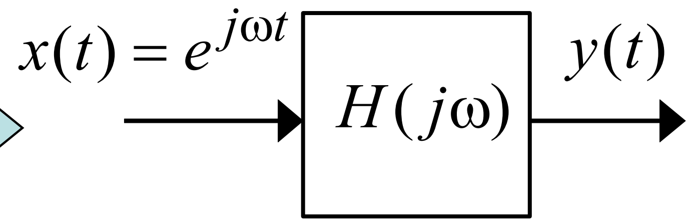
Review of Fourier Series

- Why Fourier series?
- Complex exponents are the eigenfunctions of LTI systems,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)\,d\tau$$

$$x(t) = e^{j\omega t} \Rightarrow y(t) = H(j\omega)e^{j\omega t}$$



- x(t) is presented as a linear combination of complex exponents (with different frequencies)
- LTI system response is the same linear combination of individual responses.

Periodic Signals

- Periodic signals are very important class of signals (widely used), where smallest T is a period,

$$x(t) = x(t+T), \text{ for all } t$$

- Examples: $\cos(\omega_0 t)$ & $e^{j\omega_0 t}$. Period $T = 2\pi / \omega_0$
- Introduce a set of harmonically-related complex exponents,

$$\phi_n(t) = e^{jn\omega_0 t} = e^{jn\frac{2\pi}{T}t}, \quad n = 0, \pm 1, \pm 2, ...$$

DC 1st harmonic

- Construct a periodic signal,

$$x'(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}$$

Fourier Series of Periodic Signal

- Can $x'(t)$ be made the same as $x(t)$?
- Yes, by adjusting c_n ,

$$\boxed{c_n = \frac{1}{T}\int_T x(t) e^{-j2\pi \frac{n}{T} t} dt} \Rightarrow \boxed{x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}, \omega_0 = \frac{2\pi}{T}}$$

- $\{c_n\}$ – Fourier series coefficients (or spectral coefficients, or discrete spectrum of the signal)
- c_0 – DC component or average value of x(t),

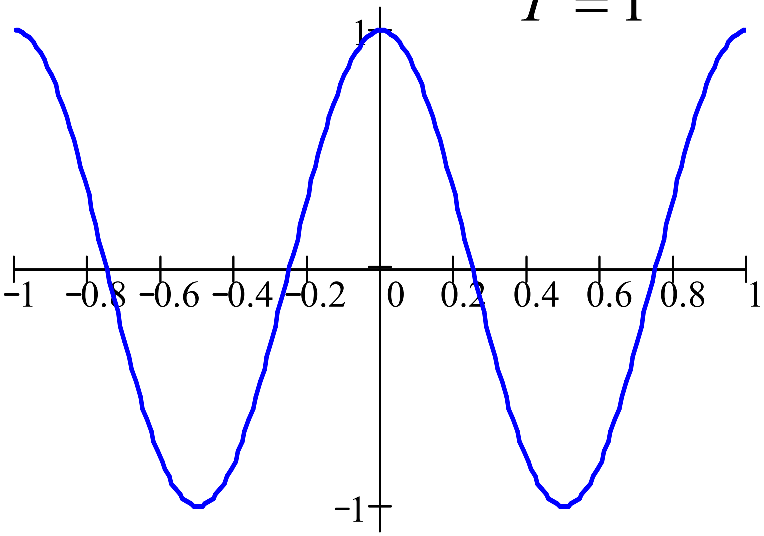
$$\boxed{c_0 = \frac{1}{T}\int_T x(t) dt}$$

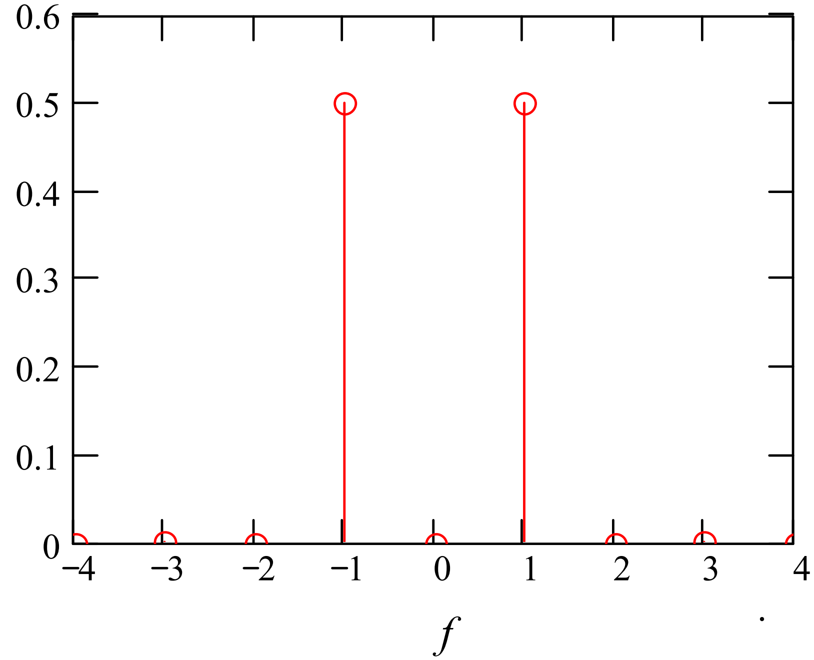
Example of Fourier Series

$$\boxed{x(t) = \cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)}$$

$$\boxed{\begin{aligned} &c_1 = \frac{1}{2}, c_{-1} = \frac{1}{2}, \\ &c_k = 0, k \neq \pm 1 \end{aligned}}$$

$T = 1$



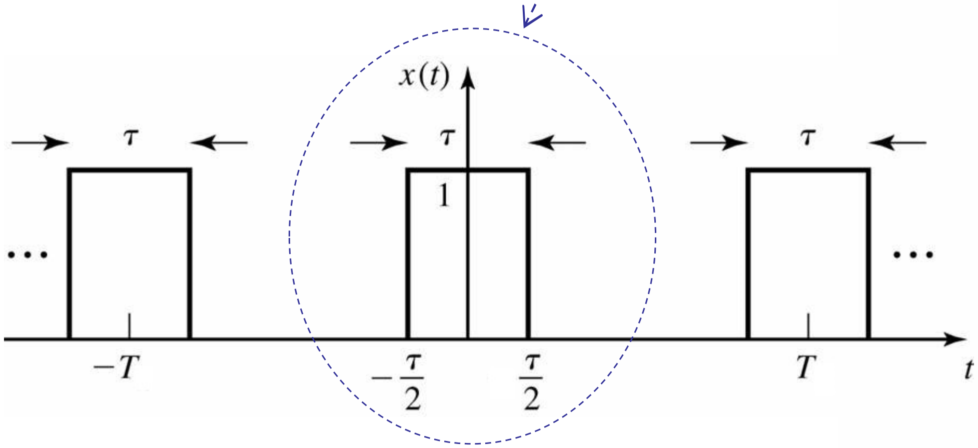


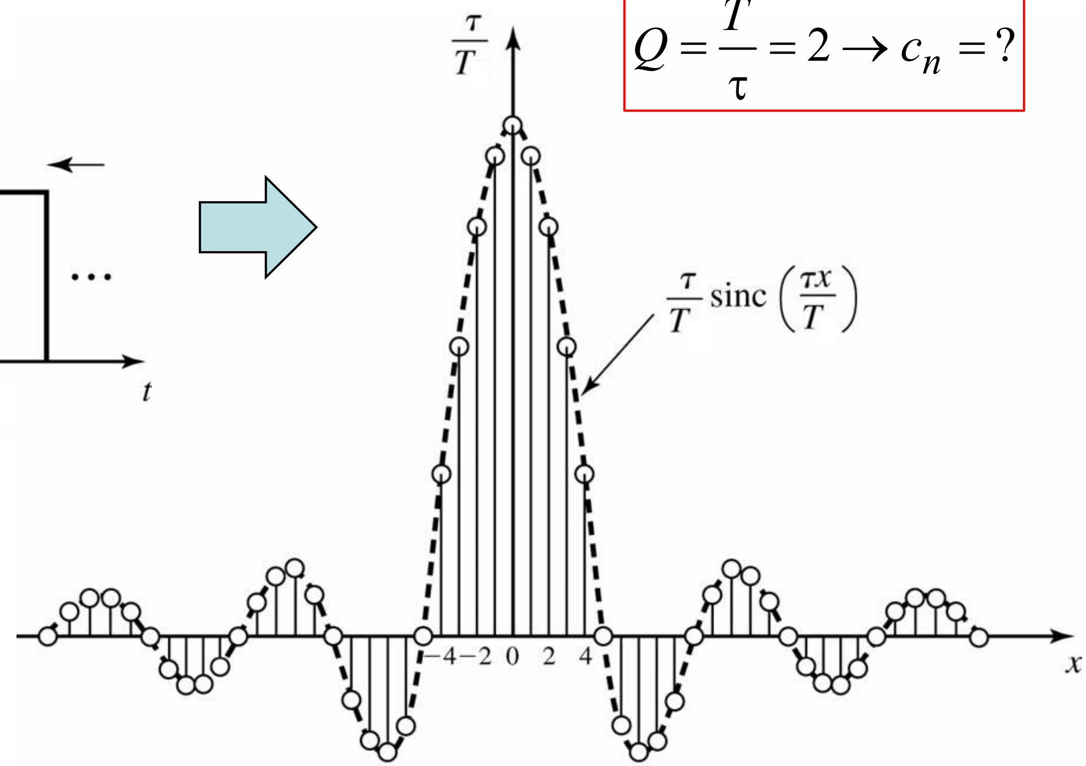
Example of Fourier Series

$$x(t) = \begin{cases} 1, |t| < \tau/2 \\ 0, \tau/2 < |t| < T/2 \end{cases}$$

$$c_n = \frac{\tau}{T} \operatorname{sinc}\left(\frac{n\tau}{T}\right),$$

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

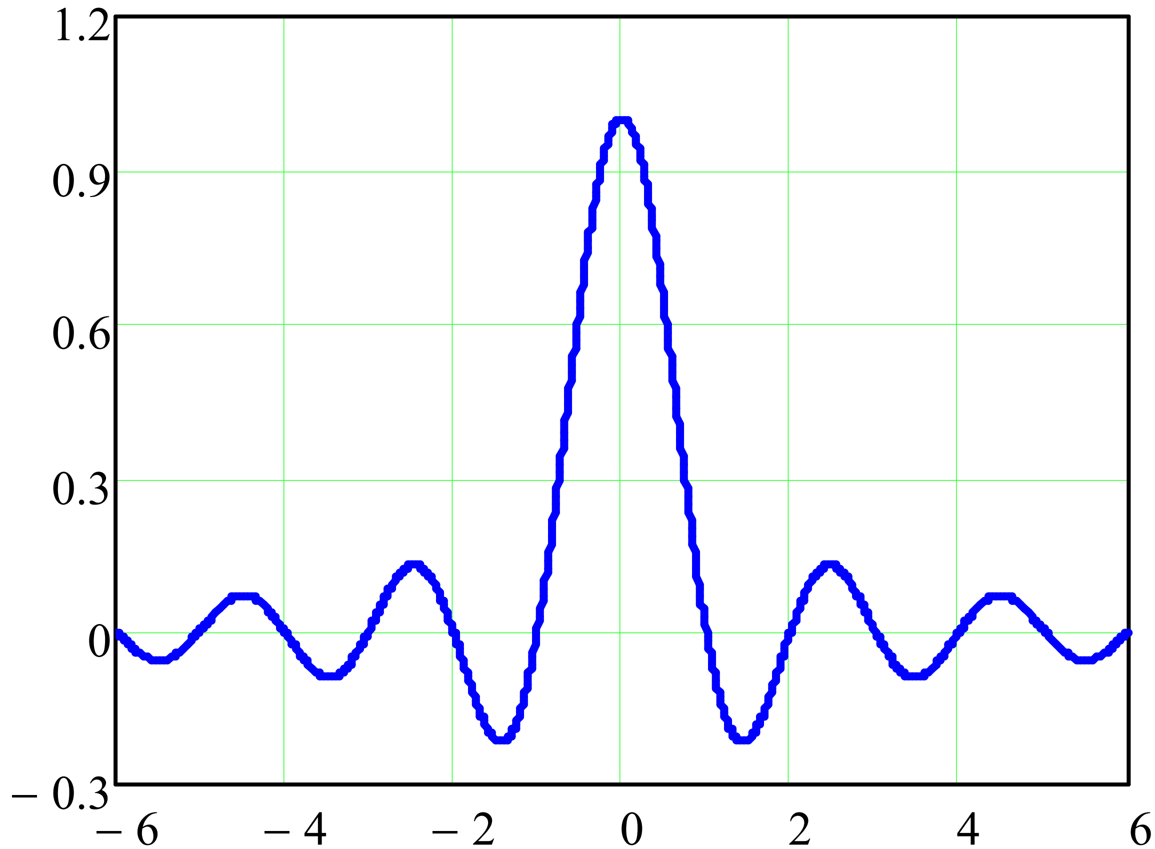




$$Q = \frac{T}{\tau} = 2 \rightarrow c_n = ?$$

Q.: How does c_n scale with the pulse amplitude? Duration? Period?

sinc(t)



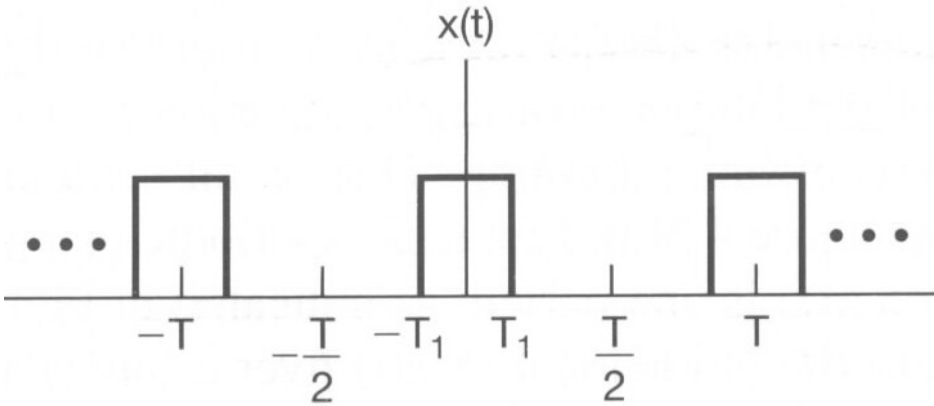
$$\mathrm{sinc}(n) = 0,\ \ n \neq 0$$

$$\mathrm{sinc}(0) = ?$$

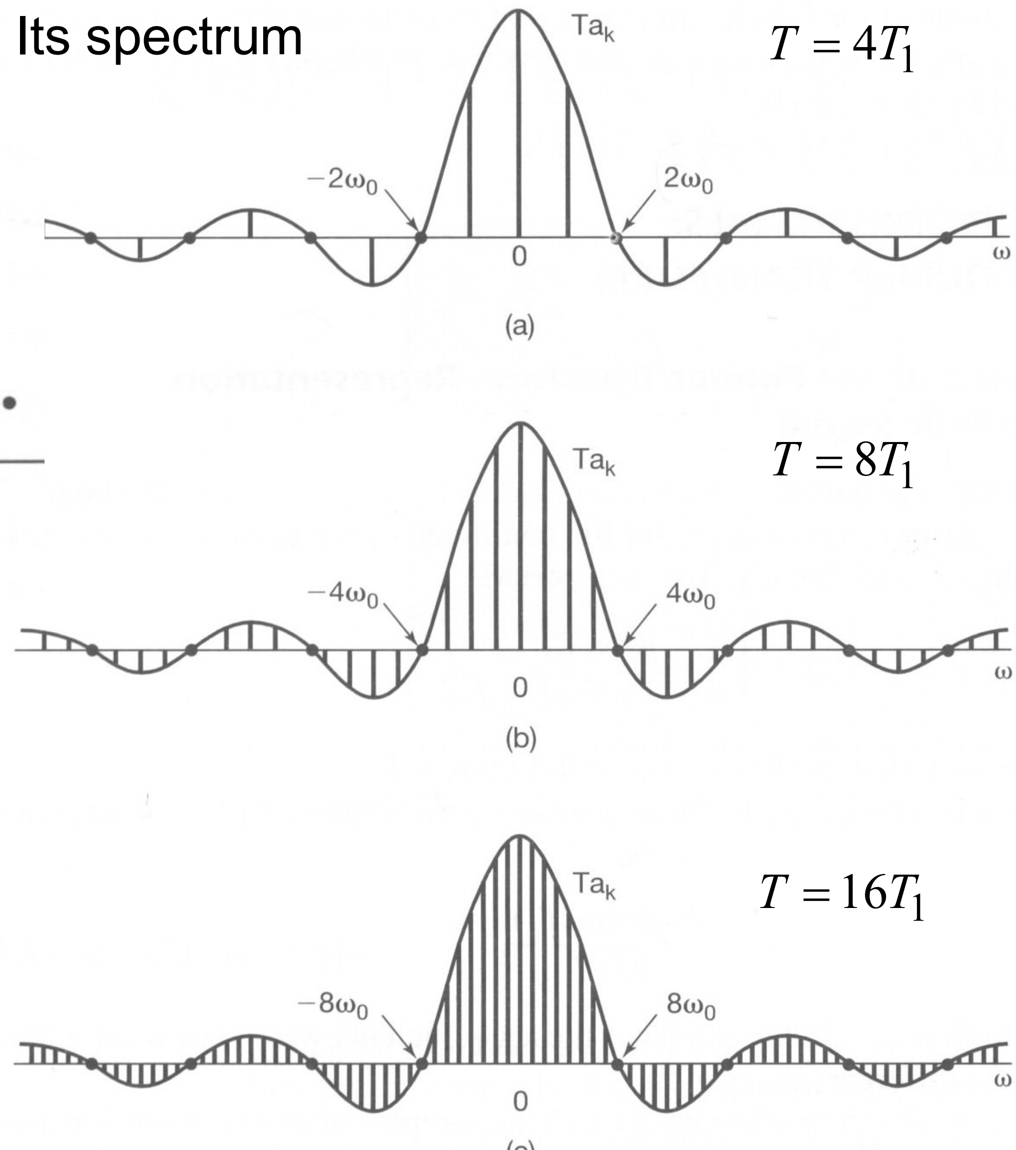
Example of Fourier Series

Its spectrum

Periodic signal



Duty cycle: $Q = \frac{T}{\tau} = \frac{T}{2T_1}$



$T = 4T_1$

$T = 8T_1$

$T = 16T_1$

Q.: How does c_n scale with the pulse amplitude? Duration? Period?

A.V. Oppenheim, A.S. Willsky, Signals and Systems, 1997.

Convergence of Fourier Series

- Dirichlet conditions:
  - x(t) must be absolutely integrable (finite power)

$$\boxed{\int_T |x(t)|\,dt < \infty}$$

  - x(t) must be of bounded variation; that is the number of maxima and minima during a period is finite
  - In any finite interval of time, there are only a finite number of discontinuities, which are finite.
- Dirichlet conditions are only sufficient, but are not necessary.
- All physical (practical) periodic signals do have Fourier series.
- Q: show that 1st condition holds if x(t) is bounded

Fourier Series of Real Signals

- For a real signal, $\operatorname{Im}\{x(t)\}=0 \Rightarrow c_{-n}=c_n^*$
- The trigonometric (sin/cos) Fourier series:

$$x(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos(n\omega_0 t)+b_n\sin(n\omega_0 t)\right],\quad \omega_0=\frac{2\pi}{T}$$

$$a_n=2\operatorname{Re}\{c_n\}=\frac{2}{T}\int_T x(t)\cos(n\omega_0 t)dt,\quad b_n=-2\operatorname{Im}\{c_n\}=\frac{2}{T}\int_T x(t)\sin(n\omega_0 t)dt$$

- Another form (cos only):

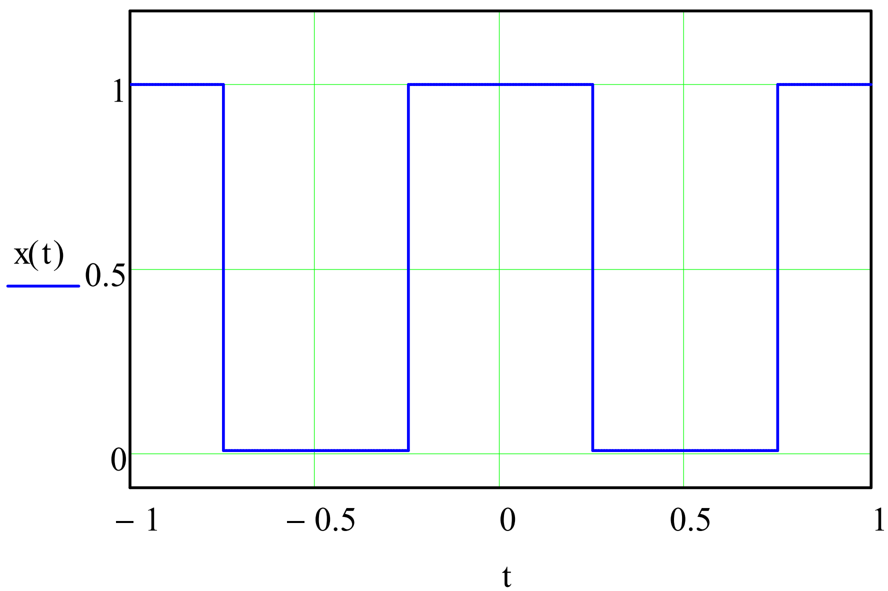
$$x(t)=x_0+\sum_{n=1}^{\infty}A_n\cos(n\omega_0 t+\varphi_n)$$

$$A_n=2|c_n|=\sqrt{a_n^2+b_n^2},\quad \varphi_n=-\arg(c_n)=-\tan^{-1}(b_n/a_n)$$

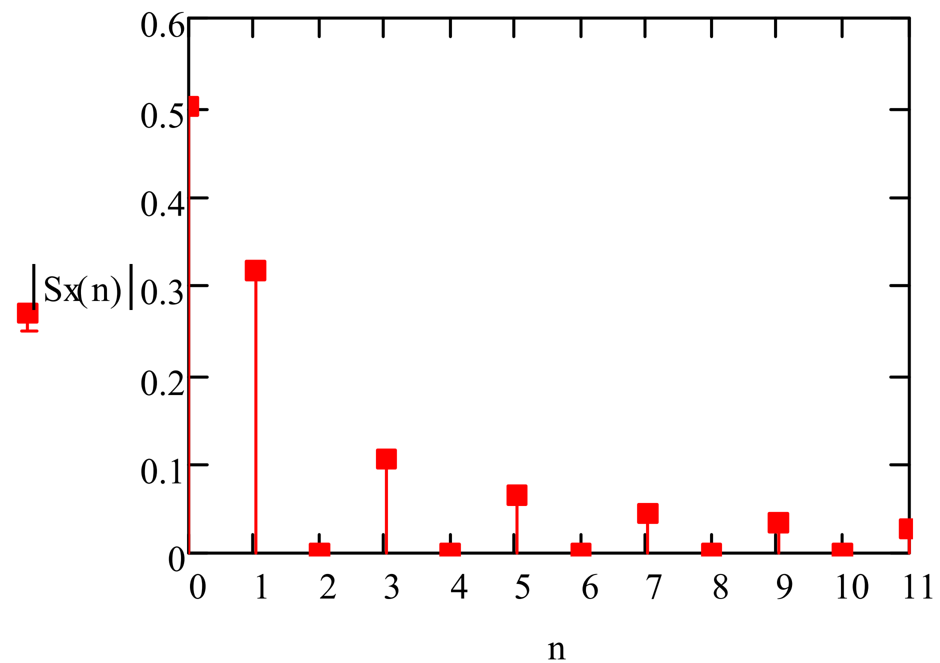
- Single-sided spectrum: only positive frequencies, on SA

Example: rectangular pulse train

Signal (OS, time domain)



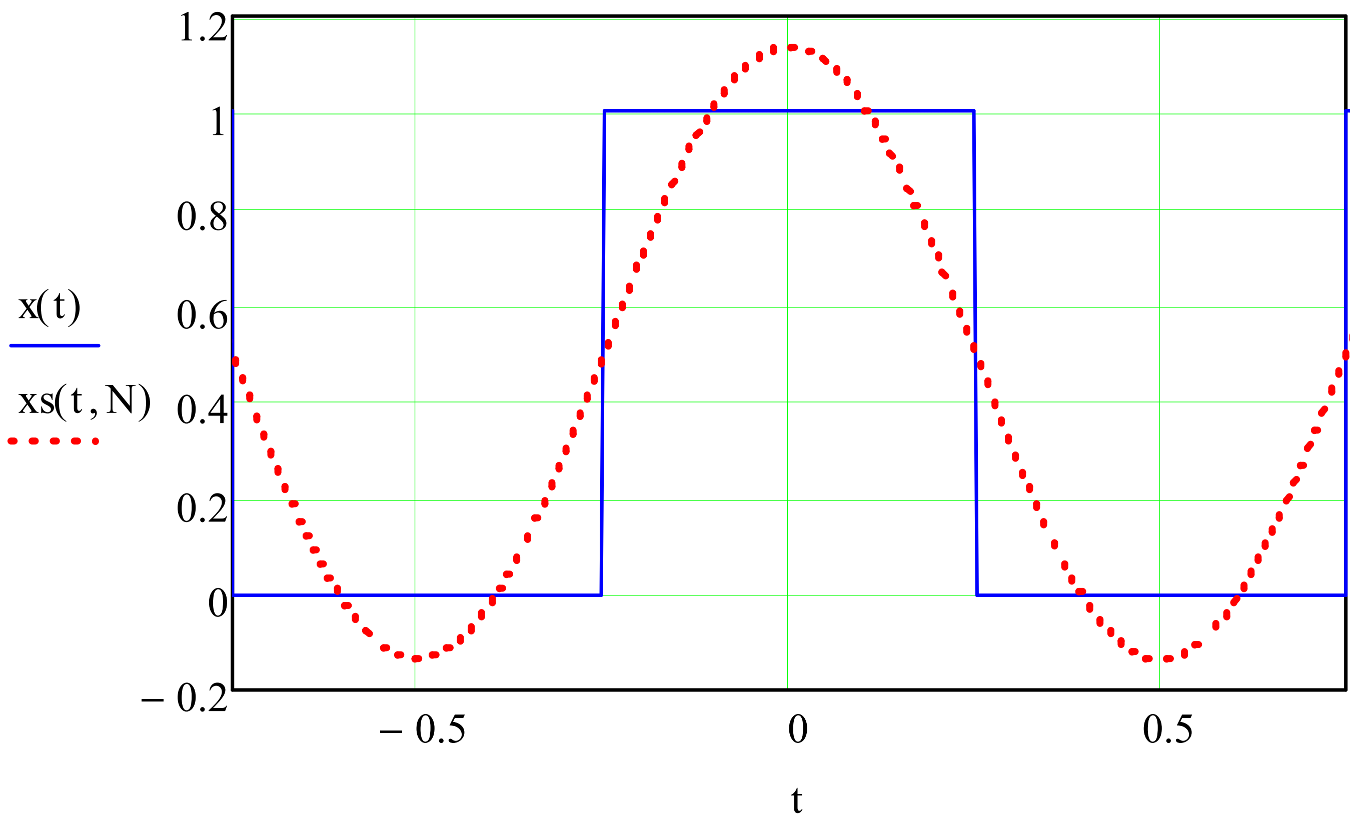
Spectrum (SA, frequency domain)



$$xs(t,N) := \frac{1}{2} + \sum_{n=1}^{N} \left(\mathrm{sinc}\left(\frac{n}{2}\right) \cdot \cos(2\pi \cdot n \cdot t) \right)$$

Convergence of Fourier Series

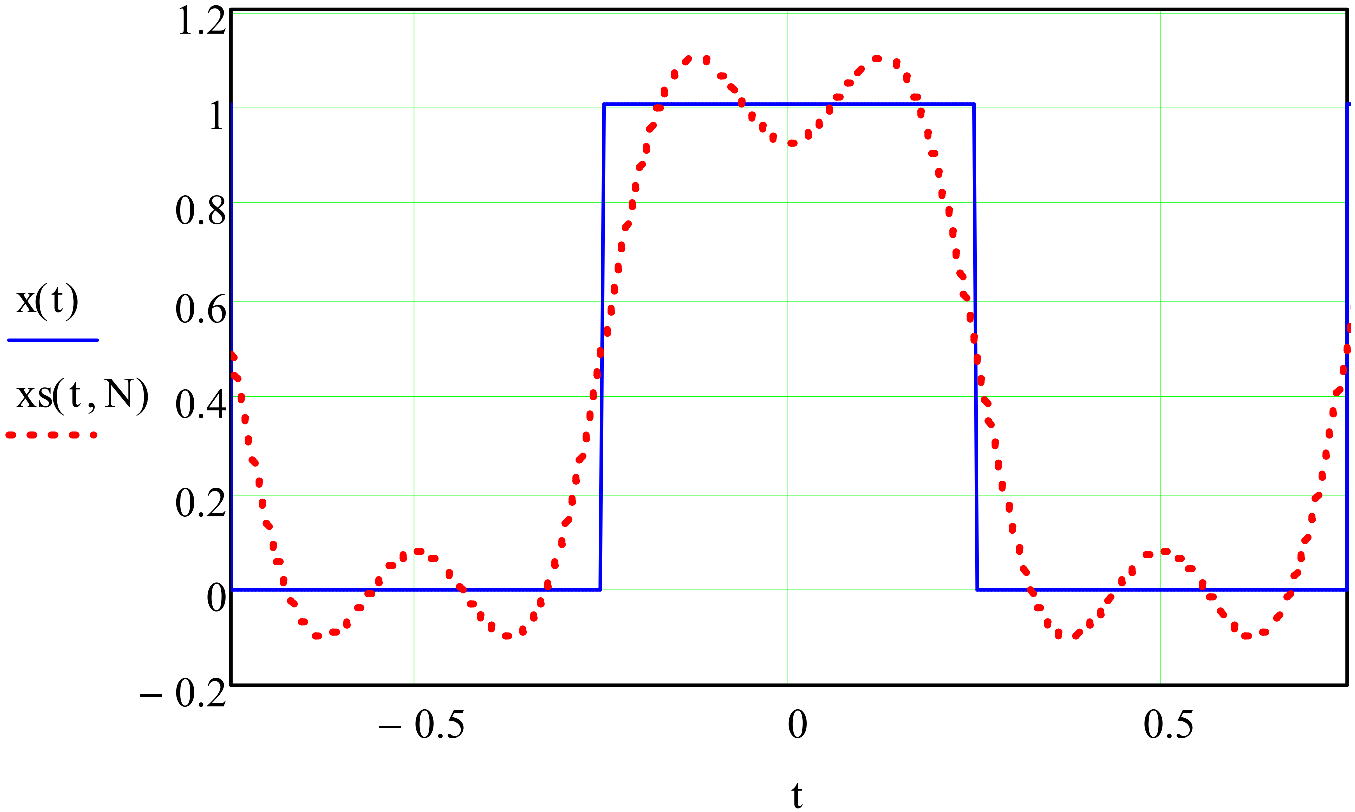
$N = 1$



$$xs(t, N) := \frac{1}{2} + \sum_{n=1}^{N} \left(\mathrm{sinc}\left(\frac{n}{2}\right) \cdot \cos(2\pi \cdot n \cdot t) \right)$$

Convergence of Fourier Series

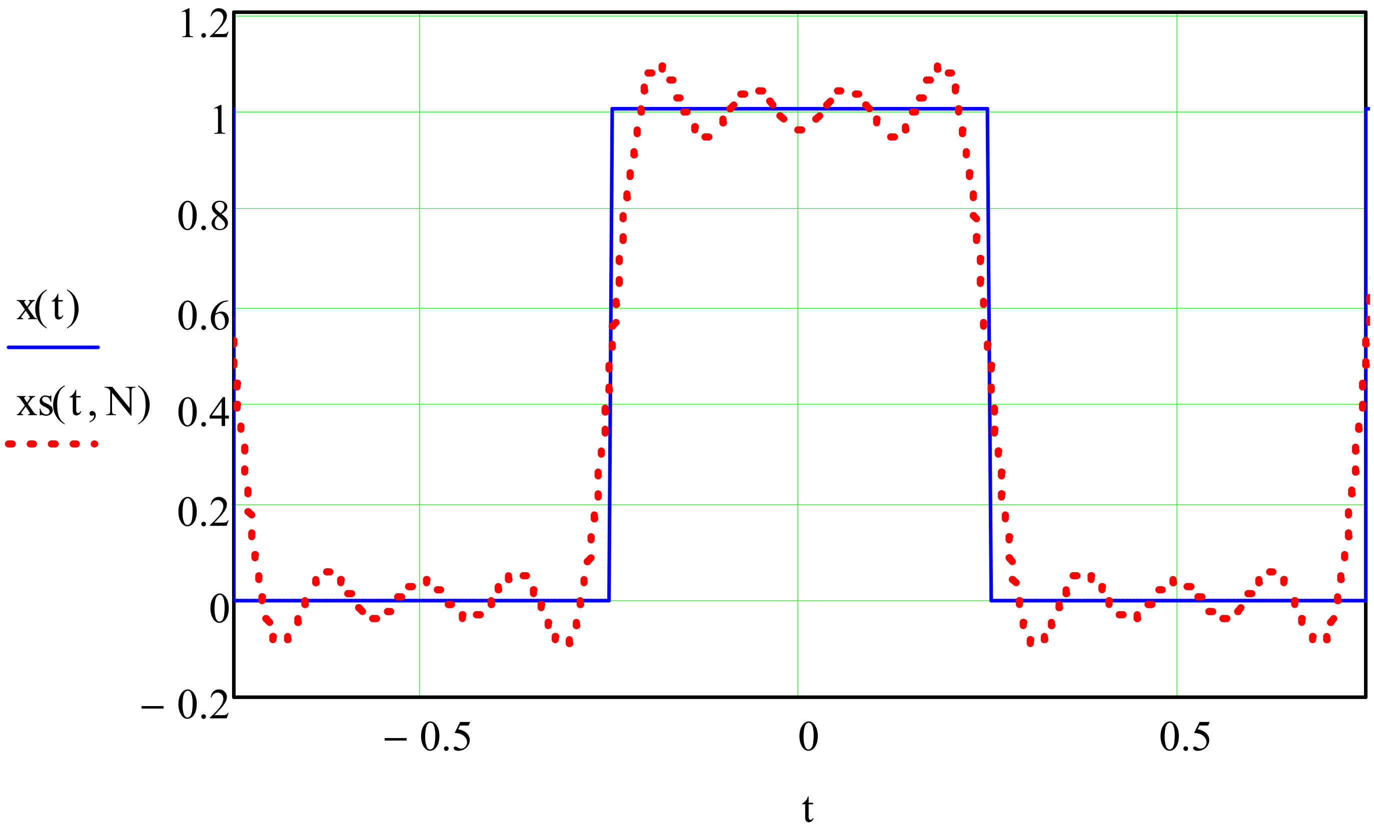
$N = 3$



$$xs(t,N) := \frac{1}{2} + \sum_{n=1}^{N} \left(\mathrm{sinc}\left(\frac{n}{2}\right) \cdot \cos(2\pi \cdot n \cdot t) \right)$$

Convergence of Fourier Series

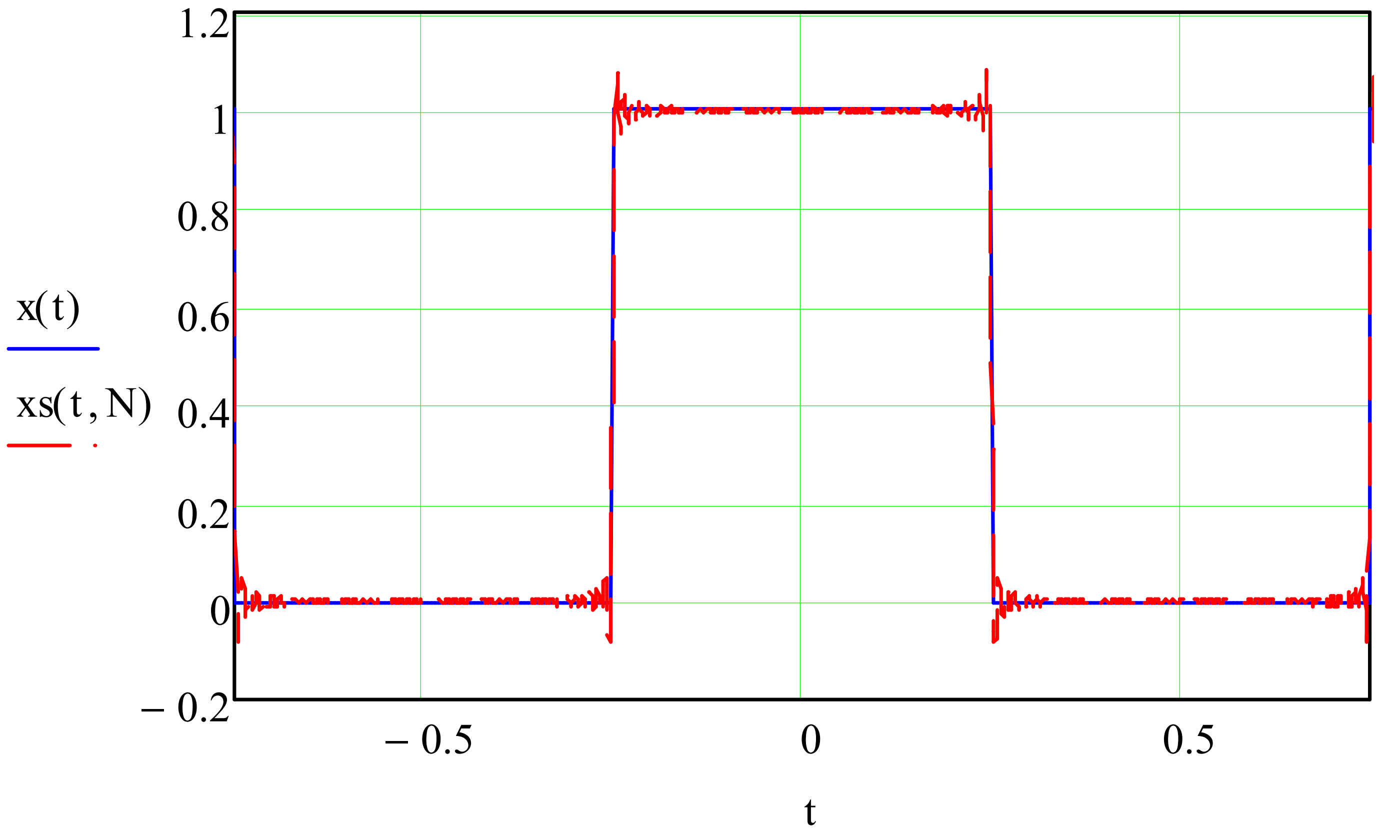
$N = 7$



$$xs(t,N) := \frac{1}{2} + \sum_{n=1}^{N} \left(\text{sinc}\left(\frac{n}{2}\right) \cdot \cos(2\pi \cdot n \cdot t) \right)$$

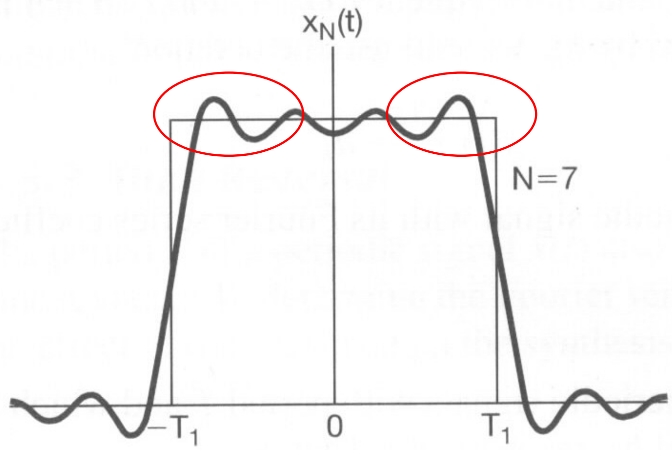
Convergence of Fourier Series

$N = 100$

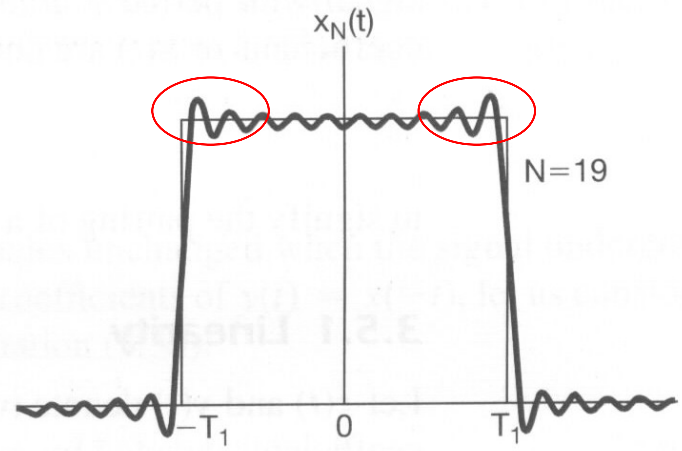


$$xs(t,N) := \frac{1}{2} + \sum_{n=1}^{N} \left(\mathrm{sinc}\left(\frac{n}{2}\right) \cdot \cos(2\pi \cdot n \cdot t) \right)$$

Gibbs Phenomenon

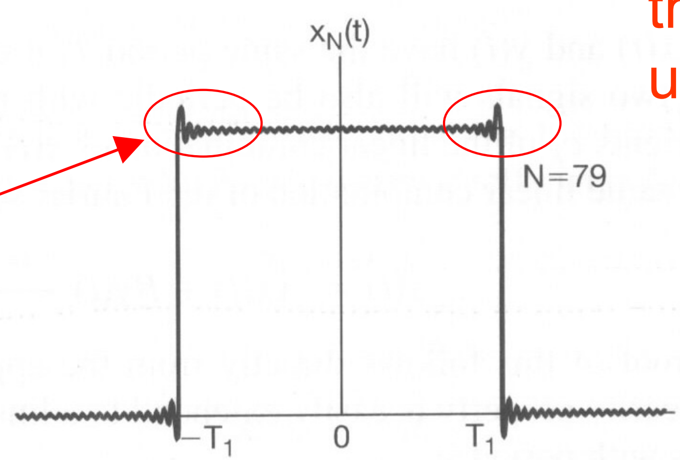


(c)



(d)

Q.: reproduce these graphs using a computer



(e)

increasing the number of terms does not decrease the ripple maximum!

Properties of Fourier Series

- Linearity: $$\boxed{\mathrm{F}\left[\alpha x_1(t)+\beta x_2(t)\right]=\alpha \mathrm{F}\left[x_1(t)\right]+\beta \mathrm{F}\left[x_2(t)\right]}$$

- Time shifting: $$\boxed{x(t)\xleftrightarrow{F} c_n \Leftrightarrow x(t-t_0)\xleftrightarrow{F} e^{-jn\omega_0 t_0}c_n}$$

- Time reversal: $$\boxed{x(t)\xleftrightarrow{F} c_n \Leftrightarrow x(-t)\xleftrightarrow{F} c_{-n}}$$

- Time scaling: $$\boxed{x(\alpha t)=\sum_{n=-\infty}^{+\infty} c_n e^{jn(\alpha\omega_0)t}}$$

Q.: prove these properties

Properties of Fourier Series

- Multiplication: $\boxed{x(t)y(t) \xleftrightarrow{F} \sum_{k=-\infty}^{\infty} c'_k c''_{n-k}}$
- Convolution: $\boxed{\int_T x(\tau)y(t-\tau)d\tau \xleftrightarrow{F} Tc'_n c''_n}$
- Differentiation: $\boxed{\frac{dx(t)}{dt} \xleftrightarrow{F} jn\omega_0 c_n}$
- Integration: $\boxed{\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{F} \frac{c_n}{jn\omega_0},}$ for $c_0 = 0$

Q.: prove these properties

Properties of Fourier Series

- Real x(t): $\boxed{c_{-n} = c_n^*}$

- Real & even x(t): $\boxed{c_{-n} = c_n, \mathrm{Im}\{c_n\} = 0}$

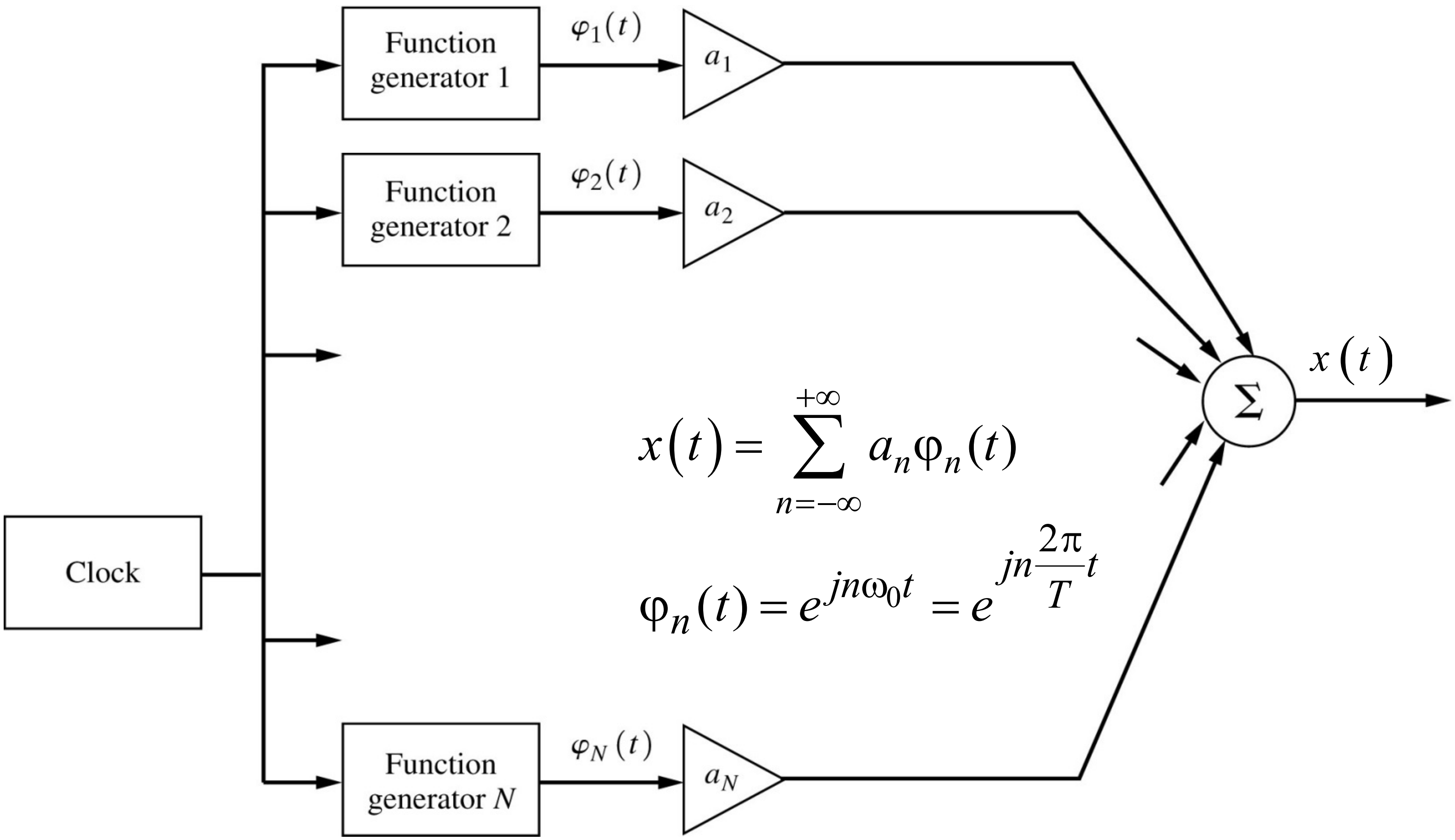
- Real & odd x(t): $\boxed{c_{-n} = -c_n, \mathrm{Re}\{c_n\} = 0}$

- Parseval's Theorem: $\boxed{\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2}$

power

Q.: prove these properties

Signal Synthesis via FS



$$x(t) = \sum_{n=-\infty}^{+\infty} a_n \varphi_n(t)$$

$$\varphi_n(t) = e^{jn\omega_0 t} = e^{jn\frac{2\pi}{T}t}$$

Couch, Digital and Analog Communication Systems, Seventh Edition.

Summary

- Review of Fourier series
- Periodic signals & complex exponents
- Series expansion of a periodic signal
- Trigonometric form of Fourier series
- Properties of Fourier series

- **Reading:** the Couch text, Sec. 2.1-2.5; Oppenheim & Willsky text, Sec. 3.0-3.5. Study carefully all the examples (including end-of-chapter study-aid examples), make sure you understand them and can solve them with the book closed.
- Do some end-of-chapter problems. Students' solution manual provides solutions for many of them.