

# 2

## Mathematical Modeling of Systems

In this chapter, we lead you through a study of mathematical models of physical systems. After completing the chapter, you should be able to

- ❑ Describe a physical system in terms of differential equations.
- ❑ Understand the way these equations are obtained.
- ❑ Realize the use of physical laws governing a particular system such as Newton's law for mechanical systems and Kirchhoff's laws for electrical systems.
- ❑ Realize that deriving mathematical models is the most important part of the entire analysis of control systems.

### Textbook

Richard C. Dorf and Robert H. Bishop, Modern Control Systems, Prentice Hall, 2001.

### 2.1 INTRODUCTION

Mathematical models may assume many different forms depending on the particular circumstances. For example, in optimal control problems, it is good to use **state-space representations**. On the other hand, for the transient-response or frequency-response analysis of single-input-single-output, linear, time-invariant systems, the transfer function representation may be more convenient than any other. Once a mathematical model of a system is obtained, various analytical and computational techniques may be used for analysis and synthesis purposes.

Because the systems under consideration are dynamic in nature, the equations are usually differential equations. If these equations can be linearized, then the Laplace transform may be utilized to simplify the method of solution.

In summary, the approach to dynamic system problems may be listed as follows:

- Define the system and its components.
- Formulate the mathematical model and list the needed assumptions.
- Write the differential equations describing the model.
- Solve the equations for the desired output variables.
- Examine the solutions and the assumptions.
- If needed, reanalyze or redesign the system.

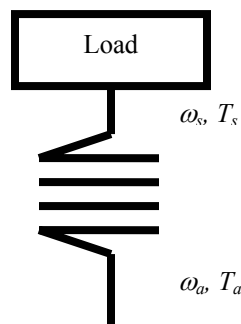
## 2.2 DIFFERENTIAL EQUATIONS OF PHYSICAL SYSTEMS

The differential equations describing the dynamic performance of a physical system are obtained by utilizing the physical laws of the process. Consider a torsional spring system with applied torque  $T_a(t)$ . Assume the torsional spring is massless. Suppose we want to measure the torque  $T_s(t)$  transmitted to the mass  $m$ . Since the spring is massless, the sum of the torques acting on the spring itself must be zero, or

$$T_a(t) - T_s(t) = 0$$

It is clear from the above equation that the torque  $T_a(t)$  applied at the end of the spring is transmitted through the spring. Such case is referred as through-variable. Similarly the angular rate difference associated with the spring is

$$\omega(t) = \omega_s(t) - \omega_a(t)$$



**Figure 1** Torsional spring mass system.

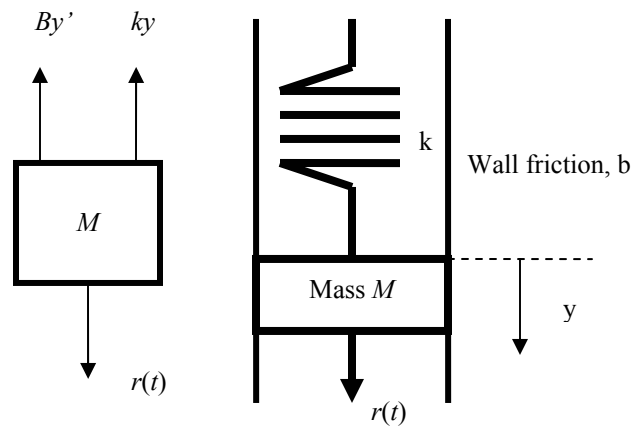
Therefore the angular rate difference is measured across the torsional spring and is referred to as an across-variable.

This approach applies equally well to mechanical, electrical, fluid, and thermodynamic systems. Table 2-1 summarizes the variables for various physical systems.

**Table 2-1** Summary of Through- and Across-Variables for Physical Systems

System	Variable 1	Variable 2	Variable 3	Variable 4
Electrical	Current	Charge	Voltage	Flux
Translational	Force	Momentum	Velocity	Displacement
Rotational	Torque	Angular Momentum	Angular Velocity	Angular Displacement
Fluid	Rate of Flow	Volume	Pressure	
Thermal	Rate of Flow	Heat energy	Temperature	

Consider the simple spring-damper mechanical system shown in Figure 2. This system may be described by Newton's second law of motion (this system may represent an automobile shock absorber).



**Figure 2** Spring-mass damper system with its free body diagram.

Summing the forces acting on M and using Newton's second law yields

$$M \frac{d^2 y(t)}{dt^2} + b \frac{dy(t)}{dt} + ky(t) = r(t)$$

## 2.3 LINEAR APPROXIMATION OF PHYSICAL SYSTEMS

A system is called linear if the principle of **superposition** applies. The principle of superposition states that the response produced by the simultaneous application of two different forcing functions is the sum of the two individual responses. Therefore, for a linear system, the response to several inputs can be calculated by treating one input at a time and adding the results.

Most of physical systems are linear with some range of the variables. However, all systems become nonlinear as the variables are increased without limit. A system is identified as linear in terms of the system excitation and response. In the case of electrical circuit, the excitation is the input current and the response is the voltage. In general, a condition of a linear system can be determined in terms of excitations  $x_n(t)$  and responses  $y_n(t)$ .

$$x_1(t) + x_2(t) + \dots + x_n(t) = y_1(t) + y_2(t) + \dots + y_n(t)$$

A system characterized by the relation  $y = x^2$  is not linear, because the superposition property is not satisfied. A system represented by the relation  $y = mx + b$  is not linear, because it does not satisfy the **homogeneity** property.

A linear system satisfies the properties of superposition and homogeneity

## 2.4 THE LAPLACE TRANSFORM

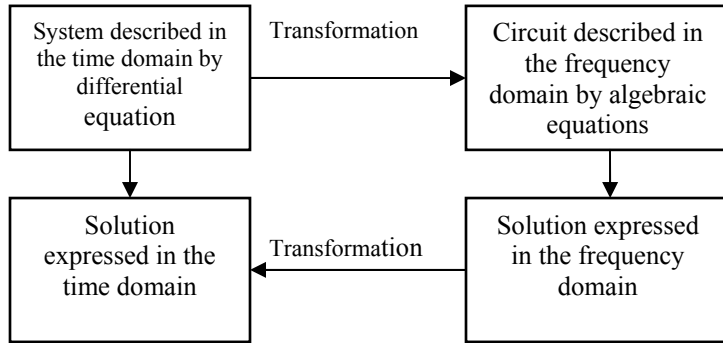
The ability to obtain linear approximation of physical systems allows considering the use of the **Laplace transformation**. A **transform** is a change in the mathematical description of a physical variable to facilitate computation [Figure 3]. The ability to obtain linear approximation of physical systems allows considering the use of the **Laplace transformation**. The Laplace transform method substitutes easily solved algebraic equations for the more difficult differential equations. The time response solution is obtained by the following operations:

- Obtain the differential equations
- Obtain the Laplace transformation of the differential equations.
- Solve the resulting algebraic transform of the variable of interest.

The Laplace transform exists for linear differential equations for which the transformation integral converges.

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt$$

Here the complex frequency is  $s = \sigma + j\omega$



**Figure 3** The transform method.

The inverse Laplace transform is written as

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds$$

The Laplace variable  $s$  can be considered to be the differential operator so that

$$s = \frac{d}{dt}$$

A table of important Laplace transform pairs is given in your textbook (Table 2.3)

## 2.5 POLES AND ZEROS

Consider the following function

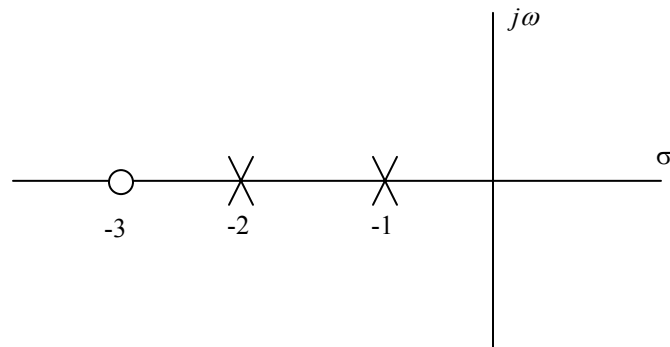
$$Y(s) = \frac{P(s)}{q(s)} = \frac{(Ms + b)y_0}{Ms^2 + bs + k}$$

The denominator polynomial  $q(s)$ , when equal to zero, is called the **characteristic equation** because the roots of this equation determine the character of the time response. The roots of this characteristic equation are also called the **poles** of the system.

The roots of the numerator polynomial  $p(s)$  are called the **zeros** of the system. Poles and zeros are critical frequencies. At the poles, the function  $Y(s)$  becomes infinite, whereas at the zeros, the function becomes zero. The complex frequency  $s$ -plane plot of poles and zeros graphically portrays the character of the natural transient response of the system. For certain case

$$Y(s) = \frac{(s + 3)y_0}{(s + 1)(s + 2)}$$

Figure 4 shows the poles and zeros of the above equation



**Figure 4** An  $s$ -plane pole and zero plot.

## 2.6 TRANSFER FUNCTION OF LINEAR SYSTEM

The **transfer function** of a linear system is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, with all initial conditions assumed to be zero. The transfer function of a system represents the relationship describing the dynamics of the system under consideration.

### 2.5.1 Transfer Function of an Operational Amplifier

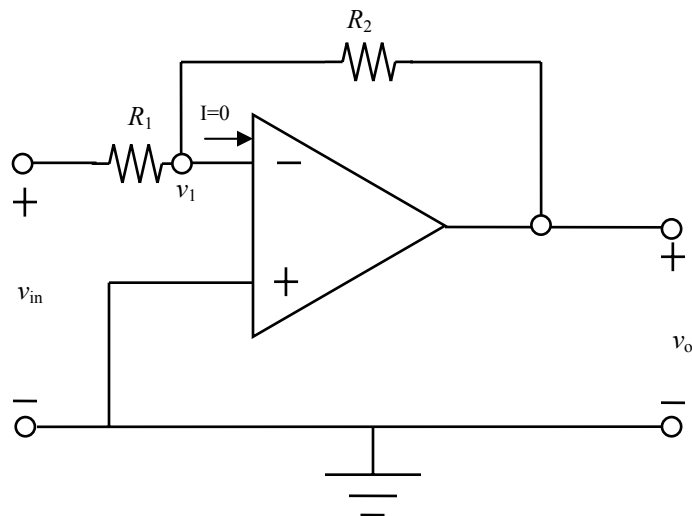
Consider the inverting amplifier shown in Figure 2-1. Under ideal conditions,  $i_1 = 0$ , so that writing the node equation at  $v_1$  yields

$$\frac{v_1 - v_{in}}{R_1} + \frac{v_1 - v_0}{R_2} = 0$$

Since  $v_2 = v_1$ , and  $v_2 = 0$ , therefore,  $v_1 = 0$

$$-\frac{v_{in}}{R_1} - \frac{v_0}{R_2} = 0, \text{ or}$$

$$\frac{v_0}{v_{in}} = -\frac{R_2}{R_1}$$



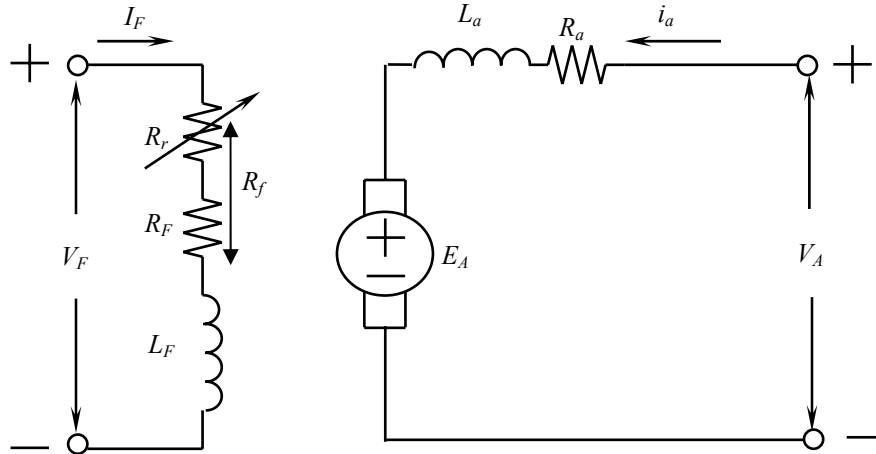
**Figure 5** An inverting amplifier operating with ideal conditions.

### 2.5.2 Transfer Function of a Separately-Excited DC Motor

There are five major types of DC motors in general use:

1. Separately-excited DC machines
2. Shunt DC machines
3. Series DC machines
4. Compound-connected DC machines
5. Permanent magnet DC machines

In this section, we will consider the first type as a power actuator that delivers energy to a load. The equivalent circuit of a separately-excited DC machine is shown in Figure 6. The armature circuit is represented by an ideal voltage source representing induced voltage  $E_A$  and a resistor  $R_A$ . This representation is really the Thevenin equivalent of the entire rotor structure, including rotor coils and poles. The field voltage is represented by  $V_F$ . The field coils, which produce the magnetic flux in the generator, are represented by inductor  $L_F$  and resistor  $R_F$ . The separate resistor  $R_r$  represents the variable resistor used to control the amount of current in the field circuit. Very often,  $R_F$  and  $R_r$  are lumped together and called  $R_F$ .



**Figure 6** Equivalent circuit of a separately excited DC motor.

The output characteristics of a DC machine can be derived from the induced voltage and torque equations of the motor plus Kirchoff's voltage law (KVL). This can be states mathematically as



$$V_A = E_A + i_A R_a$$

The internal generated voltage (induced voltage) in the machine is given by

$$E_A = K \phi \omega$$

where  $K$  is a constant on the construction of a particular DC machine (number of rotor coils, how they are interconnected, etc.). For a single rotating loop,  $K$  is  $1/\pi$ .

### The Transfer Function

- The transfer function of a DC motor will be developed for a linear approximation to an actual motor.
- The second order effects, such as hysteresis and the voltage drop across the brushes, will be neglected.
- The input voltage may be applied to the field or armature terminals.
- The air-gap flux  $\Phi$  of the motor is proportional to the field current  $i_f$ , provided the field is unsaturated, so that

$$\phi = K_f i_f$$

- The torque developed by the motor is assumed to be related linearly to the flux and the armature current as follows

$$T_m = K_1 \phi i_a(t) = K_1 K_f i_f(t) i_a(t)$$

- It is evident from the above equation that, to have a linear element, one current must be maintained constant while the other current becomes the input current (variable)

### Field Current Controlled Motor

Let us write the previous equation in Laplace transform notation

$$T_m(s) = (K_1 K_f I_a) I_f(s) = K_m I_f(s)$$

Where  $K_m$  is the motor constant. The field current is related to the field current

$$V_f(s) = (R_f + L_f s) I_f(s)$$

The motor torque  $T_m(s)$  is equal to the torque delivered to the load as

$$T_m(s) = T_L(s) + T_d(s)$$

Where  $T_L(s)$  is the load torque and  $T_d(s)$  is the disturbance torque, which is often negligible. The load torque for rotating inertia is given as

$$T_L(s) = Js^2\theta(s) + bs\theta(s)$$

Rearranging equations

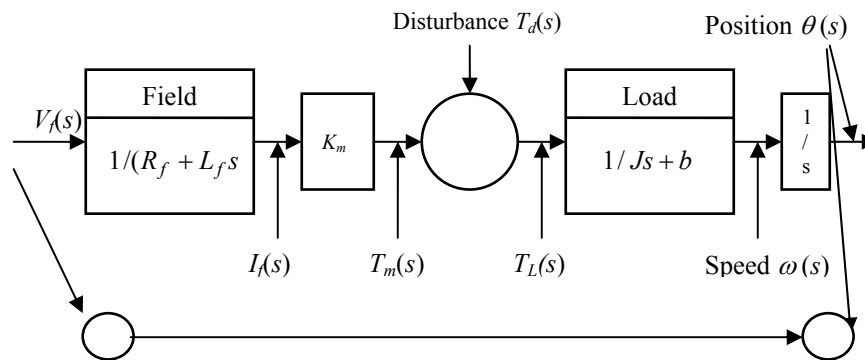
$$T_L(s) = T_m(s) - T_d(s)$$

$$T_m(s) = K_m I_f(s)$$

$$I_f(s) = \frac{V_f(s)}{R_f + L_f s}$$

The transfer function of the field-controlled DC motor is

$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(js + b)(L_f s + R_f)} = \frac{K_m / jL_f}{s(s + b/j)(s + R_f/L_f)}$$



**Figure 7** Block diagram model and signal flow graph of field-controlled DC motor.

**Complete for the armature-controlled DC motor.**

## Exercises

## E2.6

A nonlinear device is represented by the function

$$y = f(x) = x^{1/2}$$

Where the operating point for the input  $x$  is  $x_0 = 1/2$ . Determine the linear approximation in the form of Eq. (2.9) of the textbook

**Solution:** Given the linear approximation

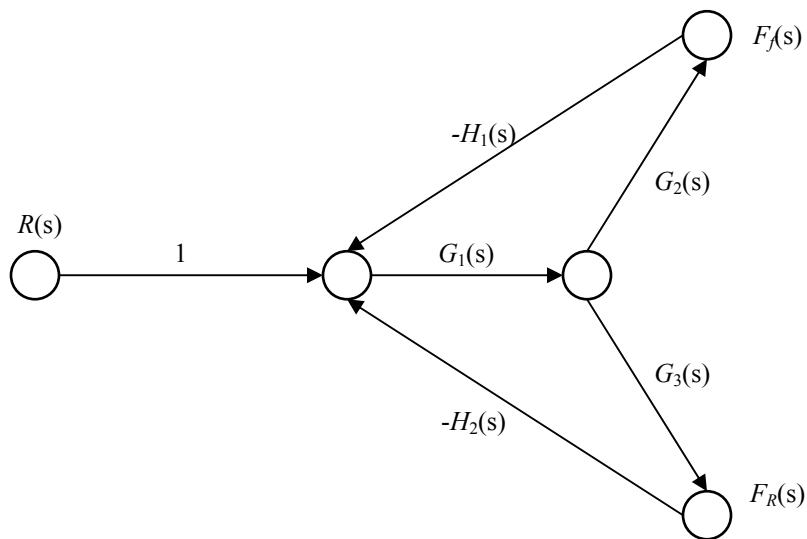
$$\Delta y = k\Delta x$$

Where

$$k = \left. \frac{df}{dx} \right|_{x_0=1/2} = \left. \frac{1}{2} x^{-1/2} \right|_{x_0=1/2} = \frac{1}{\sqrt{2}}$$

**E2.9**

A four-wheel antilock automobile braking system uses electronic feedback to control automatically the brake force on each wheel. A simplified flow graph of a brake control system is shown in the following figure, where  $F_f(s)$  and  $F_R(s)$  are the braking force of the front and rear wheels, respectively, and  $R(s)$  is the desired automobile response on an icy road. Find  $F_f(s)/R(s)$



The transfer function is

$$\frac{E_f(s)}{R(s)} = \frac{G_1(s) G_2(s)}{1 - L_1(s) - L_2(s)}$$

Where

$$L_1(s) = -G_1(s) G_2(s) H_1(s) \quad (\text{Loop 1})$$

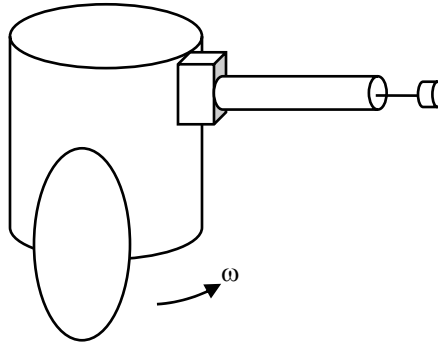
$$L_2(s) = -G_1(s) G_3(s) H_2(s) \quad (\text{Loop 2})$$

**E2.21**

The rotational velocity  $\omega$  of the satellite shown in Figure E2.21 (textbook) is adjusted by changing the length of the beam  $L$ . The transfer function between  $\omega(s)$  and the incremental change in beam length  $\Delta L(s)$  is

$$\frac{\omega(s)}{\Delta L(s)} = \frac{2.5(s+2)}{(s+5)(s+1)^2}$$

The beam length change is  $\Delta L(s) = 1/(4s)$ . Determine the response of the velocity  $\omega(s)$ .



Solution: The rotational velocity is

$$\omega(s) = \frac{2.5(s+2)}{(s+5)(s+1)^2} \frac{1}{4s}$$

By applying **partial fraction expansion**, we get

$$\omega(s) = \frac{0.25}{s} + \frac{0.0234}{s+5} - \frac{0.1562}{(s+1)^2} - \frac{0.2734}{s+1}$$

Taking the **inverse Laplace transform** yields

$$\omega(t) = 0.25 + 0.0234e^{-5t} - 0.1562e^{-t} - 0.2734e^{-t}$$

### Problems

#### P2.7

Obtain the transfer function of the differentiating circuit shown in Fig. P.2.7.

Solution:

$$V_2(s) = -\frac{Z_2(s)}{Z_1(s)}V_1(s)$$

$$Z_1 = \frac{R_1}{1 + R_1 C s}$$

$$Z_2 = R_2$$

$$V_2(s) = -\left[ \frac{R_2(1 + R_1 C s)}{R_1} \right] V_1(s).$$

#### P2.13

An electromechanical open-loop control system is shown in Figure P.2.13. The generator, driven at a constant speed, provides the field voltage for the motor. The motor has an inertial  $J_m$  and bearing friction  $b_m$ . Obtain the transfer function  $\theta_L(s) / V_f(s)$ , and draw the block diagram of the system. The generator voltage,  $v_g$ , can be assumed to be proportional to the field current,  $i_f$ .

**Solution:** The motor torque is given by

$$\begin{aligned} T_m(s) &= (J_m s^2 + b_m s)\theta_m(s) + (J_L s^2 + b_L s)n\theta_L(s) \\ &= n((J_m s^2 + b_m s)/n^2 + J_L s^2 + b_L s)\theta_L(s) \end{aligned}$$

Where

$$n = \theta_L(s) / \theta_m(s) = \text{gear ratio}$$

It is known that

$$T_m(s) = K_m I_g(s)$$

and

$$\begin{aligned} I_g(s) &= \frac{1}{(L_g + L_f)s + R_g + R_f} V_g(s), \text{ and} \\ V_g(s) &= K_g I_f(s) = \frac{K_g}{R_f + L_f s} V_f(s) \end{aligned}$$

By combining the above relationships, we get

$$\frac{\theta_L(s)}{V_f(s)} = \frac{K_g K_m}{n \Delta_1(s) \Delta_2(s)}$$

where

$$\Delta_1(s) = J_L s^2 + b_L s + \frac{J_m s^2 + b_m s}{n^2}$$

and

$$\Delta_2(s) = (L_g s + L_f s + R_g + R_f)(R_f + L_f s).$$

### P2.49

(a) The transfer function is

$$\frac{V_2(s)}{V_1(s)} = \frac{(1 + sR_1C_1)(1 + sR_2C_2)}{R_1C_2s}$$

(b)

$$\frac{V_2(s)}{V_1(s)} = \frac{0.2(s+10)(s+50)}{s}$$

### P2.50

(a) The closed-loop transfer function is

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{620s}{s^3 + 13s^2 + 128s + 6205}$$

(b) The poles of  $T(s)$  are  $s_1 = -5$  and  $s_{2,3} = -4 \pm j35$

(c) The partial fraction expansion (with a step input) is

$$Y(s) = 1 - \frac{1.0122}{s+5} + \frac{0.0061 + 0.0716j}{s+4+j35} + \frac{0.0061 - 0.0716j}{s+5-j35}$$

**Design Problems****DP2.3**

We have the output response

$$y(t) = 1 + \sin t + 2e^{-2t}$$

Taking the **Laplace transform** of the above equation

$$Y(s) = \frac{1}{s} + \frac{1}{s^2 + 1} + \frac{2}{s + 2}$$

Take the Laplace transform of the ramp input yields

$$R(s) = \frac{1}{s^2}$$

Therefore

$$G(s) = \frac{Y(s)}{R(s)} = \frac{3s^3 + 3s^2 + 5s + 2}{s^3 + 2s^2 + s + 2}$$