Review of Fourier Series

- Why Fourier series? Want to make analysis simple.
- Complex exponents are the eigenfunctions of LTI systems,

\[ y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} h(\tau)x(t - \tau) d\tau \]

\[ x(t) = e^{j\omega t} \Rightarrow y(t) = H(j\omega)e^{j\omega t} \]

- \( x(t) \) is presented as linear combination of complex exponents (with different frequencies)
- LTI system response is the same linear combination of individual responses!
Periodic Signals

- Periodic signals are very important class of signals (widely used), where smallest $T$ is a period,

$$x(t) = x(t + T), \text{ for all } t$$

- Examples: $\cos(\omega_0 t)$ & $e^{j\omega_0 t}$ . Period $T = 2\pi / \omega_0$

- Introduce a set of harmonically-related complex exponents,

$$\phi_n(t) = e^{jn\omega_0 t} = e^{jn\frac{2\pi}{T} t}, \quad n = 0, \pm 1, \pm 2, \ldots$$

- Construct a periodic signal,

$$x'(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}$$
Fourier Series of Periodic Signal

- Can $x'(t)$ be made the same as $x(t)$?
- Yes, by adjusting $c_n$,

$$c_n = \frac{1}{T} \int_{-T}^{T} x(t) e^{-j\frac{2\pi}{T}nt} dt$$

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{j\omega_0 t}, \omega_0 = \frac{2\pi}{T}$$

- $\{c_n\}$ – Fourier series coefficients (or spectral coefficients, or discrete spectrum of the signal)
- $c_0$ – DC component or average value of $x(t)$,

$$c_0 = \frac{1}{T} \int_{-T}^{T} x(t) dt$$
Example of Fourier Series

\[ x(t) = \cos(\omega_0 t) = \frac{1}{2} \left( e^{j\omega_0 t} + e^{-j\omega_0 t} \right) \]

\[ c_1 = \frac{1}{2}, \quad c_{-1} = \frac{1}{2}, \]
\[ c_k = 0, \quad k \neq \pm1 \]
Example of Fourier Series

\[ x(t) = \begin{cases} 
1, & |t| < \tau / 2 \\
0, & \tau / 2 < |t| < T / 2 
\end{cases} \]

\[ c_n = \frac{\tau}{T} \text{sinc} \left( \frac{n\tau}{T} \right), \]

\[ \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \]

Q.: How does \( c_n \) scale with the pulse amplitude? Duration? Period?
Example of Fourier Series

Periodic signal

Its spectrum

Q.: How does $c_n$ scale with the pulse amplitude? Duration? Period?

Convergence of Fourier Series

• Dirichlet conditions:
  – $x(t)$ must be absolutely integrable (finite power)
    $$\int_T |x(t)| \, dt < \infty$$
  – $x(t)$ must be of bounded variation; that is the number of maxima and minima during a period is finite
  – In any finite interval of time, there are only a finite number of discontinuities, which are finite.

• Dirichlet conditions are only sufficient, but are not necessary.

• All physically-reasonable (practical) signals meet these conditions.
Gibbs Phenomenon

increasing the number of terms does not decrease the ripple maximum!

Q.: reproduce these graphs using a computer
Fourier Series of Real Signals

• For a real signal, \( \text{Im} \{ x(t) \} = 0 \Rightarrow c_{-n} = c_n^* \)

• Then obtain the trigonometric Fourier series,

\[
x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right], \quad \omega_0 = \frac{2\pi}{T}
\]

\[
a_n = 2 \text{Re} \{ c_n \} = \frac{2}{T} \int_{T} x(t) \cos(n\omega_0 t) dt, \quad b_n = -2 \text{Im} \{ c_n \} = \frac{2}{T} \int_{T} x(t) \sin(n\omega_0 t) dt
\]

• Another form of it is

\[
x(t) = x_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \varphi_n)
\]

\[
A_n = |c_n| = \sqrt{a_n^2 + b_n^2}, \quad \varphi_n = -\arg(c_n) = -\tan^{-1}\left( b_n / a_n \right)
\]
Properties of Fourier Series

- **Linearity:**
  \[ F[\alpha x_1(t) + \beta x_2(t)] = \alpha F[x_1(t)] + \beta F[x_2(t)] \]

- **Time shifting:**
  \[ x(t) \underbrace{\rightarrow}_{F} c_n \iff x(t - t_0) \underbrace{\rightarrow}_{F} e^{-jn\omega_0 t_0} c_n \]

- **Time reversal:**
  \[ x(t) \underbrace{\rightarrow}_{F} c_n \iff x(-t) \underbrace{\rightarrow}_{F} c_{-n} \]

- **Time scaling:**
  \[ x(\alpha t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn(\alpha \omega_0) t} \]

Q.: prove these properties
Properties of Fourier Series

- Multiplication:
  \[ x(t)y(t) \xrightarrow{F} \sum_{k=-\infty}^{\infty} c'_k c''_{n-k} \]

- Convolution:
  \[ \int_T x(\tau)y(t-\tau)d\tau \xrightarrow{F} Tc'_n c''_n \]

- Differentiation:
  \[ \frac{dx(t)}{dt} \xleftarrow{F} jn\omega_0 c_n \]

- Integration:
  \[ \int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{F} \frac{c_n}{jn\omega_0}, \text{ for } c_0 = 0 \]

Q.: prove these properties
Properties of Fourier Series

- Real \( x(t) \):
  \[ c_{-n} = c_n^* \]

- Real & even \( x(t) \):
  \[ c_{-n} = c_n, \text{Im}\{c_n\} = 0 \]

- Real & odd \( x(t) \):
  \[ c_{-n} = -c_n, \text{Re}\{c_n\} = 0 \]

- Parseval’s Theorem:
  \[ \frac{1}{T} \int_T |x(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |c_n|^2 \]

Q.: prove these properties
Signal Synthesis via FS

\[ x(t) = \sum_{n=-\infty}^{+\infty} a_n \varphi_n(t) \]

Summary

- Review of Fourier series
- Periodic signals & complex exponents
- Series expansion of a periodic signal
- Trigonometric form of Fourier series
- Properties of Fourier series

**Reading:** the Couch text, Sec. 2.1-2.5; Oppenheim & Willsky text, Sec. 3.0-3.5. Study carefully all the examples (including end-of-chapter study-aid examples), make sure you understand and can solve them with the book closed.

- Do some end-of-chapter problems. Students’ solution manual provides solutions for many of them.