex circuitry.
- More complex circuitry. However, ICs allow for cost-effective implementation.
Important Properties of Angle-Modulated Signals: Summary

- FM/PM signal is a nonlinear function of the message.
- The signal’s bandwidth increases with the modulation index.
- The carrier spectral level varies with the modulation index, being 0 in some cases.
- Narrowband FM/PM: the signal’s bandwidth is twice that of the message (the same as for AM).
- The amplitude of the FM/PM signal is constant (hence, the power does not depend on the message).
Summary

- Angle modulation: PM & FM
- Narrowband (low-index) & wideband (large-index) modulation. Signal bandwidth.
- Relation between PM and FM.
- Demodulation of PM and FM signals. Slope detector & balanced discriminator. PLL detector.
- Comparison of AM and FM/PM.

**Homework**: Reading: Couch, 5.6, 4.13, 4.14. Study carefully all the examples and make sure you understand them.