MIMO: Tx & Rx antenna arrays

- multiple Tx antennas
- multiple Rx antenna
- best Tx/Rx strategies?
MIMO Channel Model

\[ y(t) = Hx(t) + \xi(t) \]  \hspace{1cm} (1)

\( x(t) \) = Tx signal (vector)
\( y(t) \) = Rx signal (vector)
\( H \) = fixed channel vector; \( h_{ij} \) = channel gain from \( j \)-th Tx antenna to \( i \)-th Rx antenna
\( \xi(t) \) = Rx noise (vector)

* Compare to the SIMO/MISO models.
MIMO Channel Model

\[ y(t) = Hx(t) + \xi(t) \]
**Tx/Rx Beamforming over the MIMO Channel**

**Tx beamforming:**

\[
x(t) = \mathbf{w}_t \cdot x(t)
\]

\(x(t) = \text{scalar Tx signal (complex amplitude, carries the Tx data)}\)

\(\mathbf{w}_t = \text{fixed Tx beamforming vector.}\)

**Rx beamforming:**

\[
y_r(t) = \mathbf{w}_r^+ y(t) = \mathbf{w}_r^+ \mathbf{Hw}_t x(t) + \mathbf{w}_r^+ \xi(t) = y_s(t) + y_n(t)
\]

\(y_s(t) = \text{signal part (no noise),}\)

\(y_n(t) = \text{noise part (no signal),}\)

\(\mathbf{w}_r = \text{(fixed) Rx beamforming vector.}\)
Tx/Rx Beamforming

How to choose \( w_t, w_r \)?

The Rx SNR \( \gamma_r \) (after the Rx beamformer) is

\[
\gamma_r = \frac{P_s}{P_n} = \frac{|y_s|^2}{|y_n|^2} = \frac{|w_r^H H w_t|^2}{|w_r|^2} \gamma_1
\]  

(4)

where \( \gamma_1 = \sigma_x^2/\sigma_0^2 \) is the Rx SNR with single Tx/Rx antenna and \( h = 1 \).

How to maximize \( \gamma_r \)?
Maximizing $\gamma_r$:

$$\gamma_r = \frac{|w_r^+ H w_t|^2}{|w_r|^2} \gamma_1 \leq |H w_t|^2 \gamma_1 \leq \sigma_1^2(H) |w_t|^2 \gamma_1 = \sigma_1^2(H) \gamma_1$$  \hspace{1cm} (5)

where $\sigma_1(H)$ is the largest singular value of $H$.

(a): how? equality?

(b): via the SVD properties,

$$|H x| \leq \sigma_1(H) |x|$$  \hspace{1cm} (6)

with equality iff $x = \alpha v_1$, where $v_1$ is the left singular vector of $H$ corresponding to its largest singular value.

(c): $|w_t| = 1$, to satisfy power constraint.
Tx/Rx Beamforming

Hence, $\gamma_r$ is maximized by

$$w_t = v_1(H), \quad w_r = u_1(H)$$  \hspace{1cm} (7)

where $u_1(H)$ is the left singular vector of $H$ corresponding to its largest singular value $\sigma_1(H)$.

The maximum Rx SNR is

$$\gamma_{r,\text{max}} = \max_{w_t, w_r} \gamma_r = \sigma_1^2(H) \gamma_1$$  \hspace{1cm} (8)
Singular Value Decomposition (SVD)\footnote{R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge Univ. Press, 2013}^2

Definition of singular value $\sigma_i$ and its left/right singular vector $v_i/u_i$ of $H$:

$$Hv_i = \sigma_i u_i, \quad u_i^+ H = \sigma_i v_i^+$$ \quad (9)

Applies to any matrix (not only square),

$$H = U\Sigma V^+ = \sum_i \sigma_i u_i v_i^+$$ \quad (10)

$U$ = unitary matrix of left singular vectors of $H$,

$V$ = likewise for its right singular vectors,

$\Sigma$ = diagonal matrix of its singular values,

$u_i, v_i$ = $i$-th column of $U, V$,

$\sigma_i \geq 0$ = $i$-th diagonal entry of $\Sigma$ = $i$-th singular value of $H$,

ordering: $\sigma_1 \geq \sigma_2 \geq \ldots$.

\footnote{https://en.wikipedia.org/wiki/Singular_value_decomposition}
Eigenvalue Decomposition (EVD)\textsuperscript{34}

EVD: applies to any square matrix.
Definition of eigenvalue $\lambda_i$ and its eigenvector $\mathbf{u}_i$ of $\mathbf{W}$:

$$\mathbf{W}\mathbf{u}_i = \lambda_i \mathbf{u}_i$$ \hspace{1cm} (11)

For Hermitian $\mathbf{W}$,

$$\mathbf{W} = \mathbf{U}\Lambda\mathbf{U}^+ = \sum_i \lambda_i \mathbf{u}_i \mathbf{u}_i^+$$ \hspace{1cm} (12)

$\mathbf{U} =$ unitary matrix of eigenvectors of $\mathbf{W}$,
$\Lambda =$ diagonal matrix of its eigenvalues,
$\mathbf{u}_i =$ $i$-th column of $\mathbf{U} =$ $i$-th eigenvector of $\mathbf{W}$,
$\lambda_i =$ $i$-th diagonal entry of $\Lambda =$ $i$-th eigenvalue of $\mathbf{W}$,

\textsuperscript{3}R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge Univ. Press, 1985
\textsuperscript{4}https://en.wikipedia.org/wiki/Eigendecomposition_of_a_matrix
Relationship of SVD and EVD

Set \( \mathbf{W} = \mathbf{HH}^+ \). Then,

\[
\lambda_i(\mathbf{W}) = \sigma_i^2(\mathbf{H}), \quad u_i(\mathbf{W}) = u_i(\mathbf{H}) \tag{13}
\]

and

\[
\mathbf{W} = \mathbf{U}\Lambda\mathbf{U}^+, \quad \mathbf{H} = \mathbf{U}\Sigma\mathbf{V}^+, \quad \Lambda = \Sigma\Sigma^+ \tag{14}
\]

i.e. the EVD can be obtained from the SVD and vice versa:

- eigenvectors of \( \mathbf{HH}^+ \) are the right singular vectors of \( \mathbf{H} \)
- eigenvectors of \( \mathbf{H}^+\mathbf{H} \) are the left singular vectors of \( \mathbf{H} \)
- eigenvalues of \( \mathbf{HH}^+ \) are the squared singular values of \( \mathbf{H} \)
The Capacity of Tx/Rx beamforming

Extended channel: the channel + Tx/Rx beamforming.

System capacity: the extended channel capacity,

\[ C = \log(1 + \gamma_{r,\text{max}}) = \log(1 + \sigma_1^2(H)\gamma_1) \]  \hspace{1cm} (15)

This is the largest rate (SE) the Tx/Rx beamforming can deliver.

Can we do better than that???

Special cases:

- SIMO channel: \( H = h, \sigma_1(H) = ? \, v_1 = ? \)
- MISO channel: \( H = h^+, \sigma_1(H) = ? \, u_1 = ? \)
- Free space: \( h_{ij} = 1 \) for all \( i, j \).
The Capacity of MIMO Channel

Can we do better than Tx/Rx beamforming ???

The capacity of MIMO channel is

\[ C = \max_{p(x)} I(X; Y) \text{ s.t. } \text{tr} \ R_x \leq P \] (16)

\( X = \) the random Tx vector,
\( Y = \) the random Rx vector.

How to find the max???
How to find the max???

Key:

\[ H(Y|X) = H(\Xi) = \log |R_\xi| + n \log(\pi e) \]  
\[ H(Y) \leq \log |R_y| + n \log(\pi e) \]  

so that

\[ I(X; Y) = H(Y) - H(\Xi) \leq \log \frac{|R_y|}{|R_\xi|} \]  

\( R_y = yy^+, \ R_\xi = \xi\xi^+ \) are covariance matrices of \( y, \xi \).

The UB is achieved by \( X \sim CN(0, R_x) \).
The Capacity of MIMO Channel

Observe that

\[ R_y = HR_xH^+ + \sigma_0^2 I \]  

(20)

so that

\[ I(X; Y) \leq \log |I + \sigma_0^{-2}WR_x| \]  

(21)

where \( W = H^+H \), and hence

\[
C = \max_{p(x)} I(X; Y) \text{ s.t. } \text{tr}R_x \leq P
\]

(22)

\[
\leq \max_{\text{tr}R_x \leq P} \log |I + \sigma_0^{-2}WR_x|
\]

(23)

Since the UB is achieved by \( X \sim CN(0, R_x) \), it is the capacity.
The Capacity of MIMO Channel

Thus, the capacity is

\[
C = \max_{\text{tr} R_x \leq P} \log |I + \sigma_0^{-2} W R_x| \tag{24}
\]

and an optimal input is \( X \sim \mathcal{CN}(0, R_x) \).

We will further normalize the noise power, \( \sigma_0^2 = 1 \).
Thus, the capacity is

\[
C = \max_{\text{tr} R_x \leq P} \log |I + \sigma_0^{-2} W R_x| \tag{24}
\]

and an optimal input is \( X \sim CN(0, R_x) \).

We will further normalize the noise power, \( \sigma_0^2 = 1 \).

But: **How to find the max??**
The Capacity of MIMO Channel

How to find the max???
The Capacity of MIMO Channel

How to find the max???

Key: Hadamard inequality.

The Capacity of MIMO Channel

The capacity is

$$C = \max_{\text{tr} R_x \leq P} \log |I + WR_x|$$

$$= \max_{\text{tr} R_x \leq P} \log |I + \Lambda W U_W^+ R_x U_W|$$

$$= \max_{\text{tr} \tilde{R}_x \leq P} \log |I + \Lambda W \tilde{R}_x|$$

$$\leq \max_{\text{tr} \tilde{D}_x \leq P} \log |I + \Lambda W \tilde{D}_x|$$

$$= \max \sum d_i \log(1 + \lambda_{wi} d_i) \text{ s.t. } d_i \geq 0, \sum d_i \leq P$$

$$\tilde{R}_x = U_W^+ R_x U_W, \ d_i = i-th \ diagonal \ entry \ of \ \tilde{D}_x$$

The UB is achieved by $U_w = U_{R_x}$, so that $d_i = \lambda_i(\tilde{R}_x) = \lambda_i(R_x)$
The Capacity of MIMO Channel

Thus, the capacity is

$$C = \max_{\lambda_i} \sum_i \log(1 + \lambda_{wi} \lambda_i) \text{ s.t. } \lambda_i \geq 0, \sum_i \lambda_i \leq P$$

and the signaling on the eigenvectors of $W = H^+ H$ (or right singular vectors of $H$) is optimal,

$$R^* = U_W \Lambda^* U_W^+ = \sum_i \lambda_i^* u_{wi} u_{wi}^+$$

(29)

where $\Lambda^* = \text{diag}\{\lambda_i^*\}$, i.e. an optimal power allocation to the channel eigenmodes.

But: How to find the max??? How to implement (29)???
The Water-Filling (WF) Algorithm

The $\max_{\lambda_i}$ is given by

$$\lambda_i^* = (\mu^{-1} - \lambda_{wi}^{-1})_+$$  \hspace{1cm} (30)

where $(x)_+ = \max(x, 0)$ is positive part; $\mu$ is the Lagrange multiplier responsible for the total power constraint $\sum_i \lambda_i \leq P$.

$\mu$ is the (unique) solution to

$$\sum_i (\mu^{-1} - \lambda_{wi}^{-1})_+ = P$$  \hspace{1cm} (31)

Numerically: via e.g. bisection method. Analytically: possible in some special cases.

This is the optimal power allocation among the eigenmodes and is known as "water-filling" (WF).
The MIMO Capacity via the WF

The MIMO capacity is

\[ C = \sum_i \log(1 + \lambda_{wi} \lambda_i^*) = \sum_{i: \lambda_{wi} > \mu} \log(\mu^{-1} \lambda_{wi}) \]  

(32)

so that active eigenmodes satisfy \( \lambda_{wi} > \mu \).

Proof of WF: via the KKT conditions for constrained optimization (Lagrange multipliers).

Q.: prove that (31) (i) always has a solution, and (ii) the solution is unique. Hint: show that the l.h.s of (31) is monotonically decreasing in \( \mu \). When \( \mu = 0? \mu = \infty? \)
WF Examples

1. Identical eigenvalues of $\mathbf{W}$: $\lambda_{wi} = \lambda_{w} \forall i$,

$$\lambda_{i}^{*} = \frac{P}{m}, \quad \mathbf{R}^{*} = \frac{P}{m} \mathbf{I}, \quad C = m \log \left(1 + \frac{P}{m} \lambda_{w}\right)$$ (33)

where $P = \gamma = \text{SNR}$ (with $m = 1$).

2. Rank-1 $\mathbf{W}$: $\lambda_{w1} = \lambda_{w}, \lambda_{w2} = \ldots = \lambda_{wm} = 0$,

$$\lambda_{1}^{*} = P, \quad \lambda_{2}^{*} = \ldots = \lambda_{m}^{*} = 0, \quad \mathbf{R}^{*} = P \mathbf{u}_1 \mathbf{u}_1^{+}$$

$$C = \log (1 + \lambda_{w} P)$$ (34)

3. Optimal Tx structure?
WF Examples

1. Identical eigenvalues of $W$: $\lambda_{wi} = \lambda_w \forall i$,
WF Properties

Q1: prove that only the strongest eigenmode is active at low SNR, while all eigenmodes are active at high SNR. Derive conditions for low/high SNR.
Q2: prove that the number of active eigenmodes increases with the SNR.
Q3: prove that stronger eigenmodes get more power, i.e. "rich get richer" or, equivalently, "capitalism is better than communism".
The Capacity of MIMO Channel

Q4: compare the MIMO channel capacity in (32) to that of the Tx-Rx beamforming in (15). Which is better (consider the most general case)? When are they equal?

Q5: consider now the MIMO channel with correlated noise, 

\[ y(t) = Hx(t) + \xi(t) \]  

where \( \xi \sim \mathcal{CN}(0, R_\xi) \), \( R_\xi \) = noise covariance matrix. Find its capacity. Correlated (”colored”) noise can model interference.

Q6: In Q5, what happens if \( R_\xi \) is singular?
Summary

- MIMO channel: Tx & Rx antenna arrays
- Tx/Rx beamforming, its capacity
- The MIMO channel capacity
- Water-filling algorithm
- Examples and special cases
Reading

- D. Tse, P. Viswanath, Fundamentals of Wireless Communications - Ch. 7.1-2, 8.1-8.3, Appendix A, B.