ELG 4157

Modern Control Engineering

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Modeling in the Frequency Domain

- Find the Laplace transform of time functions.
- Find the inverse Laplace transform.
- Find the transfer function from a differential equation.
- Block diagram transformations.

Laplace Transform

- Laplace transform transforms the system (model) which is identified by a differential equation from time domain to frequency domain.
- The Laplace transform is defined as

$$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$$

where , $s = \sigma + j\omega$ is a complex variable.

Laplace Transform

- We normally assume zero initial conditions at $t = 0$. If any of the initial conditions are non-zero, then they must be added.
- The Laplace transform (LT) allows to represent the input, output, and the system itself, as separate entities.
- **Example 1**

  Find the Laplace transform for, $f(t) = 5$,

Laplace Transform

TIME DOMAIN		FREQUENCY DOMAIN
$\delta(t)$	unit impulse	1
A	step	$\frac{A}{s}$
t	ramp	$\frac{1}{s^2}$
t^2		$\frac{2}{s^3}$
$t^n,\ n > 0$		$\frac{n!}{s^{n+1}}$
e^{-at}	exponential decay	$\frac{1}{s+a}$
$\sin(\omega t)$		$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$		$\frac{s}{s^2+\omega^2}$
te^{-at}		$\frac{1}{(s+a)^2}$
t^2e^{-at}		$\frac{2!}{(s+a)^3}$

Laplace Transform

TIME DOMAIN	FREQUENCY DOMAIN
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\left[B\cos\omega t+\left(\frac{C-aB}{\omega}\right)\sin\omega t\right]$	$\frac{Bs+C}{(s+a)^2+\omega^2}$
$2\lvert A\rvert e^{-\alpha t}\cos(\beta t+\theta)$	$\frac{A}{s+\alpha-\beta j}+\frac{A^{\text{complex conjugate}}}{s+\alpha+\beta j}$
$2t\lvert A\rvert e^{-\alpha t}\cos(\beta t+\theta)$	$\frac{A}{(s+\alpha-\beta j)^2}+\frac{A^{\text{complex conjugate}}}{(s+\alpha+\beta j)^2}$
$\frac{(c-a)e^{-at}-(c-b)e^{-bt}}{b-a}$	$\frac{s+c}{(s+a)(s+b)}$
$\frac{e^{-at}-e^{-bt}}{b-a}$	$\frac{1}{(s+a)(s+b)}$

Inverse Laplace transform

The inverse Laplace transform, which allows us to find $f(t)$ given $F(s)$, is

$$\mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}\,ds = f(t)u(t)$$

where

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

is the unit step function. Multiplication of $f(t)$ by $u(t)$ yields a time function that is zero for $t < 0$.

Properties of LT

TABLE 2.2 Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) \;= \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] \;= kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] \;= F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] \;= e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] \;= \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] \;= sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] \;= s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] \;= s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) \;= \lim_{s \to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) \;= \lim_{s \to \infty} sF(s)$	Initial value theorem[2]

Inverse Laplace transform

- **Example 2**

  Find the inverse Laplace transform for, $\mathrm{F}(s) = \frac{1}{(s+3)^2}$

- To compute the LT of a complicated function, it is better converted to a sum of simpler terms whose respective LTs are known (or easier to compute). The result is called a partial-fraction expansion.

- *Exercise:* N. S. Nise (pp. 37-44 (*review partial-fraction expansion*)).

How can we carry out system (model) analysis using Laplace Transforms ?

1. We convert the system differential equation to the s-domain using Laplace Transform by replacing ‘d/dt’ with ’s’.

2. We convert the input function to the s-domain using the transform tables.

3. We combine algebraically the input and the transfer function to find out an output function.

4. We use partial fractions to reduce the output function to simpler components.

5. We convert the output equation from the s-domain back to the time-domain to obtain the response using Inverse Laplace Transforms according to the tables.

Transfer Function

- Example3: Find the transfer function represented by:

$$\frac{dc(t)}{dt} + 2c(t) = r(t)$$

Transfer Function

- Example4:

  Find the system response for the transfer function

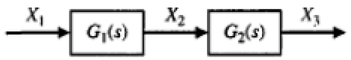
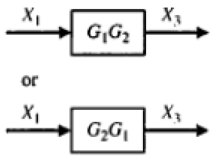
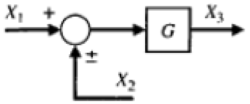
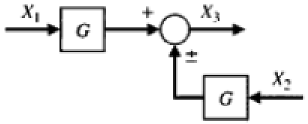
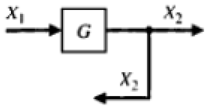
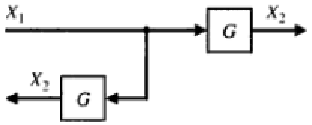
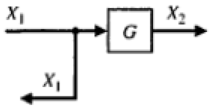
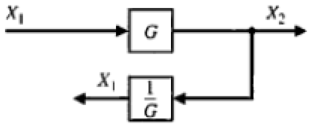
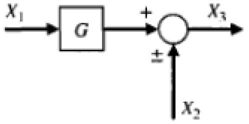
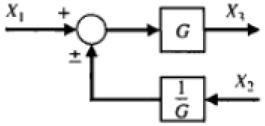
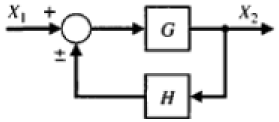
$$G(s) = \frac{1}{s+2} \quad \textit{to the input} \quad r(t) = u(t)$$

- Example5:
- Find the ramp response for the system whose transfer function is

$$G(s) = \frac{s}{(s+4)(s+8)}$$

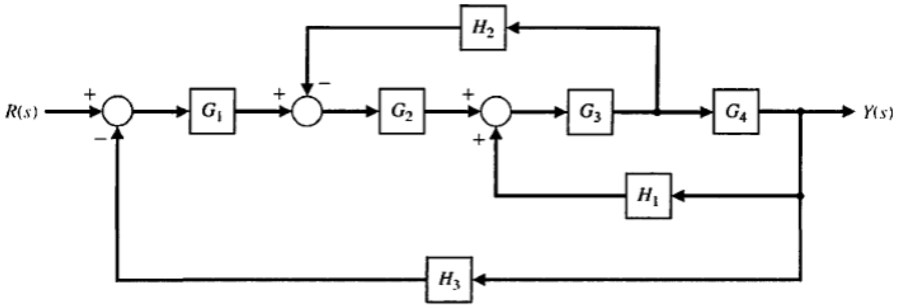
Block Diagrams

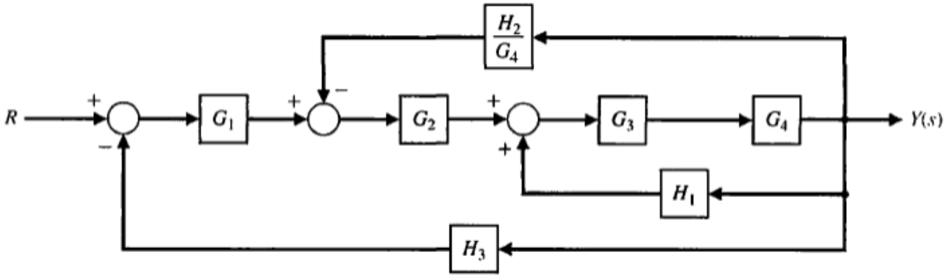
Table 2.6 Block Diagram Transformations

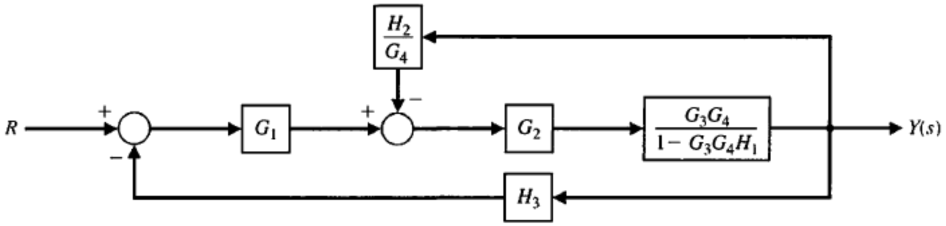
Transformation	Original Diagram	Equivalent Diagram
1. Combining blocks in cascade		
2. Moving a summing point behind a block		
3. Moving a pickoff point ahead of a block		
4. Moving a pickoff point behind a block		
5. Moving a summing point ahead of a block		
6. Eliminating a feedback loop		

Example5:

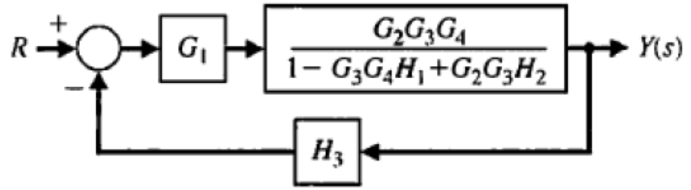
Multiple-loop feedback control system

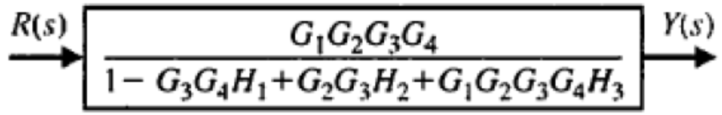






Block Diagrams





MATLAB and SIMULINK