Source Coding (Compression)

The information context of any source is given by its entropy, *H(x)*, which is the number of bits per source output.

Example: H(x) = 2 bits means that on average 2 bits are required to represent each source output symbol.
Question: How to design a good encoder?

Example:

Source entropy:

$$H(x) = -\sum_i P_i \log P_i = 1.75 bit$$

The average number of bits of code (per symbol) is

$$R = \sum_i R_i P_i = 2 > H(x) = 1.75$$

X_i	P_i	Code word	R_i
a_1	1/2	00	$R_1 = 2$
a_2	1/4	01	$R_2 = 2$
a_3	1/8	10	$R_3 = 2$
a_4	1/8	11	$R_4 = 2$

Hence, the code is not efficient (more bits are used than actually needed.)

The source-coding theorem: a source with entropy H can be encoded with arbitrary small error probability (information loss) at any rate $R>H$. If $R<H$, error probability is bounded away from 0, regardless of encoder complexity.

The source-coding theorem gives a sharp bound on the compression.

Basic idea of source coding (compression): use fewer bits for most frequent symbols. For example, less bits should be allocated for a_1 and more for a_4.

Source Codes

Fixed-length source outputs are mapped into variable-length binary sequences (codewords) based on the idea above. Synchronization may be a problem (i.e. the decoder must know when the next code word begins).

Example:

X_i	P_i	Code 1	Code 2	Code 3	Code 4
a_1	1/2	1	1	0	00
a_2	1/4	01	10	10	01
a_3	1/8	001	100	110	10
a_4	1/16	0001	1000	1110	11
a_5	1/16	00001	10000	1111	110

Source Codes

• Code 1: self-synchronizing (each word ends with 1).
• Code 2: self-synchronizing, but not instantaneous. Both of them are uniquely decodable.

• Code 4 is not uniquely decodable. Example: 110110 can be decoded as a_5a_5 or $a_4a_2a_3$. It should never be used in practice.

The prefix condition: no any code word is a prefix of another code word. Codes 1& 3 satisfy it.

Any code is uniquely decodable & instantaneous if and only if it meets the prefix condition.

Source Codes

Good code meets prefix condition (code 1 & 3), and has the smallest average word length. For code 1,

$$R = \sum_i R_i P_i = \frac{31}{16}$$

and R=30/16 for code 3. Hence code 3 is the best one: uniquely decodable & instantaneous (prefix condition), and has the least average word length.

Huffman Encoding Algorithm

Basic idea of Huffman algorithm: if we can encode each source output of P_i with $\log(1/P_i)$ bits, then

$$R = \sum_i P_i \log 1/P_i = H$$

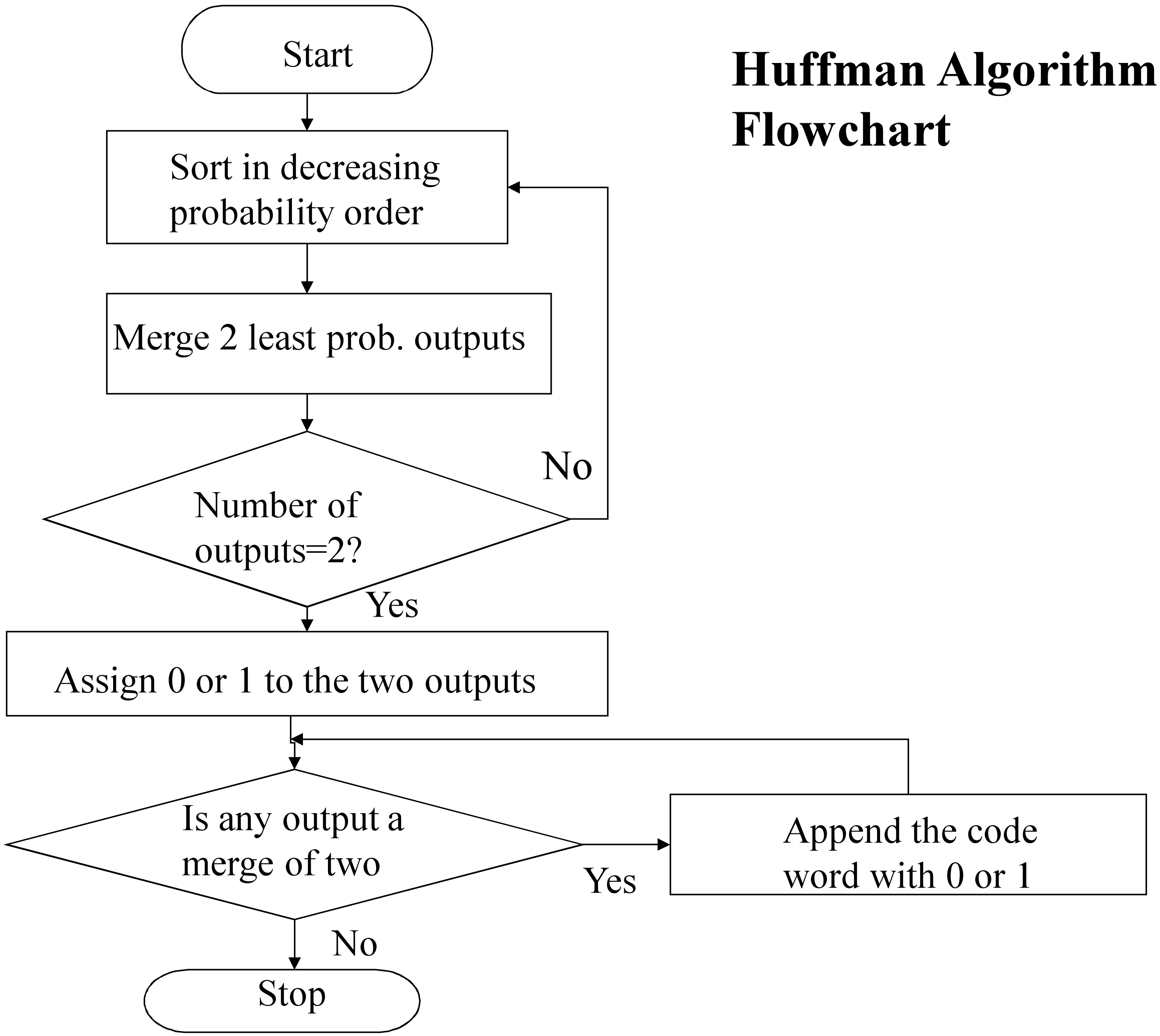
i.e. the least possible average length (from the source coding theorem).

The Huffman codes are optimum: among all the codes that satisfy the prefix conditions, they have the minimum average length.

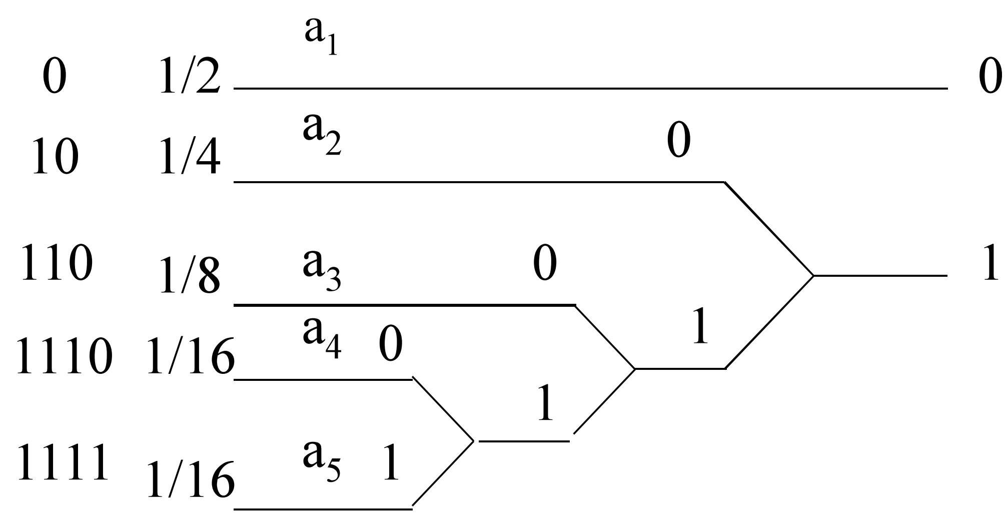
Huffman Encoding Algorithm

1. Sort source outputs in decreasing order of probabilities.
2. Merge the 2 least-probable outputs into one (its probability is the sum of indiv. prob.)
3. If the number of remaining outputs is 2, go to step 4, otherwise go to step 1
4. Assign 0 and 1 as code words for the 2 outputs
5. If an output is a merger of 2 outputs in the proceeding step, append 0 and 1 to the code word
6. Repeat 5 until there are no merged outputs

Huffman Algorithm Flowchart



Design example:



The tree diagram

The average length of a Huffman code, $R = \sum_i P(x_i) l(x_i)$ satisfies to the following: $H(x) \le R < H(x) + 1$

If the code is designed for blocks of n symbols (rather than single symbols), then

$$H(x^n) \le nR < H(x^n) + 1$$

where $R^n = nR$, and $H(x) = H(x^n)/n$. Note that $R \to H$ as $n \to \infty$. This proves the optimality of the Huffman algorithm.

Summary

- Source coding (compression).
- The source coding theorem.
- Huffman code & algorithm.

- **Homework**: Reading, Proakis and Salehi (2nd ed.), 6.2, 6.3. Study carefully all the examples, make sure you understand them and can solve with the book closed.
- Reference: T.M. Cover, J.A. Thomas, Elements of Information Theory, Wiley, 2006