Review of Fourier Transform

- Fourier series works for periodic signals only. What’s about aperiodic signals? This is very large & important class of signals
- Aperiodic signal can be considered as periodic with $T \to \infty$
- Fourier series changes to Fourier transform, complex exponents are infinitesimally close in frequency
- Discrete spectrum becomes a continuous one, also known as spectral density
Fourier Series -> Fourier Transform

Periodic signal

As T increases, spectral components are getting closer and closer, becoming the continuous spectrum at the limit

Fourier Transform

- Fourier transform (spectrum):
  \[ S_x(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt \]
  \[ S_x(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \]

- Inverse Fourier transform:
  \[ x(t) = \int_{-\infty}^{+\infty} S_x(f) e^{j2\pi ft} df \]
  \[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_x(\omega) e^{j\omega t} d\omega \]

- Dirichlet conditions (details on the next page):
  - \( x(t) \) is absolutely integrable,
  - \( x(t) \) has a finite number of maxima & minima within any finite interval,
  - \( x(t) \) has a finite number of discontinuities within any finite interval. These discontinuities must be finite.
Convergence of Fourier Transform

• Dirichlet conditions:
  – x(t) must be absolutely integrable (finite energy)
    \[ \int_{-\infty}^{\infty} |x(t)| \, dt < \infty \text{ or } \int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty \]
  – x(t) has a finite number of maxima & minima within any finite interval,
  – x(t) has a finite number of discontinuities within any finite interval. These discontinuities must be finite.

• Dirichlet conditions are only sufficient, but are not necessary.

• All physically-reasonable (practical) signals meet these conditions.
Example: Rectangular Pulse

\[ x(t) = \Pi(t / \tau) = \begin{cases} 
1, & |t| < \tau / 2 \\
0, & |t| > \tau / 2 
\end{cases} \]

\[ S_x(f) = \tau \frac{\sin \pi f \tau}{\pi f \tau} = \tau \cdot \text{sinc}(f \tau) \]
Example: \( \text{sinc}(t) \)

Shortening pulse widens its spectrum!

\[ x_1(t) \quad \text{signal} \]

\[ x_2(t) \quad \text{signal} \]

\[ S_{x_1}(f) \quad \text{spectrum} \]

\[ S_{x_2}(f) \quad \text{spectrum} \]
Generalized (Singular) Functions

• Unit step function

\[ u(t) = \begin{cases} 
1, & t > 0 \\
0, & t < 0 
\end{cases} \]

• Dirac delta function (unit impulse function)

\[ \int_{-\infty}^{\infty} \delta(t) x(t) \, dt = x(0) \]

\[ \delta(t) = \begin{cases} 
\infty, & t = 0 \\
0, & t \neq 0 
\end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t) \, dt = 1 \]
Generalized Functions

• Dirac Delta Function as a limit:

\[ \Pi_{\Delta}(t) = \begin{cases} 
1/\Delta, & |t| < \Delta/2 \\
0, & |t| > \Delta 
\end{cases} \]

\[ \delta(t) = \lim_{\Delta \to 0} \Pi_{\Delta}(t) \]

• Useful properties of delta function

\[ \int_{-\infty}^{\infty} \delta(t-t_0)x(t)dt = x(t_0) \]

\[ \int_{-\infty}^{t} \delta(t)dt = u(t) \]

\[ \frac{du(t)}{dt} = \delta(t) \]

\[ \delta(at) = \frac{1}{|a|} \delta(t) \]

\[ \delta(-t) = \delta(t) \]

\[ x(t)\delta(t) = x(0)\delta(t) \]
Fourier Transform of Periodic Signal

- FT of a complex exponent:
  \[ x(t) = e^{j\omega_0 t} \leftrightarrow 2\pi \delta(\omega - \omega_0) = \delta(f - f_0) \]

- Important property:
  \[ \delta(f) = \int_{-\infty}^{+\infty} e^{\pm j2\pi ft} df \]

- FT of a periodic signal:
  \[ x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} \leftrightarrow 2\pi \sum_{n=-\infty}^{+\infty} c_n \delta(\omega - n\omega_0) = \sum_{n=-\infty}^{+\infty} c_n \delta(f - nf_0) \]

- FT of \( \cos(\omega_0 t) \) ?
Properties of Fourier Transform*

- *Very similar* to those of Fourier series!

- **Linearity:**
  \[ F \{ \alpha x_1(t) + \beta x_2(t) \} \leftrightarrow \alpha S_{x_1}(f) + \beta S_{x_2}(f) \]

- **Time shifting:**
  \[ x(t) \leftrightarrow S_x(\omega) \Rightarrow x(t - t_0) \leftrightarrow e^{-j\omega t_0} S_x(\omega) \]

- **Time reversal:**
  \[ x(t) \leftrightarrow S_x(\omega) \Rightarrow x(-t) \leftrightarrow S_x(-\omega) \]

- **Time scaling:**
  \[ x(at) \leftrightarrow \frac{1}{|a|} S_x \left( \frac{\omega}{a} \right) \]

*properties are useful for evaluating Fourier transform in a simple way

Prove these properties.
Properties of Fourier Transform

- **Conjugation:**
  \[ x(t) \iff S_x(\omega) \Rightarrow x^*(t) \iff S_x^*(-\omega) \]

- **Differentiation:**
  \[ x(t) \iff S_x(\omega) \Rightarrow \frac{dx(t)}{dt} \iff j\omega S_x(\omega) \]

- **Integration:**
  \[ \int_{-\infty}^{t} x(t)dt \iff \frac{1}{j\omega} S_x(\omega) + \pi S_x(0)\delta(\omega) \]

- **Multiplication:**
  \[ x(t)y(t) \iff \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega')S_y(\omega - \omega')d\omega' = S_x(\omega) * S_y(\omega) \]

- **Frequency shift (modulation):**
  \[ x(t)e^{j\omega_0t} \iff S(\omega - \omega_0) \]
Duality of Fourier Transform

\[ x(t) \leftrightarrow S_x(\omega) \Rightarrow S_x(\omega) \leftrightarrow 2\pi x(-\omega) \]

\[ x(t) \leftrightarrow S_x(f) \Rightarrow S_x(f) \leftrightarrow x(-f) \]

Convolution Property

• This property is of great importance

\[ y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \iff S_x(\omega)H(\omega) = S_y(\omega) \]

Cascade connection of LTI blocks

• Example: FT of a triangular pulse by convolution
Parseval Theorem

- Total energy in time domain is the same as the total energy in frequency domain:

\[
E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |S_x(f)|^2 \, df = \frac{1}{2\pi} \int_{-\infty}^{\infty} |S_x(\omega)|^2 \, d\omega
\]

- \( E(f) = |S_x(f)|^2 \) - energy spectral density (ESD) of \( x(t) \). Represents the amount of energy per Hz of bandwidth
- Counterpart of Parseval theorem for periodic signals
- Autocorrelation property:

\[
R_x(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau) \, dt \leftrightarrow |S_x(\omega)|^2
\]

\( R_x(0) = E \)
Parseval Theorem: Example

\[ \int_{-\infty}^{+\infty} \left( \frac{\sin t}{t} \right)^2 dt = ? \]
Fourier Transform of Real Signal

- if \( x(t) \) is real, \( \text{Im}\{x(t)\} = 0 \Rightarrow S_x(-\omega) = S_x^*(\omega) \)

- Fourier transform can be presented as

\[
x(t) = 2\int_{0}^{\infty} |S_x(f)| \cos(2\pi f + \varphi(f)) df,
\]

\[
\varphi(f) = \tan^{-1}\left(\frac{\text{Im}[S_x(f)]}{\text{Re}[S_x(f)]}\right)
\]

No negative frequencies!
Signal Bandwidth & Negative Frequencies

- What negative frequency means?
- It means that there is $e^{-j2\pi ft}$ term in signal spectrum
- Convenient mathematical tool. Do not exist in practice (i.e., cannot be measured on spectrum analyzer)
- What is the signal bandwidth? There are many definitions.
- Defined for positive frequencies only.
- Determines the frequency band over which a substantial part of the signal power/energy is concentrated.
- For band-limited signals

$$\Delta f = f_{\text{max}} - f_{\text{min}}, \quad f_{\text{max}}, f_{\text{min}} \geq 0$$
Power and Energy

- Power $P_x$ & energy $E_x$ of signal $x(t)$ are:

$$P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

- Energy-type signals: $E_x < \infty$
- Power-type signals: $0 < P_x < \infty$
- Signal *cannot* be both energy & power type!
- Signal energy: if $x(t)$ is voltage or current, $E_x$ is the energy dissipated in 1 Ohm resistor
- Signal power: if $x(t)$ is voltage or current, $P_x$ is the power dissipated in 1 Ohm resistor.
Energy-Type Signals (summary)

- Signal energy in time & frequency domains:
  \[ E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |S_x(f)|^2 df = \frac{1}{2\pi} \int_{-\infty}^{\infty} |S_x(\omega)|^2 d\omega \]

- Energy spectral density (energy per Hz of bandwidth):
  \[ E_x(f) = |S_x(f)|^2 \]

- ESD is FT of autocorrelation function:
  \[ R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x^*(t-\tau)dt \leftrightarrow E_x(f) \]
  \[ R_x(0) = E_x \]
Power-Type Signals: PSD

• Definition of the power spectral density (PSD) (power per Hz of bandwidth):

\[
P_x(f) = \lim_{T \to \infty} \frac{|S_T(f)|^2}{T} \Rightarrow P_x = \int_{-\infty}^{\infty} P_x(f) df < \infty
\]

• where \( x_T(t) \) is the truncated signal (to \([-T/2, T/2])\),

\[
x_T(t) = x(t) \Pi \left( \frac{t}{T} \right) = \begin{cases} 
  x(t), & -T/2 \leq t \leq T/2 \\
  0, & \text{otherwise} 
\end{cases}
\]

• and \( S_T(f) \) is its spectrum (FT),

\[
S_T(f) = FT \{ x_T(t) \}
\]
Power-Type Signals

• Time-average autocorrelation function:

\[ R_x(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)x^*(t-\tau) \, dt \]

• Power of the signal:

\[ P_x = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt = R_x(0) \]

• Wiener-Khintchine theorem:

\[ P_x = \int_{-\infty}^{\infty} P_x(f) \, df \Rightarrow P_x(f) = FT\{R_x(\tau)\} \]
Periodic Signals

- **Power of a periodic signal:**

\[ P_x = \frac{1}{T} \int_T |x(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |c_n|^2 = R_x(0) \quad \leftarrow x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j n \omega_0 t} \]

- **Autocorrelation function:**

\[ R_x(\tau) = \frac{1}{T} \int_T x(t) x^*(t - \tau) \, dt = \sum_{n=-\infty}^{\infty} |c_n|^2 e^{j n \omega_0 \tau} \]

- **Power spectral density (PSD):**

\[ P_x(f) = FT \{ R_x(\tau) \} = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta \left( f - \frac{n}{T} \right) \]

Prove these properties!
Relation Between Fourier Transform & Series

- consider a periodic signal \( x(t) = x(t+T) \)
- truncate it (one period only):
  \[
  x_T(t) = \begin{cases} 
  x(t), & -T/2 < t \leq T/2 \\
  0, & \text{otherwise}
  \end{cases}
  \]
- find FT of the truncated signal \( x_T(t) \): \( x_T(t) \leftrightarrow S_{x_T}(\omega) \)
- Fourier series of the original periodic signal \( x(t) \) is
  \[
  c_n = \frac{1}{T} S_{x_T}(n\omega_0)
  \]
  \(\Rightarrow\) prove this
- Continuous spectrum is the envelope of discrete spectrum (see slide 2)!
Fourier Transform of Single Pulse ~ Envelope of Fourier Series of Pulse Train

Summary

• Definition of Fourier Transform
• Properties of Fourier Transform
• Signal bandwidth
• Signal power & energy. Energy and power-type signals
• Fourier transform of periodic signals
• Relation between Fourier series & transform

• Reading: Couch, 2.1-2.6; Oppenheim & Willsky, Ch. 1, 3 & 4. Study carefully all the examples (including end-of-chapter study-aid examples), make sure you understand and can solve them with the book closed.
• Do some end-of-chapter problems. Students’ solution manual provides solutions for many of them.