ELG4157: Digital Control Systems

Discrete Equivalents

Z-Transform

Stability Criteria

Steady State Error

Design of Digital Control Systems

Advantages and Disadvantages

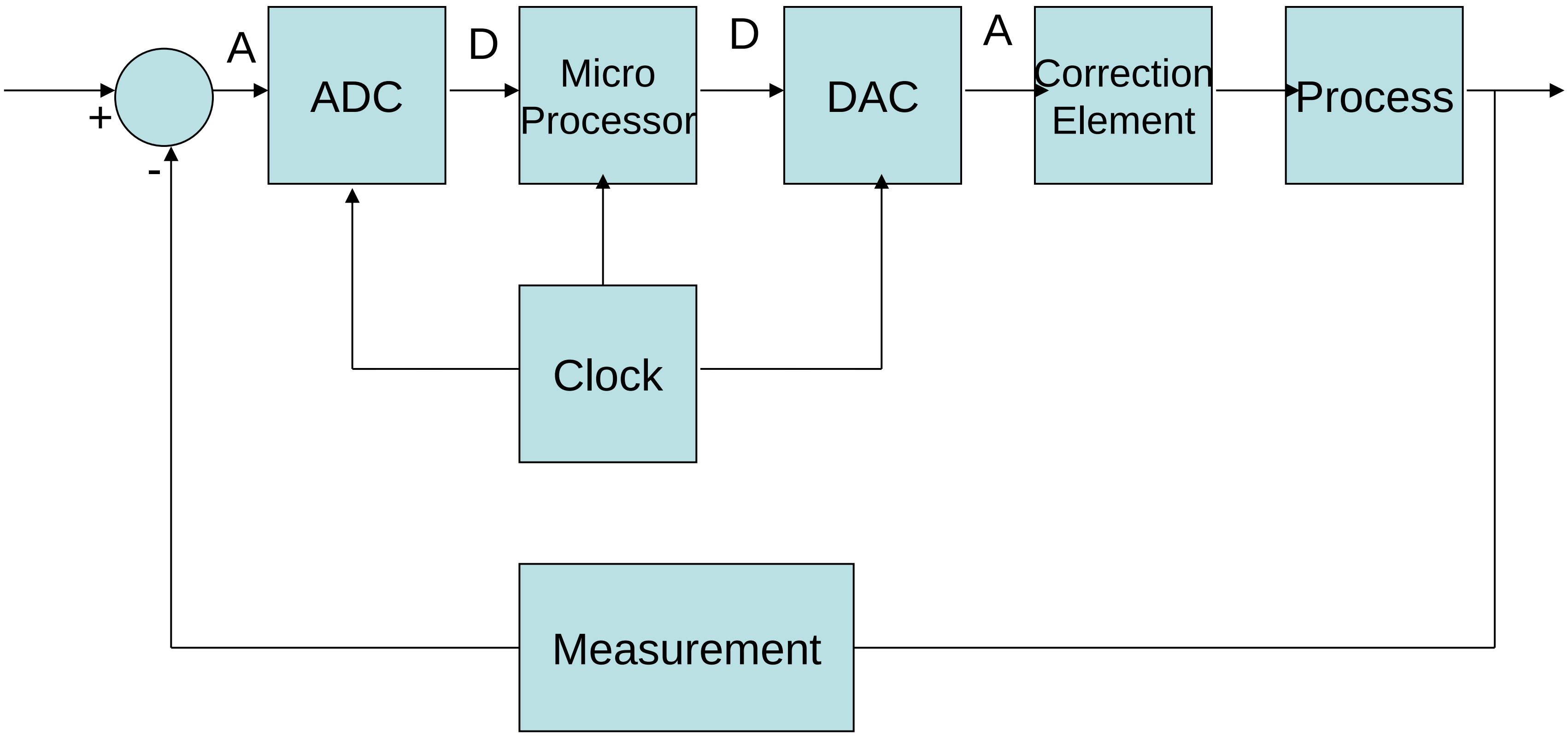
- Improved sensitivity.
- Use digital components.
- Control algorithms easily modified.
- Many systems inherently are digital.

- Develop complex math algorithms.
- Lose information during conversions due to technical problems.
- Most signals continuous in nature.

Digitization

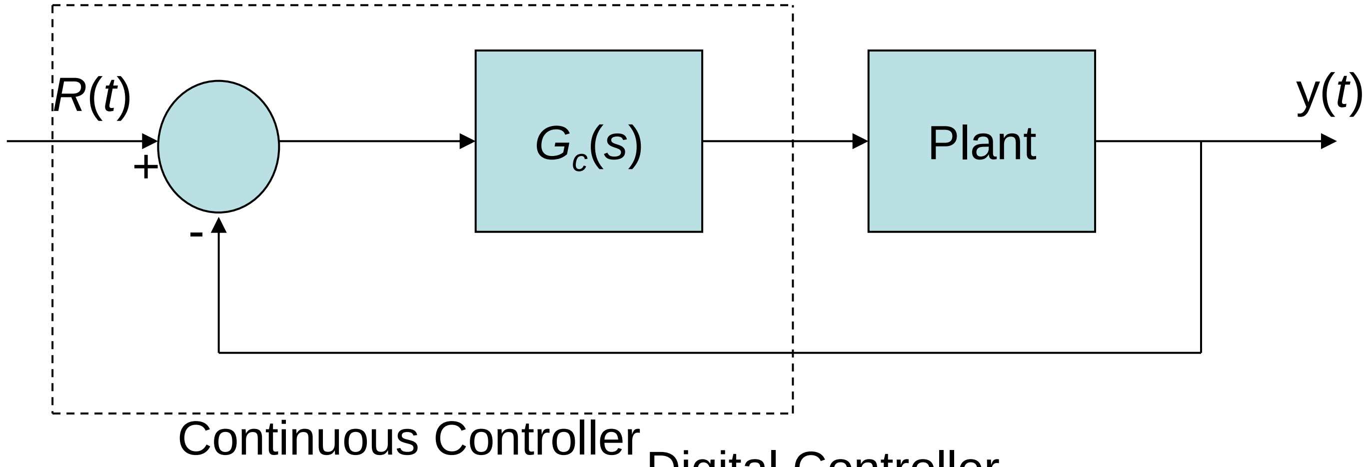
- The difference between the continuous and digital systems is that the digital system operates on samples of the sensed plant rather than the continuous signal and that the control provided by the digital controller D(s) must be generated by algebraic equations.
- In this regard, we will consider the action of the analog-to-digital (A/D) converter on the signal. This device samples a physical signal, mostly voltage, and convert it to binary number that usually consists of 10 to 16 bits.
- Conversion from the analog signal $y(t)$ to the samples $y(kt)$, occurs repeatedly at instants of time T seconds apart.
- A system having both discrete and continuous signals is called sampled data system.
- The sample rate required depends on the closed-loop bandwidth of the system. Generally, sample rates should be about 20 times the bandwidth or faster in order to assure that the digital controller will match the performance of the continuous controller.

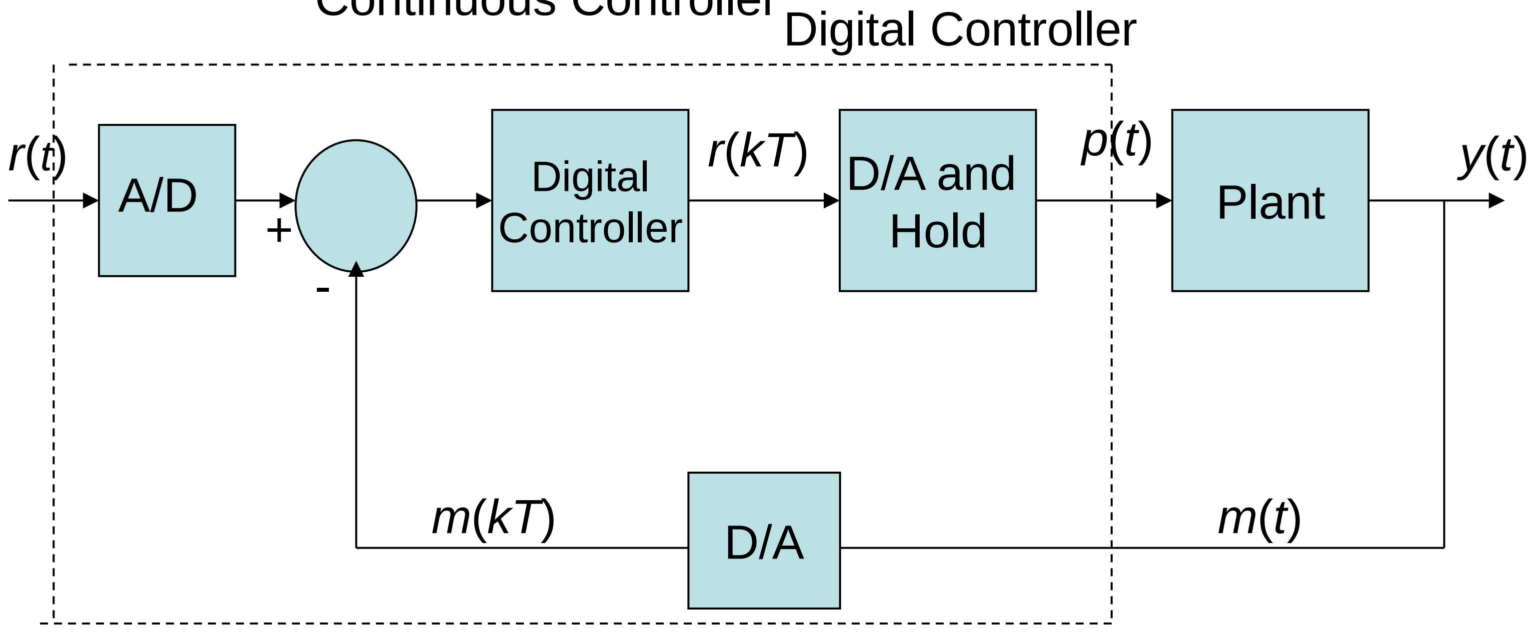
Digital Control System



A: Analog
D: Digital

Continuous Controller and Digital Control



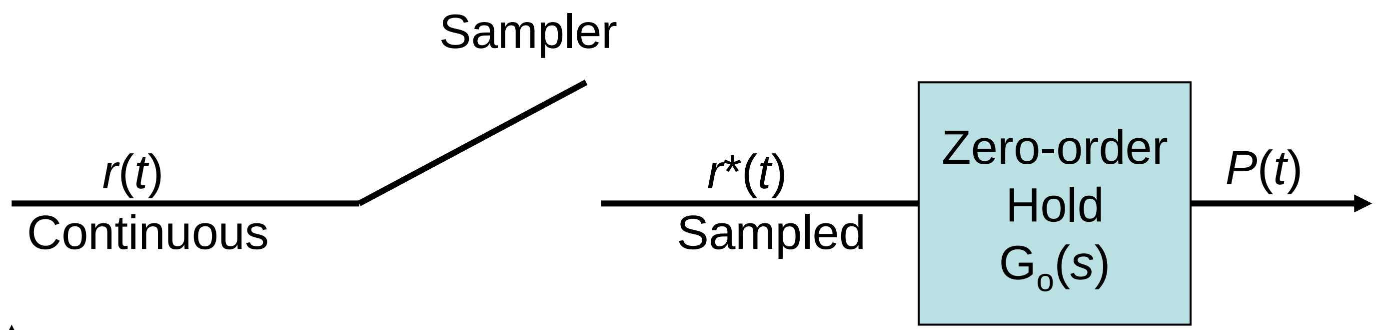


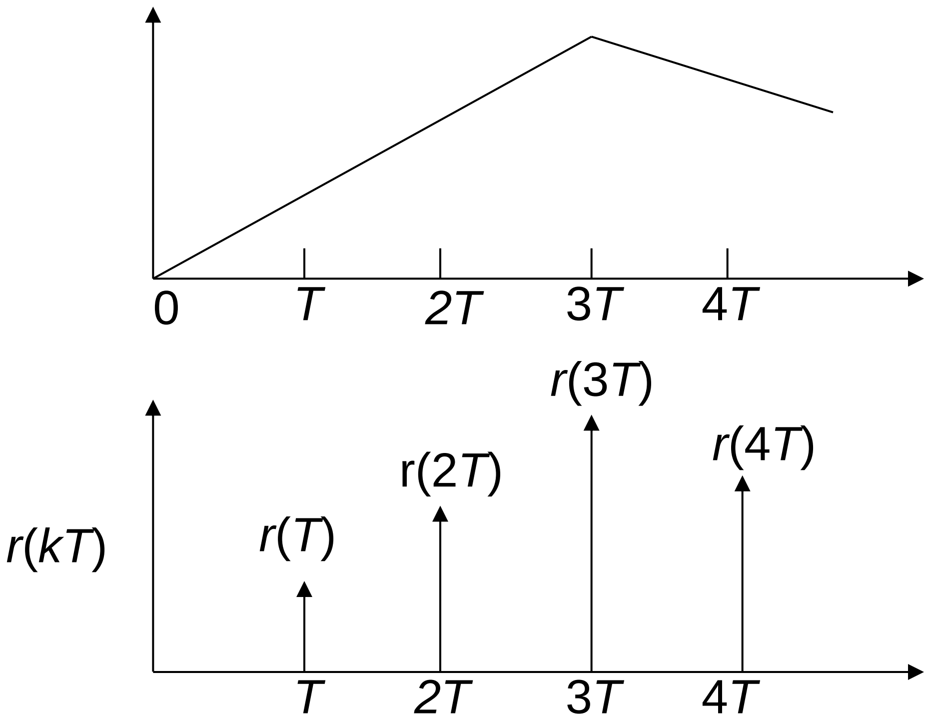
Applications of Automatic Computer Controlled Systems

- Most control systems today use digital computers (usually microprocessors) to implement the controllers). Some applications are:
- Machine Tools
- Metal Working Processes
- Chemical Processes
- Aircraft Control
- Automobile Traffic Control
- Automobile Air-Fuel Ratio
- Digital Control Improves Sensitivity to Signal Noise.

Digital Control System

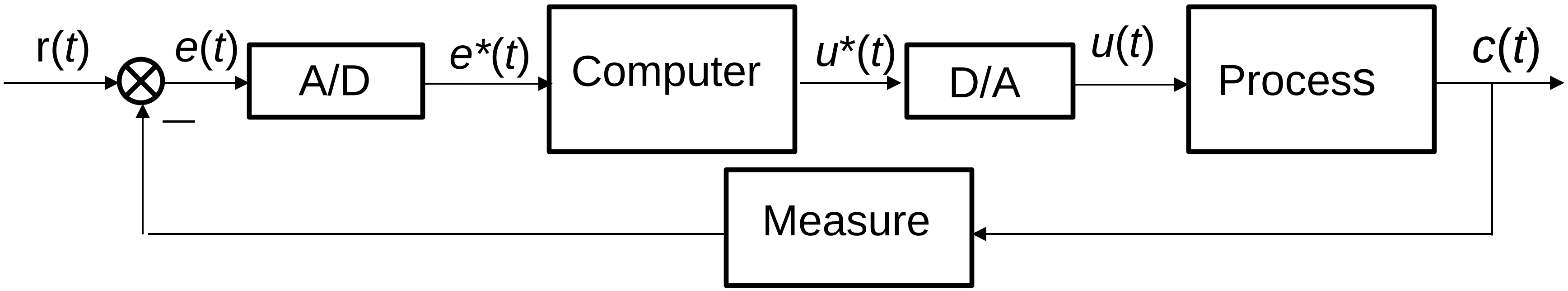
- Analog electronics can integrate and differentiate signals. In order for a digital computer to accomplish these tasks, the differential equations describing compensation must be approximated by reducing them to algebraic equations involving addition, division, and multiplication.
- A digital computer may serve as a compensator or controller in a feedback control system. Since the computer receives data only at specific intervals, it is necessary to develop a method for describing and analyzing the performance of computer control systems.
- The computer system uses data sampled at prescribed intervals, resulting in a series of signals. These time series, called sampled data, can be transformed to the *s*-domain, and then to the z-domain by the relation $z = e^{zt}$.
- Assume that all numbers that enter or leave the computer has the same fixed period *T*, called the sampling period.
- A sampler is basically a switch that closes every *T* seconds for one instant of time.





$$G_0(s) = \frac{1}{s} - \frac{1}{s}e^{-sT} = \frac{\left(1 - e^{-sT}\right)}{s}$$

Modeling of Digital Computer



Sampling analysis

Expression of the sampling signal

$$x^*(t) = x(t)\cdot\delta_T(t) = x(t)\cdot\sum_{k=0}^{\infty}\delta(t-kT) = \sum_{k=0}^{\infty}x(kT)\delta(t-kT)$$

Analog to Digital Conversion: Sampling

An input signal is converted from continuous-varying physical value (e.g. pressure in air, or frequency or wavelength of light), by some electro-mechanical device into a continuously varying electrical signal. This signal has a range of amplitude, and a range of frequencies that can present. This continuously varying electrical signal may then be converted to a sequence of digital values, called samples, by some analog to digital conversion circuit.

- There are two factors which determine the accuracy with which the digital sequence of values captures the original continuous signal: the maximum rate at which we sample, and the number of bits used in each sample. This latter value is known as the quantization level

Zero-Order Hold

- The Zero-Order Hold block samples and holds its input for the specified sample period.
- The block accepts one input and generates one output, both of which can be scalar or vector. If the input is a vector, all elements of the vector are held for the same sample period.
- This device provides a mechanism for discretizing one or more signals in time, or resampling the signal at a different rate.
- The sample rate of the Zero-Order Hold must be set to that of the slower block. For slow-to-fast transitions, use the unit delay block.

The *z*-Transform

The z-Transform is used to take discrete time domain signals into a complex-variable frequency domain. It plays a similar role to the one the Laplace transform does in the continuous time domain. The z-transform opens up new ways of solving problems and designing discrete domain applications. The z-transform converts a discrete time domain signal, which is a sequence of real numbers, into a complex frequency domain representation.

$$r^*(t) = \sum_{k=0}^{\infty} r(kT)\,\delta(t-kT)$$

For a signal $t \rangle 0$, Using the Laplace transforms, we have

$$\Im\{r^*(t)\} = \sum_{k=0}^{\infty} r(kT)e^{-ksT}$$

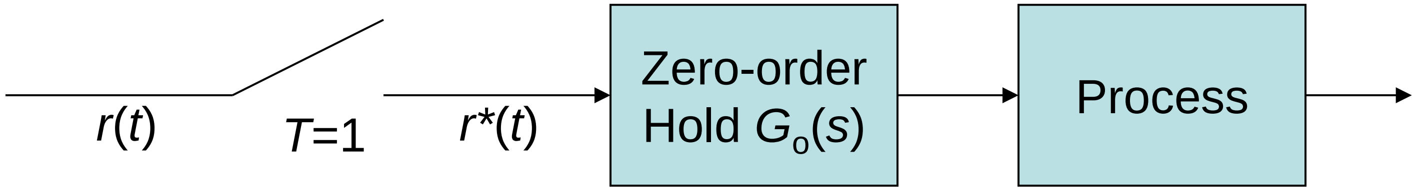
$$z = e^{sT}$$

$$Z\{r(t)\} = Z\{r^*(t)\} = \sum_{k=0}^{\infty} r(kT)z^{-k}$$

$$U(z) = \frac{z}{z-1}$$

$$Z\{f(t)\} = F(z) = \sum_{k=0}^{\infty} f(kT)z^{-k}$$

Transfer Function of Open-Loop System



$$G_o(s) = \frac{(1-e^{-st})}{s}; G_p(s) = \frac{1}{s(s+1)}$$

$$\frac{Y(s)}{R*(s)} = G_o(s)G_p(s) = G(s) = \frac{1-e^{-st}}{s^2(s+1)}$$

$$\text{Expanding into partial fraction}: G(s) = \left(1-e^{-st}\right)\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right)$$

$$G(z) = \frac{0.3678z + 0.2644}{z^2 - 1.3678z + 0.3678}$$

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1.	–	–	Kronecker delta $\delta_0(k)$ $1 \quad k = 0$ $0 \quad k \neq 0$	1
2.	–	–	$\delta_0(n-k)$ $1 \quad n = k$ $0 \quad n \neq k$	z^{-k}
3.	$\frac{1}{s}$	$1(t)$	$1(k)$	$\frac{1}{1-z^{-1}}$
4.	$\frac{1}{s+a}$	e^{-at}	e^{-akT}	$\frac{1}{1-e^{-aT}z^{-1}}$
5.	$\frac{1}{s^2}$	t	kT	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
6.	$\frac{2}{s^3}$	t^2	$(kT)^2$	$\frac{T^2z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$
7.	$\frac{6}{s^4}$	t^3	$(kT)^3$	$\frac{T^3z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$
8.	$\frac{a}{s(s+a)}$	$1-e^{-at}$	$1-e^{-akT}$	$\frac{(1-e^{-aT})z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}$
9.	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z^{-1}}{(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})}$
10.	$\frac{1}{(s+a)^2}$	te^{-at}	kTe^{-akT}	$\frac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
11.	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	$(1-akT)e^{-akT}$	$\frac{1-(1+aT)e^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
12.	$\frac{2}{(s+a)^3}$	t^2e^{-at}	$(kT)^2e^{-akT}$	$\frac{T^2e^{-aT}(1+e^{-aT}z^{-1})z^{-1}}{(1-e^{-aT}z^{-1})^3}$
13.	$\frac{a^2}{s^2(s+a)}$	$at-1+e^{-at}$	$akT-1+e^{-akT}$	$\frac{[(aT-1+e^{-aT})+(1-e^{-aT}-aTe^{-aT})z^{-1}]z^{-1}}{(1-z^{-1})^2(1-e^{-aT}z^{-1})}$
14.	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\frac{z^{-1}\sin \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
15.	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\frac{1-z^{-1}\cos \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
16.	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin \omega t$	$e^{-akT}\sin \omega kT$	$\frac{e^{-aT}z^{-1}\sin \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
17.	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$e^{-akT}\cos \omega kT$	$\frac{1-e^{-aT}z^{-1}\cos \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
18.	–	–	a^k	$\frac{1}{1-az^{-1}}$
19.	–	–	a^{k-1} $k = 1, 2, 3, \ldots$	$\frac{z^{-1}}{1-az^{-1}}$
20.	–	–	ka^{k-1}	$\frac{z^{-1}}{(1-az^{-1})^2}$
21.	–	–	k^2a^{k-1}	$\frac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$
22.	–	–	k^3a^{k-1}	$\frac{z^{-1}(1+4az^{-1}+a^2z^{-2})}{(1-az^{-1})^4}$
23.	–	–	k^4a^{k-1}	$\frac{z^{-1}(1+11az^{-1}+11a^2z^{-2}+a^3z^{-3})}{(1-az^{-1})^5}$
24.	–	–	$a^k \cos k\pi$	$\frac{1}{1+az^{-1}}$

Z-Transform

Z-transform method: Partial-fraction expansion approaches

$$If: \quad X(s) = \frac{A(s)}{(s+a_1)(s+a_2)\cdots(s+a_n)} = \frac{K_1}{s+a_1} + \frac{K_2}{s+a_2} + \cdots + \frac{K_n}{s+a_n}$$

$$Then: \quad X(z) = \sum_{i=1}^{n} \frac{K_i z}{z - e^{-a_i T}}$$

Example: $$Z\left[\frac{5(s+4)}{s(s+1)(s+2)}\right] = Z\left[\frac{10}{s} - \frac{15}{s+1} + \frac{5}{s+2}\right] = \frac{10z}{z-1} - \frac{15z}{z-e^{-T}} + \frac{5z}{z-e^{-2T}}$$

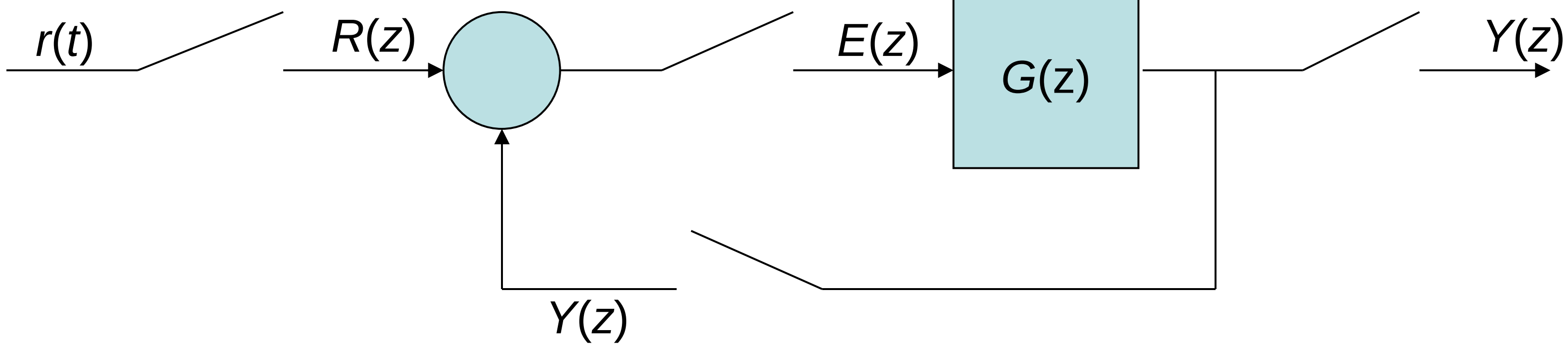
Inverse Z-transform method: Partial-fraction expansion approaches

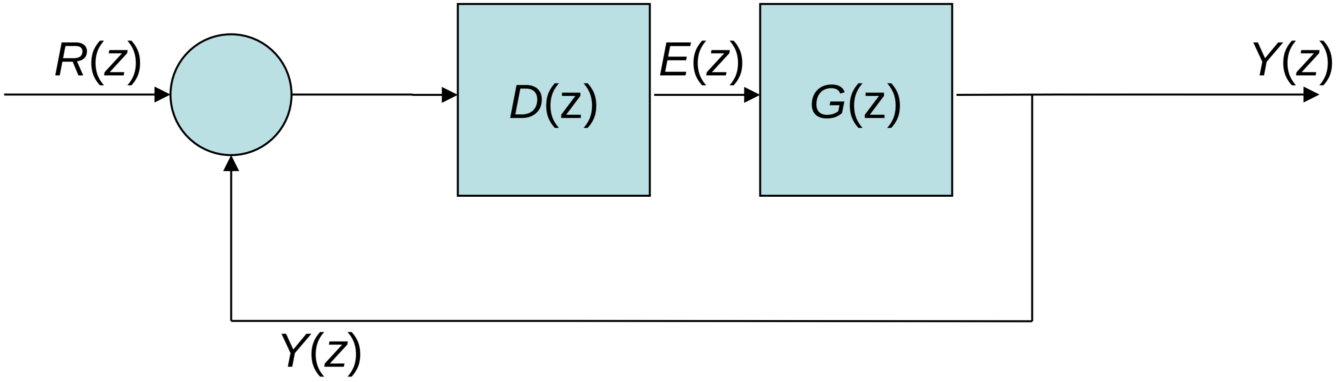
$$If: \quad X(z) = \frac{A(z)}{(z-e^{-a_1 T})(z-e^{-a_2 T})\cdots(s-e^{-a_n T})} = \frac{K_1 z}{z-e^{-a_1 T}} + \frac{K_2 z}{s-e^{-a_2 T}} + \cdots$$

$$then: \; X(kT) = \sum_{i=1}^{n} K_i e^{-a_i kT}$$

Example: $$x(kT) = Z^{-1}\left[\frac{z(1-e^{-2T})}{(z-1)(z-e^{-2T})}\right] = Z^{-1}\left[\frac{z}{z-1} - \frac{z}{z-e^{-2T}}\right] = 1 - e^{-2kT}$$

Closed-Loop Feedback Sampled-Data Systems





$$\frac{Y(z)}{R(z)} = T(z) = \frac{G(z)}{1+G(z)} = \frac{G(z)D(z)}{1+G(z)D(z)}$$

Now Let us Continue with the Closed-Loop System for the Same Problem

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1+G(z)} = \frac{0.3678z + 0.2644}{z^2 - z + 0.6322}$$

$$\text{Assume an a unit step input}: R(z) = \frac{z}{z-1}$$

$$Y(z) = \frac{z(0.3678z + 0.2644)}{(z-1)(z^2 - z + 0.6322)} = \frac{0.3678z^2 + 0.2644z}{z^3 - 2z^2 + 1.6322z - 0.6322}$$

$$Y(z) = 0.3678z^{-1} + z^{-2} + 1.4z^{-3} + 1.4z^{-4} + 1.147z^{-5}$$

Stability

- The difference between the stability of the continuous system and digital system is the effect of sampling rate on the transient response.
- Changes in sampling rate not only change the nature of the response from overdamped to underdamped, but also can turn the system to an unstable.
- Stability of a digital system can be discussed from two perspectives:
- *z*-plane
- *s*-plane

Stability Analysis in the *z*-Plane

A linear **continuous** feedback control system is stable if all poles of the closed-loop transfer function $T(s)$ lie in the left half of the *s*-plane.

In the left-hand *s*-plane, $\sigma < 0$; therefore, the related magnitude of *z* varies between 0 and 1. Accordingly the imaginary axis of the *s*-plane corresponds to the unit circle in the *z*-plane, and the inside of the **unit circle** corresponds to the left half of the *s*-plane.

A sampled system is stable if all the poles of the closed-loop transfer function $T(z)$ lie within the unit circle of the *z*-plane.

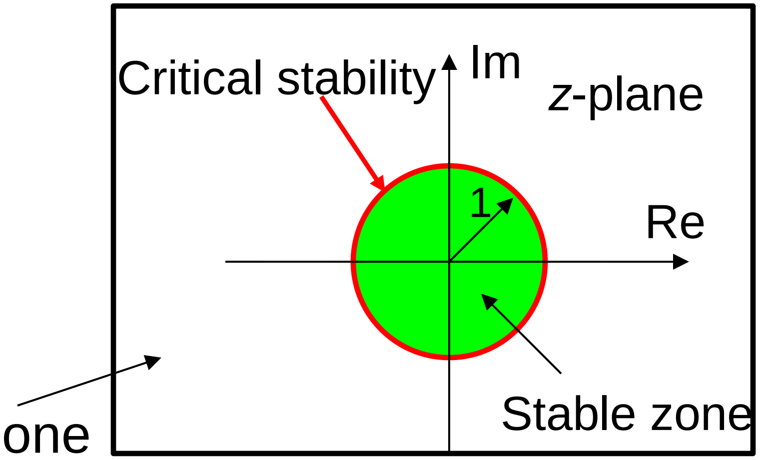
$$z = e^{sT} = e^{(\sigma + j\omega)T}$$

$$|z| = e^{\sigma T}$$

$$\angle z = \omega T$$

The Stability Analysis

The graphic expression of the stability condition for the sampling control systems



The stability criterion

In the characteristic equation 1+*GH*(*z*)=0, substitute *z* with

$$z = \frac{s+1}{s-1}$$ —— Bilinear transformation

We can analyze the stability of the sampling control systems the same as we did in chapter 3 (Routh criterion in the *s*-plane) .

$$\left(\begin{array}{l} Proof: \ suppose \ w = \alpha + j\beta, z = x + jy, then: \\ s = \alpha + j\beta = \frac{z+1}{z-1} = \frac{x+jy+1}{x+jy-1} \cdot \left(\frac{x-1-jy}{x-1-jy} \right) = \frac{x^2+y^2-1}{(x-1)^2+y^2} - j\frac{2y}{(x-1)^2+y^2} \\ \alpha \le 0 \quad \Rightarrow \quad x^2+y^2-1 \le 0 \Rightarrow x^2+y^2 \le 1 \\ (for\ the\ left\ half\ of\ the\ s\text{-}plane) \Rightarrow (inside\ the\ unit\ circle\ of\ the\ z\text{-}plane) \end{array} \right)$$

The Stability Analysis

$$1+G(z)=1+\frac{0.632Kz}{z^2-1.368z+0.368}=0$$

Determine *K* for the stable system

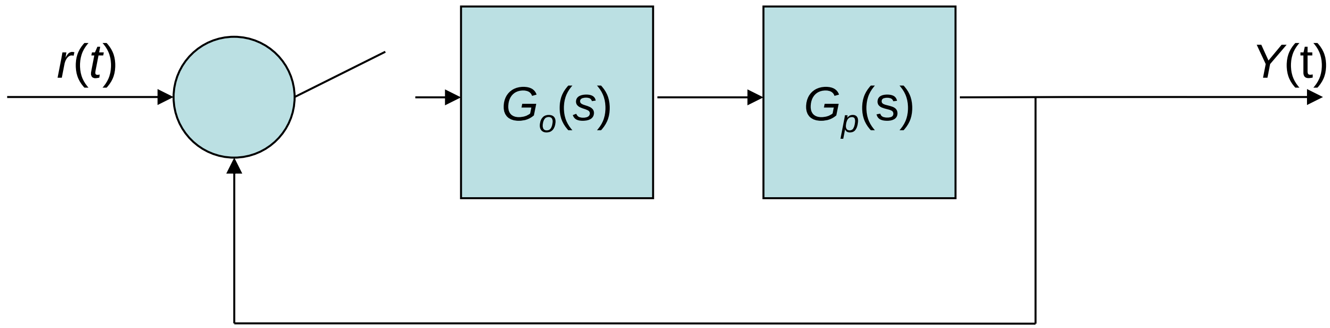
Solution: Make $z=\frac{s+1}{s-1}$

$$1+\frac{0.632Kz}{z^2-1.368z+0.368}=0 \Rightarrow 0.632Ks+1.264s+(2.736-0.632K)=0$$

$$\begin{array}{lll} & 0.632K & 2.736-0.632K \\ \textit{In terms of the Routh criterion:} & 1.264 & \\ & 2.736-0.632K & \end{array}$$

We have: $0 < K < 4.33$

Example: Stability of a closed-loop system



$$G_p(s) = \frac{K}{s(s+1)}; G(z) = \frac{K(0.3678z + 0.2644)}{z^2 - 1.3678z + 0.3678} = \frac{K(az+b)}{z^2 - (1+a)z + a}$$

The poles of the losed-loop transfer function t(z) are the roots of the equation

$$[1 + \mathrm{G(z)}] = 0 : z^2 - (1+a)z + a + Kaz + Kb = 0$$

$$K = 1; \quad z^2 - z + 0.6322 = (z - 0.5 + j0.6182)(z - 0.5 - j0.6182) = 0$$

The system is stable because the roots lie within the unit circle, When $K = 10$

$$z^2 + 2.310z + 3.012 = (z + 1.115 + j1.295)\,(z + 1.115 - j1.295)\ \text{(unstable)}$$

This system is stable for : $0 \langle K \langle 2.39$

Second-order sampled system is unstable for increased gain where the continuous is stable for all values of gain.

Example

Find the stable range of the gain K for the unity feedback digital cruise control system of Example 3.2 with the analog plant transfer function

$$G(s) = \frac{K}{s+3}$$

and with digital-to-analog converter (DAC) and analog-to-digital converter (ADC) if the sampling period is 0.02 s.

Solution

The transfer function for analog subsystem ADC and DAC is

$$\begin{aligned} G_{ZAS}(z) &= \left(1 - z^{-1}\right) \mathscr{Z}\left\{ \mathscr{L}^{-1}\left[\frac{G(s)}{s}\right]\right\} \\ &= \left(1 - z^{-1}\right) \mathscr{Z}\left\{ \mathscr{L}^{-1}\left[\frac{K}{s(s+3)}\right]\right\} \end{aligned}$$

Using the partial fraction expansion

$$\frac{K}{s(s+3)} = \frac{K}{3}\left[\frac{1}{s} - \frac{1}{s+3}\right]$$

we obtain the transfer function

$$G_{ZAS}(z) = \frac{1.9412 \times 10^{-2} K}{z - 0.9418}$$

For unity feedback, the closed-loop characteristic equation is

$$1 + G_{ZAS}(z) = 0$$

which can be simplified to

$$z - 0.9418 + 1.9412\times 10^{-2} \quad K = 0$$

The stability conditions are

$$0.9418 - 1.9412\times 10^{-2} \quad K < 1$$

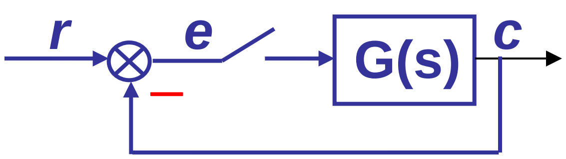
$$-0.9418 + 1.9412\times 10^{-2} \quad K < 1$$

Thus, the stable range of K is

$$-3 < K < 100.03$$

The Steady State Error Analysis

$$e_{ss} = \lim_{z \to 1}(z-1)E(z)$$

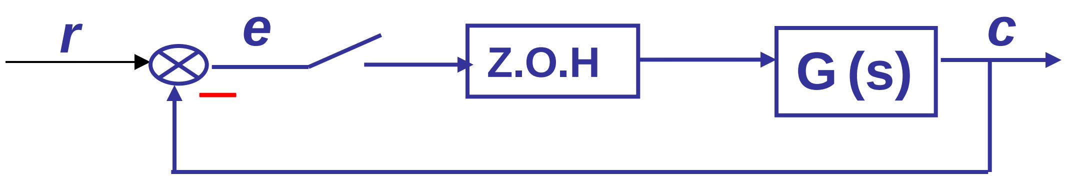


$$E(z) = R(z) - c(z) = R(z) - \frac{R(z)G(z)}{1+G(z)} = \frac{R(z)}{1+G(z)}$$

$$e_{ss} = \lim_{z \to 1}(z-1)E(z) = \lim_{z \to 1}(z-1)\frac{R(z)}{1+G(z)}$$

$$= \begin{cases} \dfrac{1}{1+K_p^*} & r(t) = 1(t) \Leftrightarrow R(z) = \dfrac{z}{z-1}; \quad K_p^* = \lim\limits_{z \to 1} G(z) \\ \dfrac{T}{K_v^*} & r(t) = t \Leftrightarrow R(z) = \dfrac{Tz}{(z-1)^2}; \quad K_v^* = \lim\limits_{z \to 1}(z-1)G(z) \\ \dfrac{T^2}{K_a^*} & r(t) = t^2 \Leftrightarrow R(z) = \dfrac{T^2 z(z+1)}{(z-1)^3}; \quad K_a^* = \lim\limits_{z \to 1}(z-1)^2 G(z) \end{cases}$$

Example



$$T = 1s \qquad G(s) = \frac{K}{s(s+5)}$$

1) Determine *K* for the stable system.

2) If $r(t) = 1+t$, determine e_{ss}=?

Solution

1)

$$G(z) = Z\left[\frac{1-e^{-Ts}}{s}\cdot\frac{K}{s(s+5)}\right]$$

$$= (1-e^{-Ts})Z\left[\frac{K}{s^2(s+5)}\right]$$

$$= (1-e^{-Ts})Z\left[\frac{K/5}{s^2} + \frac{-K/5}{s} + \frac{K/25}{s+5}\right]$$

$$= (1-z^{-1})\left(\frac{KTz/5}{(z-1)^2} - \frac{Kz/5}{z-1} + \frac{Kz/25}{z-e^{-5T}}\right)\Bigg|_{T=1}$$

$$\approx -\frac{K}{5}\cdot\frac{z^2 - 2.2067z + 0.2135}{(z-1)(z-0.0067)}$$

The charecteristic equation of the system:

$$1+G(z)=1-\frac{K}{5}\cdot\frac{z^2-2.2067z+0.2135}{(z-1)(z-0.0067)}=0$$

$$(5\text{-}K)z^2+(2.2067K-5.0335)z+(0.0335-0.2135K)=0$$

$$z=\frac{s+1}{s-1}\Rightarrow\quad 0.9932w+(9.993-1.573K)w+(10.067-2.4202K)=0$$

$$0<K<4.16$$

2)

$$K_p^*=\lim_{z\to1}G(z)=\lim_{z\to1}\frac{K}{5}\cdot\frac{z^2-2.2067z+0.2135}{(z-1)(z-0.0067)}=\infty$$

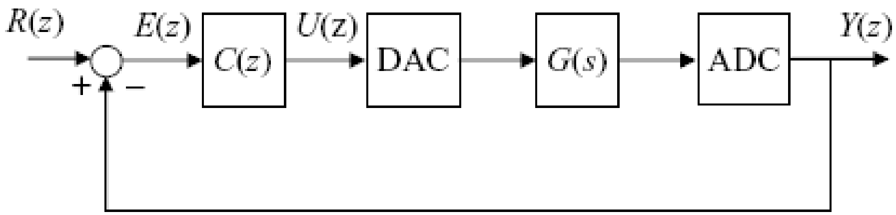
$$K_v^*=\lim_{z\to1}(z-1)G(z)=\lim_{z\to1}-\frac{K}{5}\cdot\frac{z^2-2.2067z+0.2135}{(z-0.0067)}\approx0.2K$$

$$e_{ss}=\frac{1}{1+K_p^*}+\frac{T}{K_v^*}=0+\left.\frac{T}{0.2K}\right|_{\mathrm{T=1}}=\frac{5}{K}$$

Steady State Error and System Type

System	Steady-state errors in response to		
	Step input $r(t) = 1$	Ramp input $r(t) = t$	Acceleration input $r(t) = \frac{1}{2}t^2$
Type 0 system	$\frac{1}{1 + K_p}$	∞	∞
Type 1 system	0	$\frac{1}{K_v}$	∞
Type 2 system	0	0	$\frac{1}{K_a}$

1) For unity feedback in figure below,



$$G(s) = \frac{s+8}{s+5}$$

$$C(z) = \frac{0.35z}{z-1}$$

a sampling period of 0.02 s.

(a) Find the z-transfer function for the analog subsystem with DAC and ADC.
(b) Find the closed-loop transfer function and characteristic equation.
(c) Find the steady-state error for a sampled unit step and a sampled unit ramp. Comment on the effect of the controller on steady-state error.

2) Find the stable range of the gain K for the vehicle position control system with the analog plant transfer function

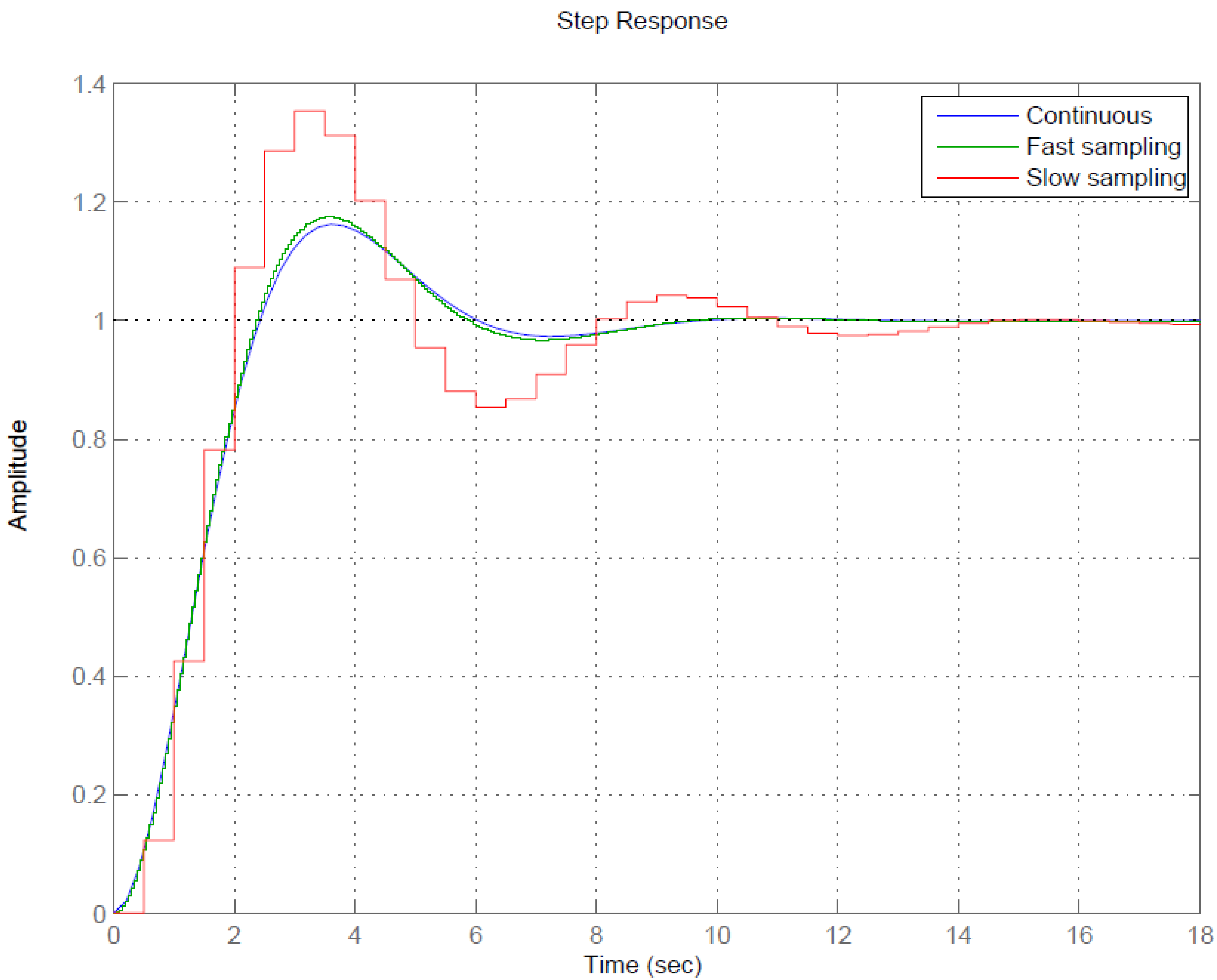
$$G(s) = \frac{K}{s(s+10)}$$

and with DAC and ADC if the sampling period is 0.05 s.

Design of Digital Control Systems

The Procedure:

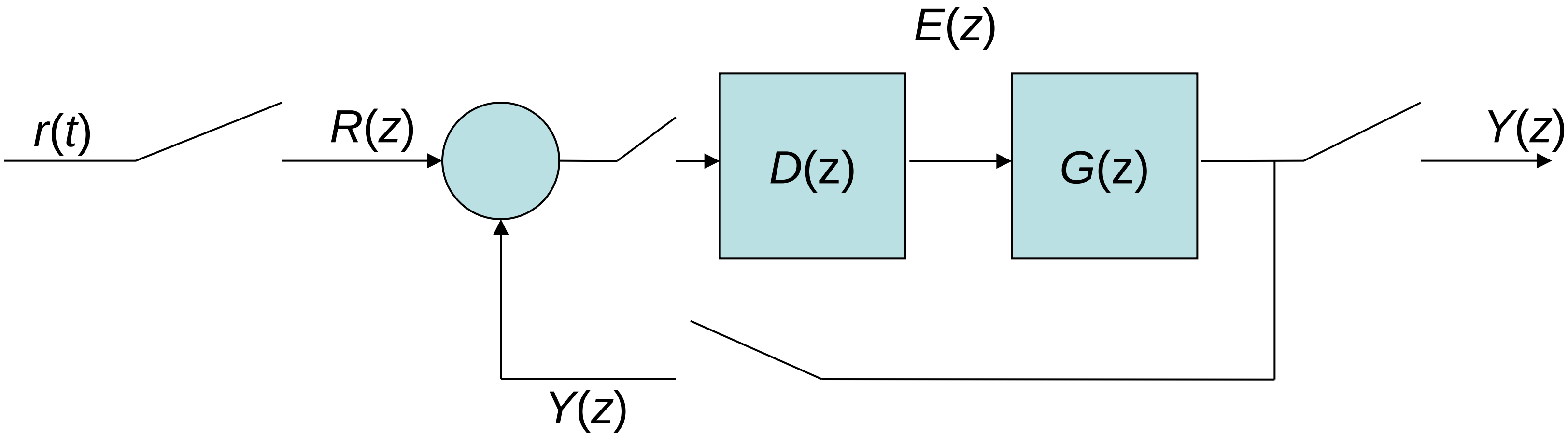
- Start with continuous system.
- Add sampled-data system elements.
- Chose sample period, usually small but not too small. Use sampling period T = 1 / 10 f_B, *where* $f_B = \omega_B / 2\pi$ and ω_B is the bandwidth of the closed-loop system.
  - Practical limit for sampling frequency: 20 < ω_s / ω_B >40
- Digitize control law.
- Check performance using discrete model or SIMULINK.



Step Response
Continuous
Fast sampling
Slow sampling
Amplitude
Time (sec)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
2
4
6
8
10
12
14
16
18

Start with a Continuous Design

D(*s*) may be given as an existing design or by using root locus or bode design.



Add Samples Necessary for Digital Control

- Transform $D(s)$ to $D(z)$: We will obtain a discrete system with a similar behavior to the continuous one.

- Include D/A converter, usually a zero-order-device.

- Include A/D converter modeled as an ideal sampler.

- And an antialiasing filter, a low pass filter, unity gain filter with a sharp cutoff frequency.

- Chose a sample frequency based on the closed-loop bandwidth ω_B of the continuous system.

Closed-Loop System with Digital Computer Compensation

$$\frac{Y(z)}{R(z)} = T(z) = \frac{G(z)D(z)}{1+G(z)D(z)}$$

$$\text{The tranfer function of the computer is } \frac{U(z)}{E(z)} = D(z)$$

Consider the second order system with a zero-order hold and a plant

$$Gp(s) = \frac{1}{s(s+1)} \text{ when } T = 1; G(z) = \frac{0.3678(z+0.7189)}{(z-1)(z-0.3678)}; \text{If we select } D(z) = \frac{k(z-0.3678)}{(z+r)}$$

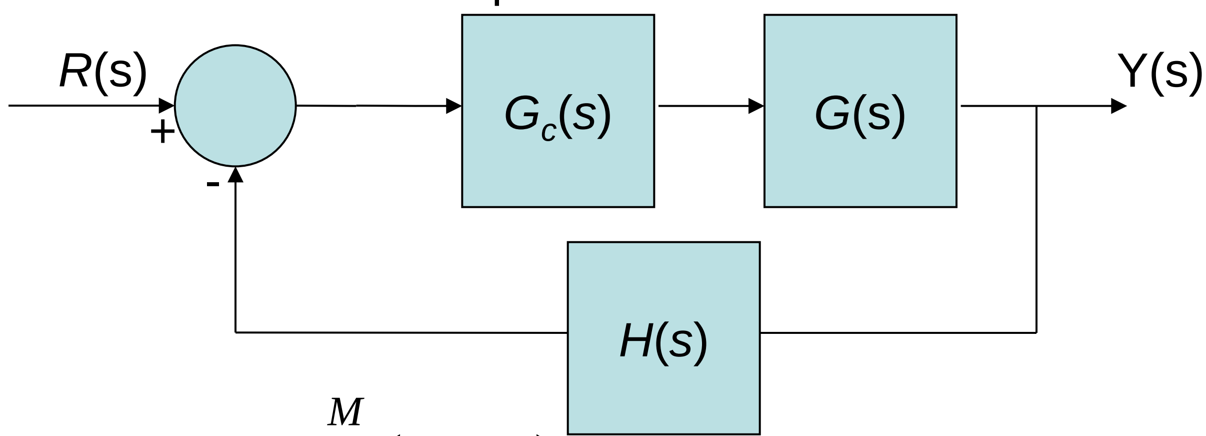
We cancer the pole of $G(z)$ at $z = 0.3678$ and have the two parameters r and K.

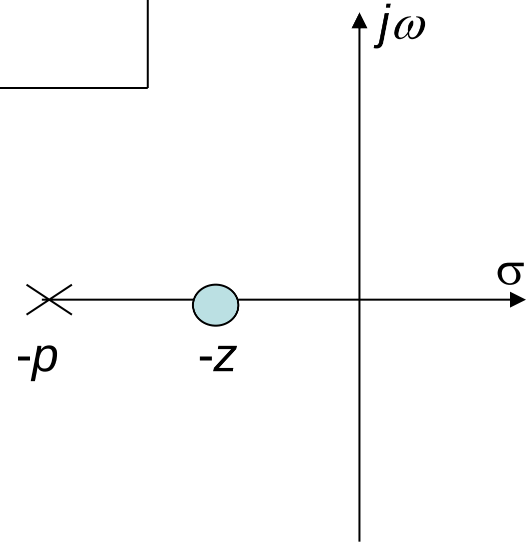
$$D(z) = \frac{1.359(z-0.7189)}{(z+0.240)}; G(z)D(z) = \frac{0.5(z+0.7189)}{(z-1)(z+0.240)}$$

$$G_c(s) = K\frac{s+a}{s+b};\ D(z) = C\frac{z-A}{z-B}; Z\{G_c(s)\} = D(z); A = e^{-aT}; B = e^{-bT};\ C\frac{(1-A)}{(1-B)} = K\frac{a}{b}$$

Compensation Networks (10.3; page 747)

The compensation network, $G_c(s)$ is cascaded with the unalterable process $G(s)$ in order to provide a suitable loop transfer function $G_c(s)G(s)H(s)$.





$$G_c(s) = \frac{K\prod_{i=1}^{M}(s+z_i)}{\prod_{j=1}^{N}(s+p_i)}$$

$$G_c(s) = \frac{K(s+z)}{(s+p)}\left(\text{First}-\text{order compensator}\right)$$

When $z \langle p$, the network is called a phase-lead network

Closed-Loop System with Digital Computer Compensation

There are two methods of compensator design:

(1) $G_c(s)$-to-$D(z)$ conversion method, and

(2) Root locus z-plane method.

The $G_c(s)$-to-$D(z)$ Conversion Method

$$G_c(s) = K\frac{s+a}{s+b} \text{ (First-Order Compensator)}$$

$$D(z) = C\frac{z-A}{z-B} \text{ (Digital Controller)}$$

$$Z\{G_c(s)\} = D(z) \text{ (z-transform)}$$

$$A = e^{-aT};\, B = e^{-bT};\; C\frac{(1-A)}{(1-B)} = K\frac{a}{b} \text{ when } s = 0$$

The Frequency Response

The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal.

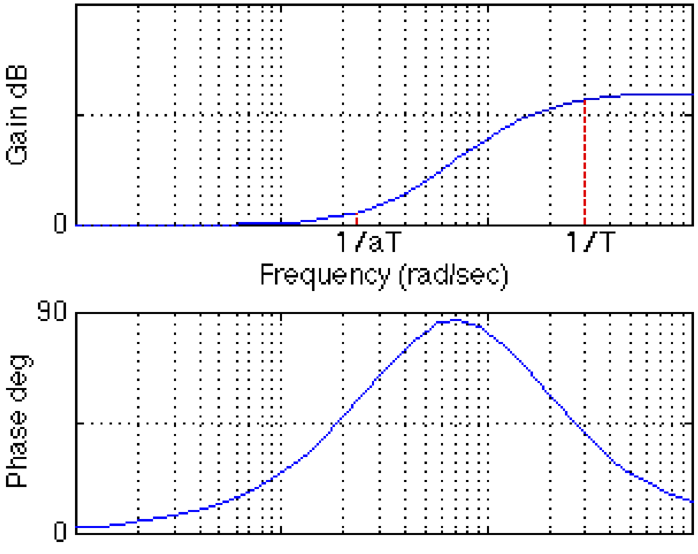
The sinusoid is a unique input signal, and the resulting output signal for a linear system, as well as signals throughout the system, is sinusoidal in the steady-state; it differs form the input waveform only in amplitude and phase.

Phase-Lead Compensator Using Frequency Response

A first-order phase-lead compensator can be designed using the frequency response. A lead compensator in frequency response form is given by

$$G_c(s) = \frac{(1 + \alpha\cdot\tau\cdot s)}{\alpha\cdot(1 + \tau\cdot s)} \qquad p = \frac{1}{\tau} \qquad z = \frac{1}{\alpha\tau} \qquad \omega_m = \sqrt{z\cdot p} \qquad \sin(\phi_m) = \frac{\alpha - 1}{\alpha + 1}$$

In frequency response design, the phase-lead compensator adds positive phase to the system over the frequency range. A bode plot of a phase-lead compensator looks like the following



Phase-Lead Compensator Using Frequency Response

Additional positive phase increases the phase margin and thus increases the stability of the system. This type of compensator is designed by determining alfa from the amount of phase needed to satisfy the phase margin requirements.

Another effect of the lead compensator can be seen in the magnitude plot. The lead compensator increases the gain of the system at high frequencies (the amount of this gain is equal to alfa. This can increase the crossover frequency, which will help to decrease the rise time and settling time of the system.

Phase-Lag Compensator Using Root Locus

A first-order lag compensator can be designed using the root locus. A lag compensator in root locus form is given by

$$G_c(s) = \frac{(s + z)}{(s + p)}$$

where the magnitude of z is greater than the magnitude of p. A phase-lag compensator tends to shift the root locus to the right, which is undesirable. For this reason, the pole and zero of a lag compensator must be placed close together (usually near the origin) so they do not appreciably change the transient response or stability characteristics of the system.

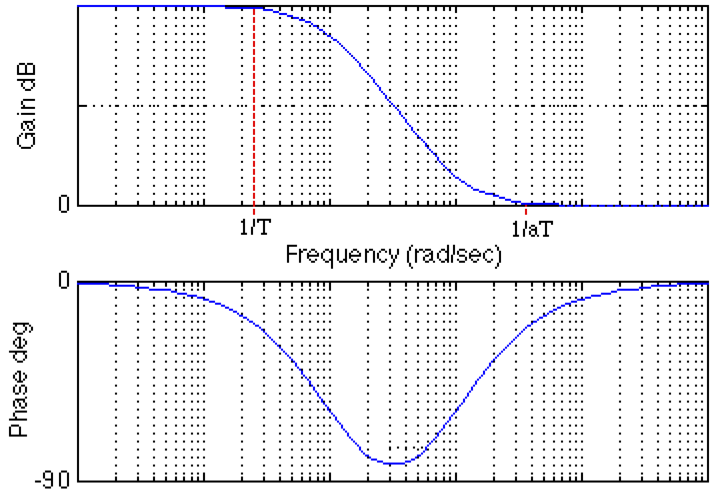
When a lag compensator is added to a system, the value of this intersection will be a smaller negative number than it was before. The net number of zeros and poles will be the same (one zero and one pole are added), but the added pole is a smaller negative number than the added zero. Thus, the result of a lag compensator is that the asymptotes' intersection is moved closer to the right half plane, and the entire root locus will be shifted to the right.

Lag or Phase-Lag Compensator using Frequency Response

A first-order phase-lag compensator can be designed using the frequency response. A lag compensator in frequency response form is given by

$$G_c(s) = \frac{(1 + \alpha\cdot\tau\cdot s)}{\alpha\cdot(1 + \tau\cdot s)}$$

The phase-lag compensator looks similar to a phase-lead compensator, except that a is now less than 1. The main difference is that the lag compensator adds negative phase to the system over the specified frequency range, while a lead compensator adds positive phase over the specified frequency. A bode plot of a phase-lag compensator looks like the following



Example: Design to meet a Phase Margin Specification

Based on Chapter 10 (Dorf): Example 13.7

$G_p(s) = \dfrac{1740}{s(0.25s+1)}$. We will attempt to design $G_c(s)$ so that we achieve

a phase margin of 45^{o} with a crossover frequeny $\omega_{\mathrm{c}} = 125$ rad/s (Fig 10.10).

Using the Bode diagram of $G_p(s)$, we find that the phase margin is 2^{o} (Eq 10.24).

Based on 10.4, we find that the required pole - zero ratio is $\alpha = 6.25$ (Eq 10.18).

$$\omega_c = (ab)^{\frac{1}{2}};\ a = 50;\ \text{and}\ b = 312;\ G_c(s) = \frac{K(s+50)}{(s+312)}$$

We select K in order to yield $|GG_c(j\omega)| = 1$

When $\omega = \omega_c = 125$ rad/s. Then $K = 5.6$.

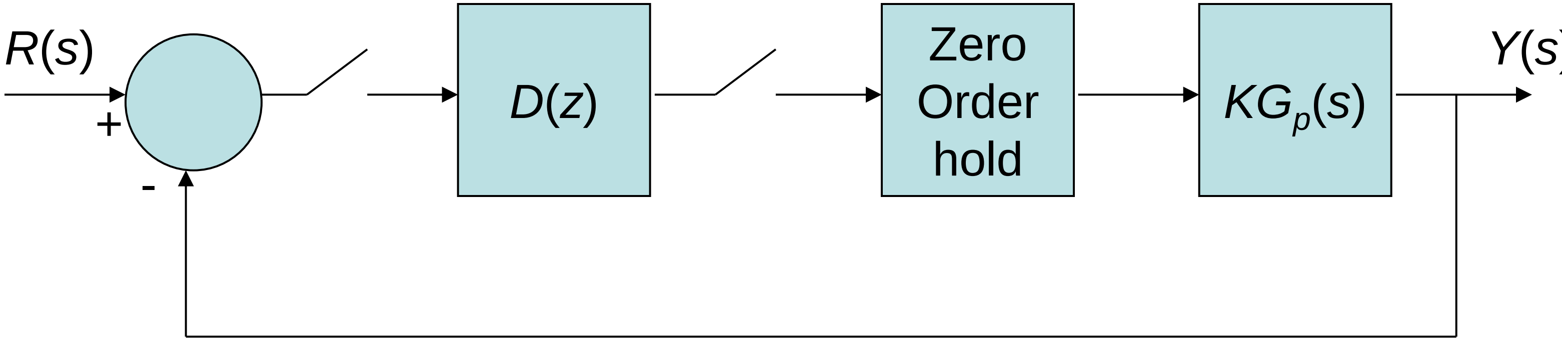
Now the compensator $G_c(s)$ is to be realized by $D(z)$.

Set $T = 0.001$ second. We have

$$A = e^{-0.05} = 0.95,\ B = \mathrm{e}^{-0.312} = 0.73,\ \text{and}\ C = 4.85;\ D(z) = \frac{4.85(z-0.95)}{(z-0.73)}$$

If we select another value for the sampling period, the the coefficient of $D(z)$ would differ!

The Root Locus of Digital Control Systems



$$\frac{Y(z)}{R(z)} = \frac{KG(z)D(z)}{1+KG(z)D(z)};\ 1+KG(z)D(z)=0 \text{ (Characteristic equation)}$$

Plot the root locus for the characteristic equation of the sampled system as K varies.

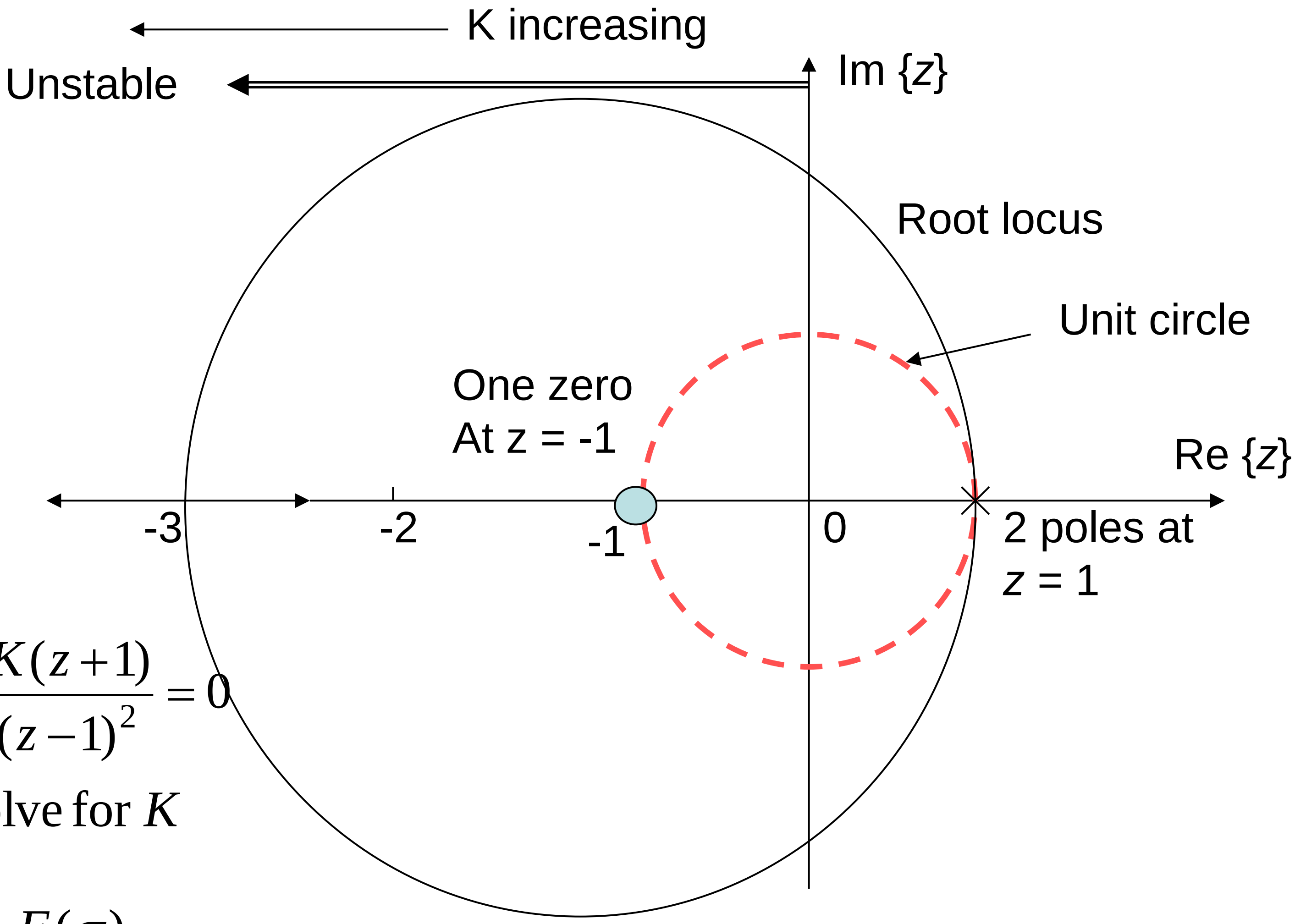
1. The root locus starts at the poles and progresses to the zeros.

2. The root locus lies on a section of the real axis to the left of an odd number of poles and zeros.

3. The root locus is symmetrical with respect to the horizontal real axis.

4. $1+KG(z)D(z)=0$ or $|KG(z)D(z)|=1$ and $\angle KG(z)D(z)=180^o \pm k360^o$

Root Locus of a Second Order System



$$1 + KG(z) = 1 + \frac{K(z+1)}{(z-1)^2} = 0$$

Let $z = \sigma$ and solve for K

$$K = -\frac{(\sigma - 1)^2}{(\sigma + 1)} = F(\sigma)$$

$$\frac{dF(\sigma)}{d\sigma} = 0; \sigma_1 = -3; \sigma_2 = 1$$

Design of a Digital Controller

In order to achieve a specified response utilizing a root locus method,

we will select a controller $D(z) = \frac{(z-a)}{(z-b)}$

Use (z - a) to cancel one pole at G(z) that lies on the positive real axis of the z - plane.

Select (z - b) so that the locus of the compensated system will give a set of complex roots at a desired point within the unit circle on the z - plane.

Example: Design of a digital compensator

Let us design a compensator D(z) that will result in a stable system when $G_p(s)$ is as described in Example 13.8.

With $D(z)=1$, we have unstable system. Select $D(z)=\dfrac{z-a}{z-b}$

$$KG(z)D(z)=\frac{K(z+1)(z-a)}{(z-1)^2(z-b)}$$

If we select $a=1$ and $b=0.2$,

we have $KG(z)D(z)=\dfrac{k(z+1)}{(z-1)(z-0.2)}$

Using the equation for $F(\sigma)$, we obtain the entry point as z = -2.56.

The root locus is on the unit circle at $K=0.8$.

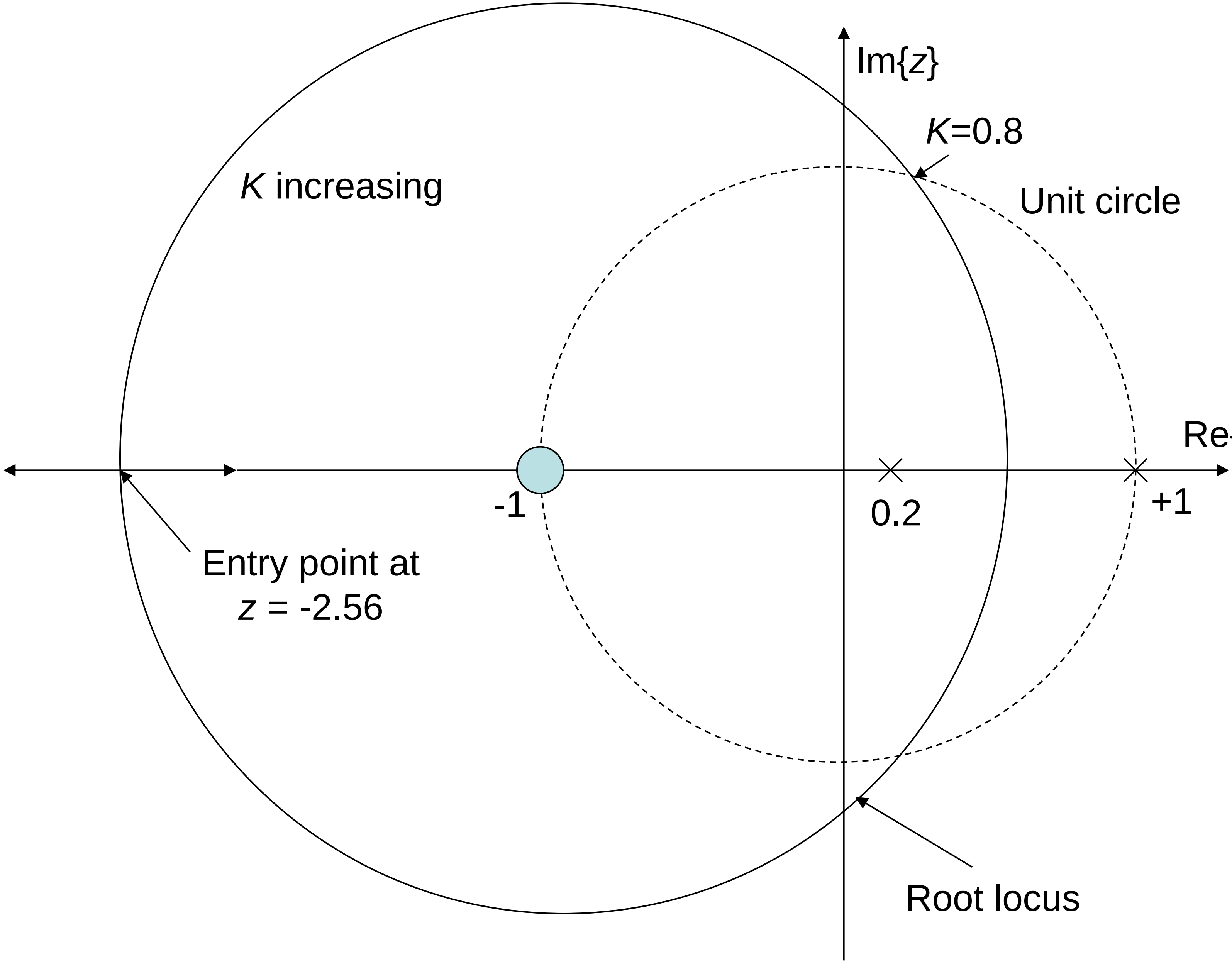
Thus the system is stable for K$\langle$0.8.

If the system performance were inadequate, we would improve the root locus by selecting $a = 1$ and $b = -0.98$ so that

$$KG(z)D(z) = \frac{K(z+1)}{(z-1)(z+0.98)} \cong \frac{K}{(z-1)}$$

Then the root locus would lie on the real axis of the z-plane.

When $K = 1$, the root of the characteristic equation is at the origin.


Im{z}
K=0.8
K increasing
Unit circle
Re{z}
-1
0.2
+1
Entry point at
z = -2.56
Root locus

P13.10 Dorf

$$G_p(s) = \frac{1}{s(s+10)}; T = 0.1; D(z) = K$$

(a) The transfer function $G(z)D(z) = K \dfrac{0.0037z + 0.0026}{z^2 - 1.368z + 0.3679}$

(b) The closed-loop system characteristic equation is $1 + K \dfrac{0.0037z + 0.0026}{z^2 - 1.368z + 0.3679} = 0$

(c) Using root locus method, maximum value of K is 239.

(d) Using Figure 13.19 for $T/\tau = 1$ and maximum overshoot of 0.3, we find $K = 75$.

(e) When $K = 75$; $T(z) = \dfrac{0.2759z + 0.1982}{z^2 - 1.092z + 0.5661}$

(f) When $K = 119.5$, the poles are $z = 0.4641 \pm j0.6843$. The overshoot is 0.55.

P13.11 Dorf

(a) $G_c(s) = K\dfrac{s+a}{s+b}$

By using Bode Plot, we may select $a = 0.7$, $b = 0.1$, and $K = 150$.

The compensated system overshoot and safety-state tracking error (for a ramp input) are $PO = 30\%$ and $e_{ss} \langle\, 0.01$.

(b) Use $G_c(s)$ to $D(z)$ method $(T = 0.1)$: $D(z) = C\dfrac{z-A}{z-B} = 155.3\dfrac{z-0.9324}{z-0.99}$

$$A = e^{-aT}; B = e^{-bT}; C\frac{1-A}{1-B} = K\frac{a}{b}$$

$$A = e^{-0.007} = 0.9324; B = e^{-0.01} = 0.99; C = 155.3$$

(d) Use $G_c(s)$ to $D(z)$ method $(T = 0.01)$: $D(z) = C\dfrac{z-A}{z-B} = 150\dfrac{z-0.993}{z-0.999}$

$$A = e^{-aT}; B = e^{-bT}; C\frac{1-A}{1-B} = K\frac{a}{b}$$

$$A = e^{-0.07} = 0.993; B = e^{-0.01} = 0.999; C = 150$$