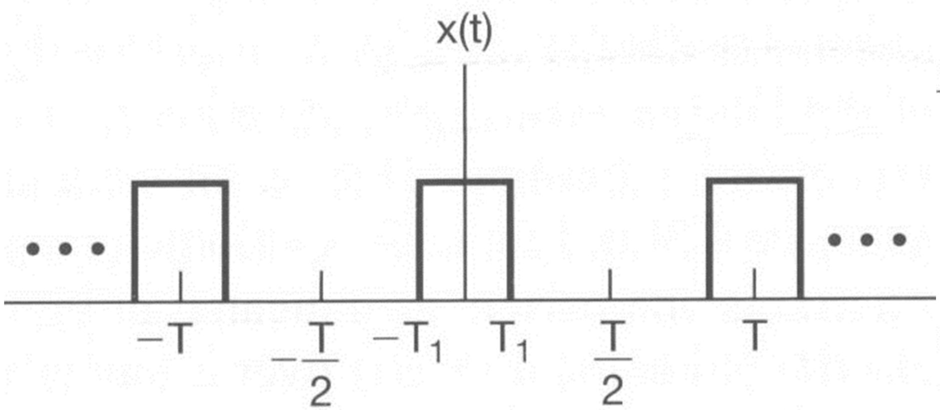
Review of Fourier Transform

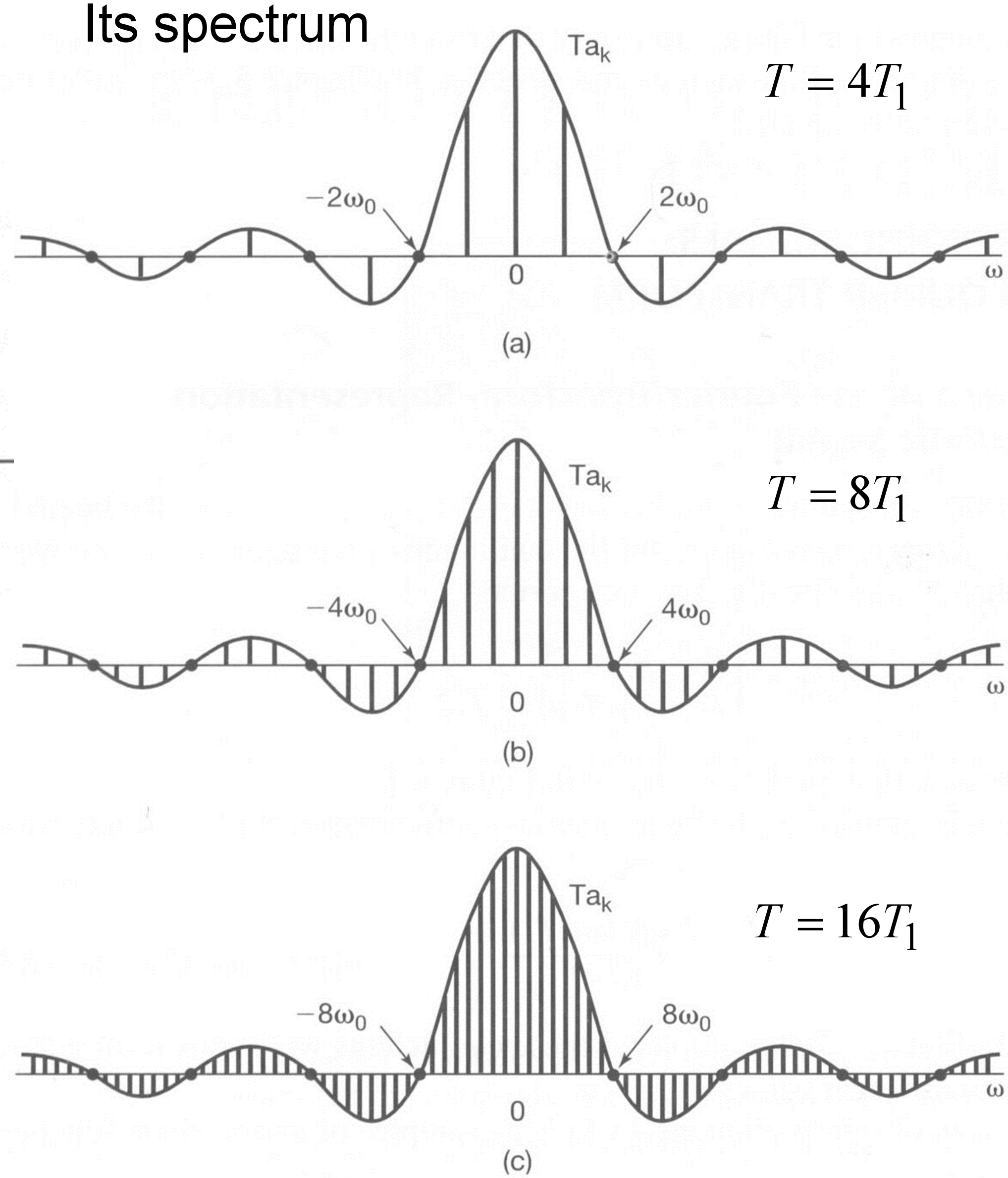
- Fourier series works for periodic signals only. What's about aperiodic signals? This is very large & important class of signals
- Aperiodic signal can be considered as periodic with $T \to \infty$
- Fourier series changes to Fourier transform, complex exponents are infinitesimally close in frequency
- Discrete spectrum becomes a continuous one, also known as spectral density

Fourier Series -> Fourier Transform

Periodic signal



Its spectrum



A.V. Oppenheim, A.S. Willsky, Signals and Systems, 1997.

As T increases, spectral components are getting closer and closer, becoming the continuous spectrum at the limit

Fourier Transform

- Fourier transform (spectrum):

$$S_x(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt$$

radial frequency

$$S_x(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

- Inverse Fourier transform:

$$x(t) = \int_{-\infty}^{+\infty} S_x(f) e^{j2\pi ft} df$$

radial frequency

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_x(\omega) e^{j\omega t} d\omega$$

- Existence:
  - Dirichlet conditions (details on the next page)
  - Bounded (polynomial at most) growth

Convergence of Fourier Transform

- Dirichlet conditions:
  - x(t) must be absolutely integrable or finite energy

$$\boxed{\int_{-\infty}^{\infty} \left|x(t)\right| dt < \infty} \quad \text{or} \quad \boxed{\int_{-\infty}^{\infty} \left|x(t)\right|^2 dt < \infty}$$

  - x(t) has a finite number of maxima, minima & discontinuities within any finite interval (discontinuities must be finite).
- Dirichlet conditions are only sufficient, but are not necessary.
- If |x(t)| grows not faster with |t| than a power -> OK.
  - singular functions are needed for FT in this case
- All physical (practical) signals meet these conditions.

Convergence of Fourier Transform

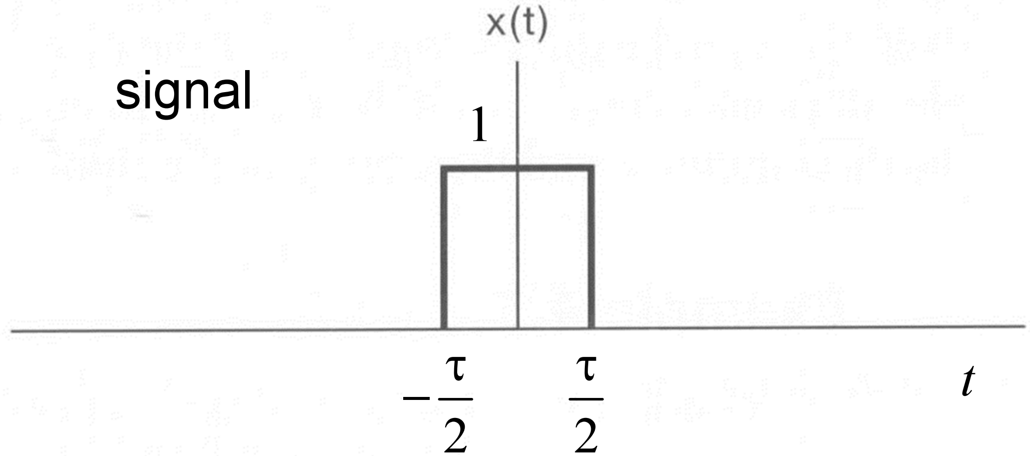
- Engineering/physics: Nature takes care of it for us.

- "…we may be confident that no one can generate a waveform without a spectrum or construct an antenna without a radiation pattern. … The question of the existence of transforms may safely be ignored when the function to be transformed is an accurately specified description of a physical quantity. Physical possibility is a valid sufficient condition for the existence of a transform." *R. Bracewell, The Fourier Transform and Its Applications, McGraw-Hill, 1999.*

Example: Rectangular Pulse

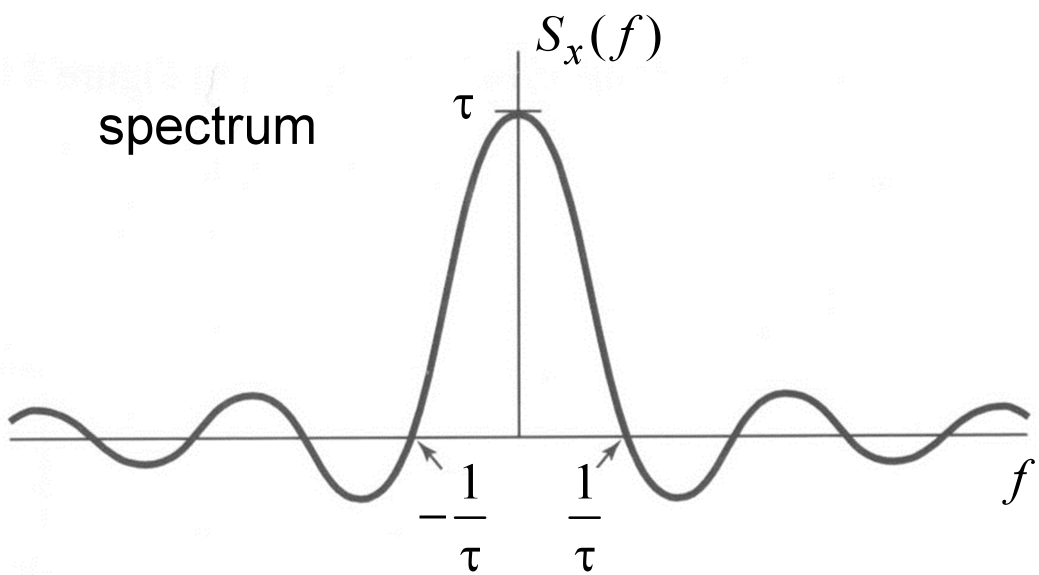
$$x(t) = \Pi(t/\tau) = \begin{cases} 1, |t| < \tau/2 \\ 0, |t| > \tau/2 \end{cases}$$

signal



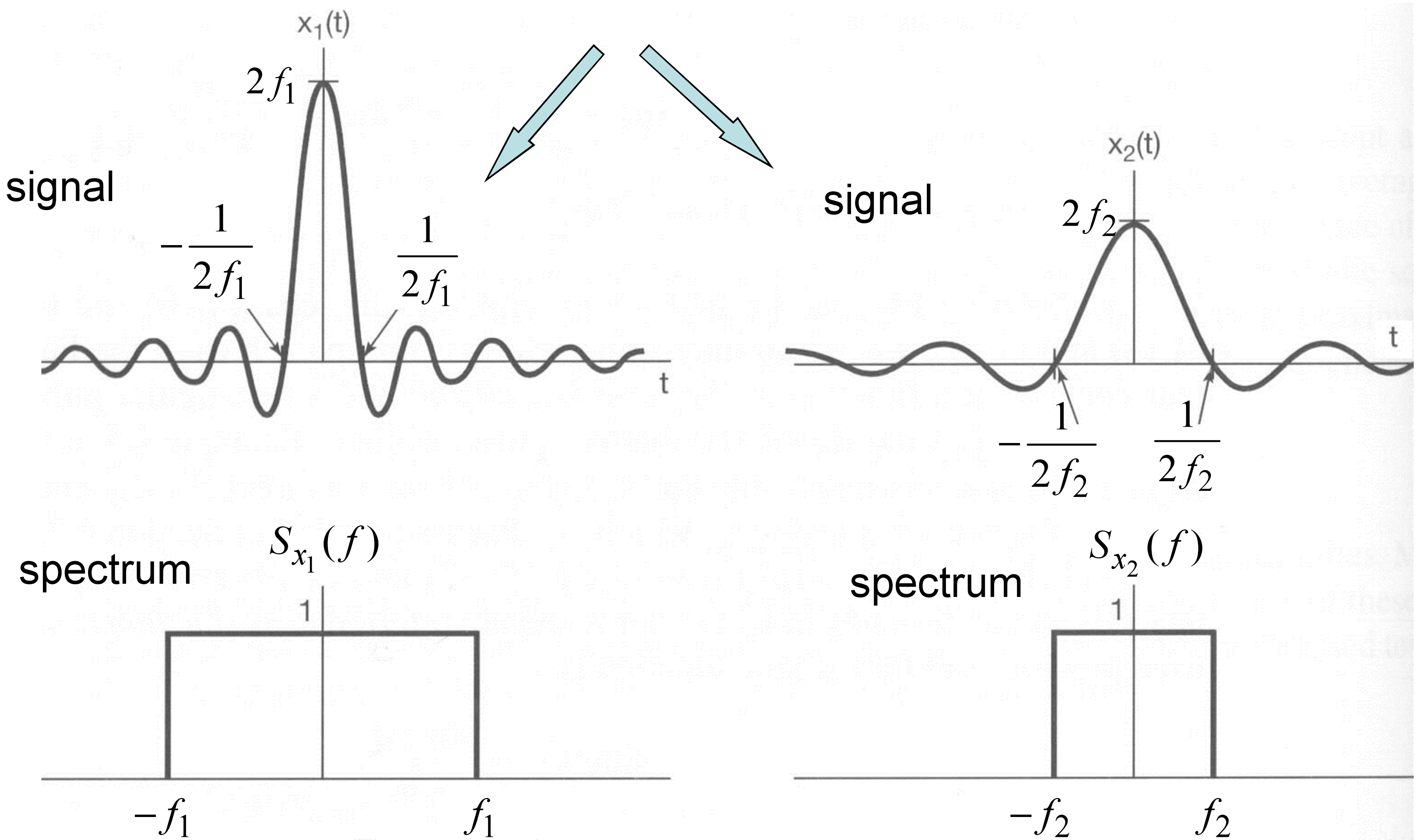
$$S_x(f) = \tau \frac{\sin \pi f \tau}{\pi f \tau} = \tau \cdot \mathrm{sinc}(f\tau)$$

spectrum



Example: sinc(t)

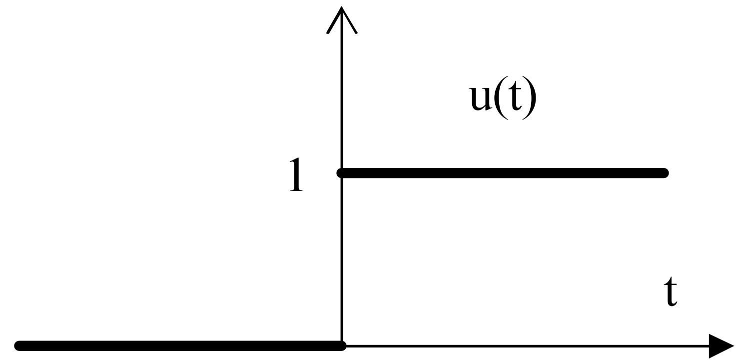
Shortening pulse widens its spectrum!



Generalized (singular) Functions: Why?

- Unit step function:

$$u(t)=\begin{cases}1, t>0\\ 0, t<0\end{cases}$$

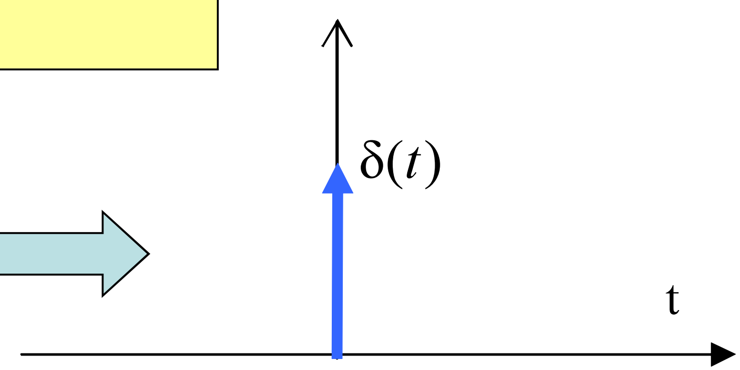


- Dirac delta function: *defined by its action, not values*

$$\int_{-\infty}^{\infty}\delta(t)x(t)dt=\int_{-\varepsilon}^{\varepsilon}\delta(t)x(t)dt=x(0)\quad\forall\varepsilon>0$$

Fourier transform?

$$\delta(t)=\begin{cases}0, t\neq 0\\ \infty, t=0\end{cases}\quad \& \quad \int_{-\infty}^{\infty}\delta(t)dt=1$$

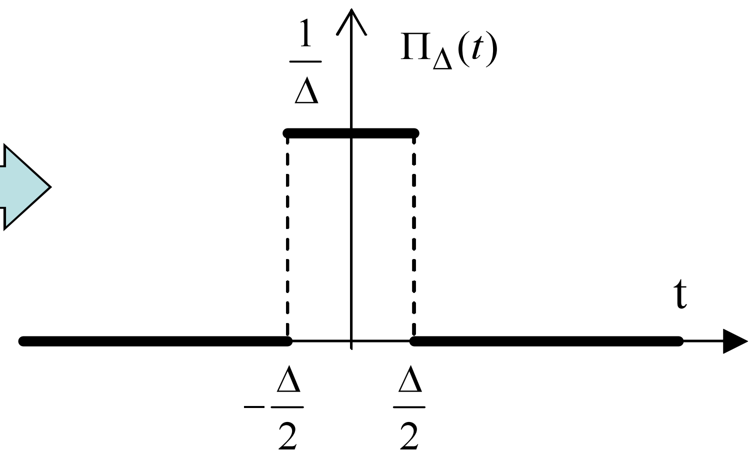


Generalized Functions

- Dirac Delta Function as a limit:

$$\Pi_\Delta(t) = \begin{cases} 1/\Delta, & |t| < \Delta/2 \\ 0, & |t| > \Delta \end{cases}$$

$$\boxed{\delta(t) = \lim_{\Delta \to 0} \Pi_\Delta(t)}$$



- In practice: small but non-zero, $\boxed{0 < \Delta \ll T \to \delta(t) \approx \Pi_\Delta(t)}$
- Examples: humans, systems/circuits (e.g. computer, cell phone etc.)

Useful properties of delta function

- Convolution property:

$$\boxed{\delta(t) * x(t) = \int_{-\infty}^{\infty} \delta(\tau) x(t-\tau) dt = x(t)}$$

- Integration & differentiation:

$$\boxed{\int_{-\infty}^{t} \delta(t) dt = u(t)} \qquad \boxed{\frac{du(t)}{dt} = \delta(t)}$$

- Scaling, symmetry, product:

$$\boxed{\delta(at) = \frac{1}{|a|}\delta(t)} \qquad \boxed{\delta(-t) = \delta(t)} \qquad \boxed{x(t)\delta(t) = x(0)\delta(t)}$$

Fourier Transform of Periodic Signal

- FT of a complex exponent:

$$\boxed{x(t) = e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0) = \delta(f - f_0)}$$

- Important property:

$$\boxed{\delta(f) = \int_{-\infty}^{+\infty} e^{\pm j2\pi ft} dt}$$

← prove this property

- FT of a periodic signal:

$$\boxed{x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} \overset{FT}{\leftrightarrow} 2\pi \sum_{n=-\infty}^{+\infty} c_n \delta(\omega - n\omega_0) = \sum_{n=-\infty}^{+\infty} c_n \delta(f - nf_0)}$$

- FT of $\cos(\omega_0 t)$?

Properties of Fourier Transform*

- *Very similar* to those of Fourier series!

- Linearity:

$$\boxed{\alpha x_1(t)+\beta x_2(t) \overset{F}{\leftrightarrow} \alpha S_{x_1}(f)+\beta S_{x_2}(f)}$$

- Time shifting:

$$\boxed{x(t) \leftrightarrow S_x(\omega) \Rightarrow x(t-t_0) \leftrightarrow e^{-j\omega t_0} S_x(\omega)}$$

- Time reversal:

$$\boxed{x(t) \leftrightarrow S_x(\omega) \Rightarrow x(-t) \leftrightarrow S_x(-\omega)}$$

- Time scaling:

$$\boxed{x(at) \leftrightarrow \frac{1}{|a|} S_x\left(\frac{\omega}{a}\right)}$$

Prove these properties.

*properties are useful for evaluating Fourier transform in a simple way

Properties of Fourier Transform

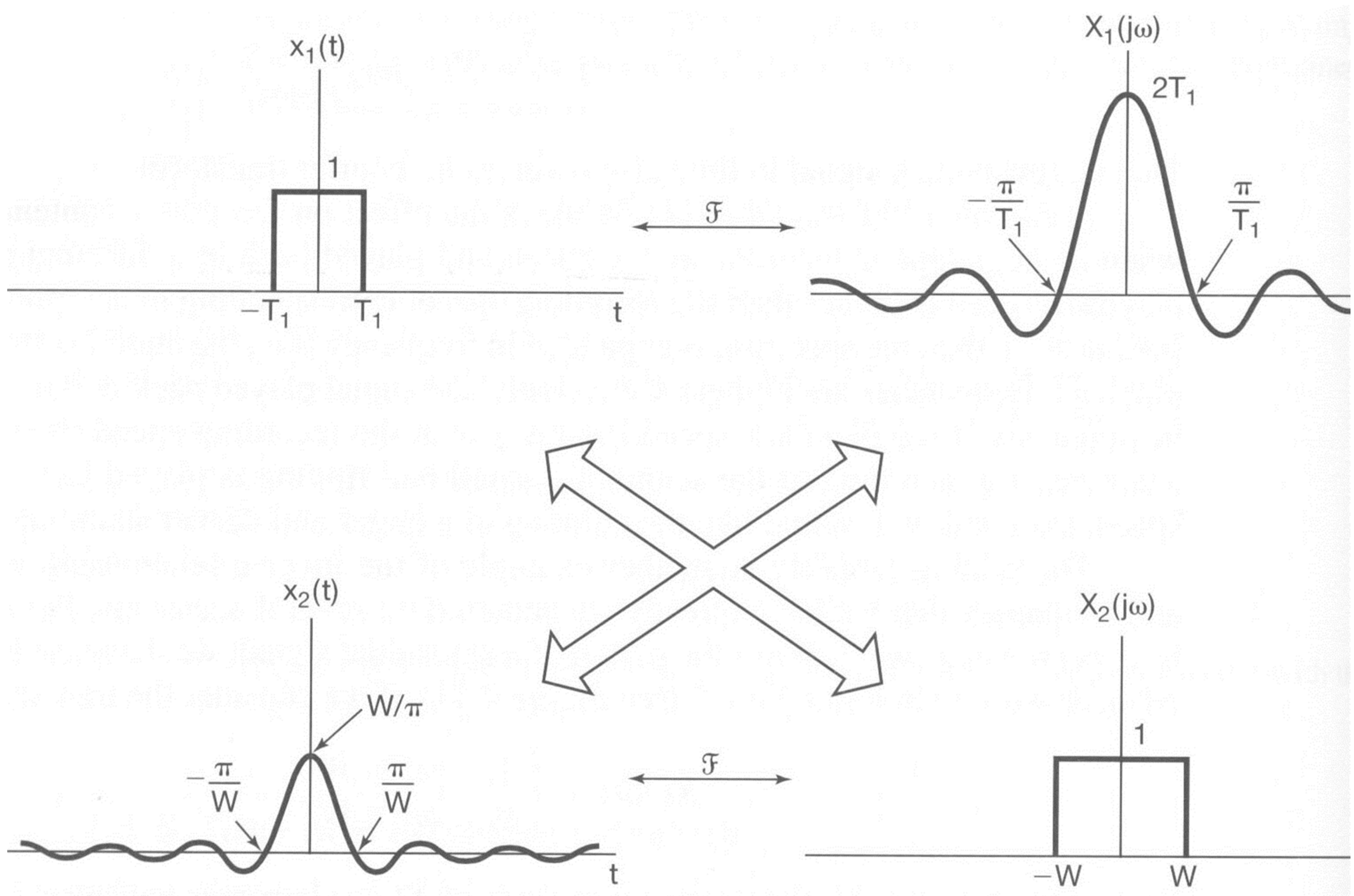
- Conjugation: $$x(t) \leftrightarrow S_x(\omega) \Rightarrow x^*(t) \leftrightarrow S_x^*(-\omega)$$
- Differentiation: $$x(t) \leftrightarrow S_x(\omega) \Rightarrow \frac{dx(t)}{dt} \leftrightarrow j\omega S_x(\omega)$$
- Integration: $$\int_{-\infty}^{t} x(t)dt \leftrightarrow \frac{1}{j\omega} S_x(\omega) + \pi S_x(0)\delta(\omega)$$
- Multiplication: $$x(t)y(t) \leftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega')S_y(\omega-\omega')d\omega' = \\ = S_x(\omega) * S_y(\omega)$$
- Frequency shift (modulation): $$x(t)e^{j\omega_0 t} \leftrightarrow S(\omega-\omega_0)$$

Prove these properties

Duality of Fourier Transform

$$\boxed{x(t) \leftrightarrow S_x(\omega) \Rightarrow S_x(t) \leftrightarrow 2\pi x(-\omega)}$$

$$\boxed{x(t) \leftrightarrow S_x(f) \Rightarrow S_x(t) \leftrightarrow x(-f)}$$

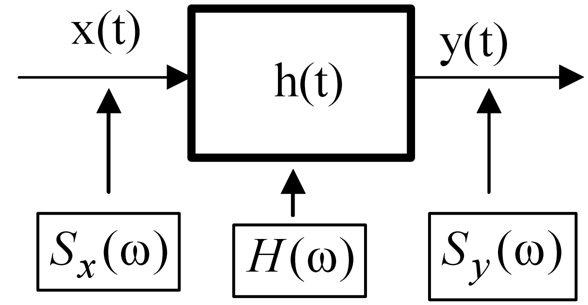


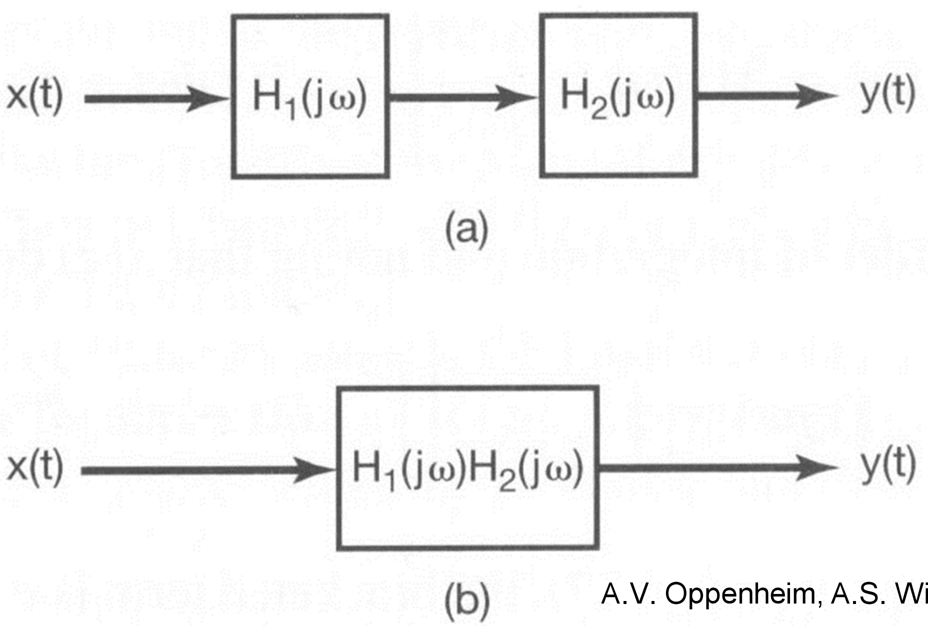
A.V. Oppenheim, A.S. Willsky, Signals and Systems, 1997.

Convolution Property

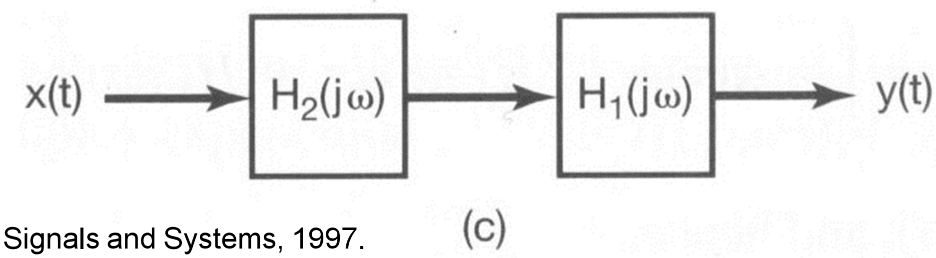
- This property is of great importance

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \leftrightarrow S_x(\omega)H(\omega) = S_y(\omega)$$





Cascade connection of LTI blocks



A.V. Oppenheim, A.S. Willsky, Signals and Systems, 1997.

- Example: FT of a triangular pulse by convolution

Parseval Theorem

- Total energy in time domain is the same as the total energy in frequency domain:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |S_x(f)|^2 df = \frac{1}{2\pi} \int_{-\infty}^{\infty} |S_x(\omega)|^2 d\omega$$

- $E(f) = |S_x(f)|^2$ - energy spectral density (ESD) of x(t). Represents the amount of energy per Hz of bandwidth
- Counterpart of Parseval theorem for periodic signals
- Autocorrelation property:

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)\, x^*(t-\tau)\, dt \leftrightarrow |S_x(\omega)|^2$$

$$R_x(0) = E$$

Parseval Theorem: Example

$$\int_{-\infty}^{+\infty} \left(\frac{\sin t}{t} \right)^2 dt = ?$$

Fourier Transform of Real Signal

- if x(t) is real, $\boxed{\mathrm{Im}\left\{x(t)\right\}=0 \Rightarrow S_x(-\omega)=S_x{}^*(\omega)}$

- Fourier transform can be presented as

$$\boxed{\begin{aligned} x(t) &= 2\int_0^{\infty}\left|S_x(f)\right|\cos\left(2\pi f+\varphi(f)\right)df, \\ \varphi(f) &= \tan^{-1}\left(\frac{\mathrm{Im}\left[S_x(f)\right]}{\mathrm{Re}\left[S_x(f)\right]}\right) \end{aligned}}$$

No negative frequencies!

Signal Bandwidth & Negative Frequencies

- What is negative frequency ?
- It means that there is $e^{-j2\pi ft}$ term in the signal spectrum
- Convenient mathematical tool. Do not exist in practice (i.e., cannot be measured on spectrum analyzer)
- What is the signal bandwidth? There are many definitions.
- Defined for positive frequencies only.
- Determines the frequency band over which a substantial part of the signal power/energy is concentrated.
- For band-limited signals

$$\boxed{\Delta f = f_{\max} - f_{\min}, \quad f_{\max}, f_{\min} \geq 0}$$

Power and Energy

- Power P_x & energy E_x of signal x(t) are:

$$\boxed{P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt}$$

$$\boxed{E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt}$$

- Energy-type signals: $E_x < \infty$
- Power-type signals: $0 < P_x < \infty$
- Signal *cannot* be both energy & power type!
- Signal energy: if x(t) is voltage or current, E_x is the energy dissipated in 1 Ohm resistor
- Signal power: if x(t) is voltage or current, P_x is the power dissipated in 1 Ohm resistor.

Energy-Type Signals (summary)

- Signal energy in time & frequency domains:

$$\boxed{E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |S_x(f)|^2 df = \frac{1}{2\pi}\int_{-\infty}^{\infty} |S_x(\omega)|^2 d\omega}$$

- Energy spectral density (energy per Hz of bandwidth):

$$\boxed{E_x(f) = |S_x(f)|^2}$$

- ESD is FT of autocorrelation function:

$$\boxed{R_x(\tau) = \int_{-\infty}^{+\infty} x(t)\, x^*(t-\tau)\, dt \leftrightarrow E_x(f)} \qquad \boxed{R_x(0) = E_x}$$

Power-Type Signals: PSD

- Definition of the power spectral density (PSD) (power per Hz of bandwidth):

$$\boxed{P_x(f) = \lim_{T\to\infty} \frac{|S_T(f)|^2}{T} \Rightarrow P_x = \int_{-\infty}^{\infty} P_x(f)\,df < \infty}$$

- where $x_T(t)$ is the truncated signal (to [-T/2,T/2]),

$$x_T(t) = x(t)\Pi\left(\frac{t}{T}\right) = \begin{cases} x(t), & -T/2 \le t \le T/2 \\ 0, & \text{otherwise} \end{cases}$$

- and $S_T(f)$ is its spectrum (FT),

$$S_T(f) = FT\{x_T(t)\}$$

Power-Type Signals

- Time-average autocorrelation function:

$$\boxed{R_x(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x^*(t-\tau)\, dt}$$

- Power of the signal:

$$\boxed{P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt = R_x(0)}$$

- Wiener-Khintchine theorem :

$$\boxed{P_x = \int_{-\infty}^{\infty} P_x(f)\, df \Rightarrow P_x(f) = FT\{R_x(\tau)\}}$$

Periodic Signals

- Power of a periodic signal:

$$\boxed{P_x = \frac{1}{T}\int_T |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2 = R_x(0)} \leftarrow x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}$$

- Autocorrelation function:

$$\boxed{R_x(\tau) = \frac{1}{T}\int_T x(t)\,x^*(t-\tau)\,dt = \sum_{n=-\infty}^{\infty} |c_n|^2 e^{jn\omega_0 \tau}}$$

- Power spectral density (PSD):

$$\boxed{P_x(f) = FT\{R_x(\tau)\} = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta\left(f - \frac{n}{T}\right)}$$

Prove these properties!

Relation Between Fourier Transform & Series

- consider a periodic signal x(t)=x(t+T)
- truncate it (one period only):

$$x_T(t) = \begin{cases} x(t), -T/2 < t \le T/2 \\ 0, \quad \text{otherwise} \end{cases}$$

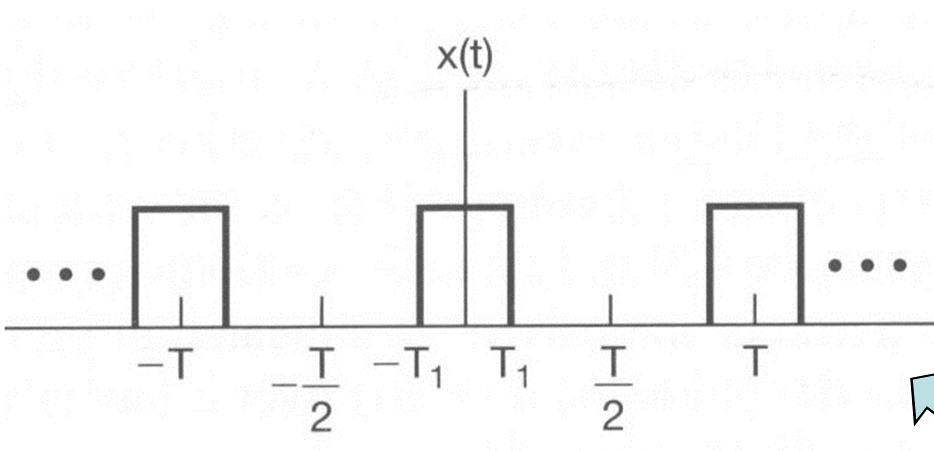
- find FT of the truncated signal $x_T(t)$: $x_T(t) \leftrightarrow S_{x_T}(\omega)$
- Fourier series of the original periodic signal x(t) is

$$\boxed{c_n = \frac{1}{T} S_{x_T}(n\omega_0)}$$

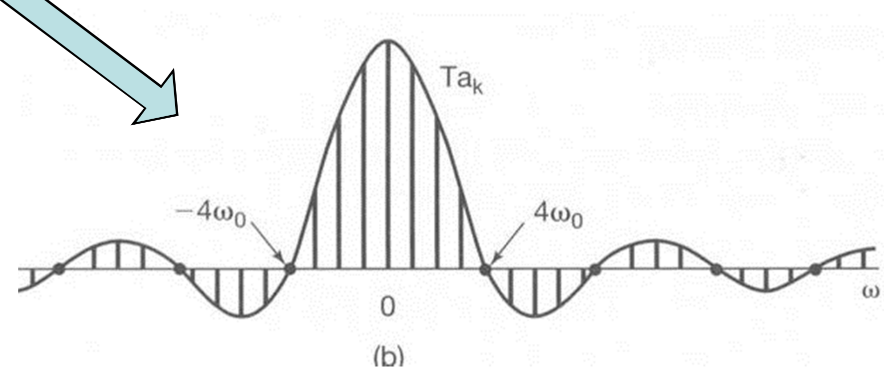
← prove this

- Continuous spectrum is the envelope of discrete spectrum (see slide 2)!

Fourier Transform of Single Pulse ~ Envelope of Fourier Series of Pulse Train



FT/S



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Summary

- Definition of Fourier Transform
- Properties of Fourier Transform
- Signal bandwidth
- Signal power & energy. Energy and power-type signals
- Fourier transform of periodic signals
- Relation between Fourier series & transform

- **Reading**: Couch, 2.1-2.6; Oppenheim & Willsky, Ch. 1, 3 & 4. Study carefully all the examples (including end-of-chapter study-aid examples), make sure you understand them and can solve them with the book closed.
- Do some end-of-chapter problems. Students' solution manual provides solutions for many of them.