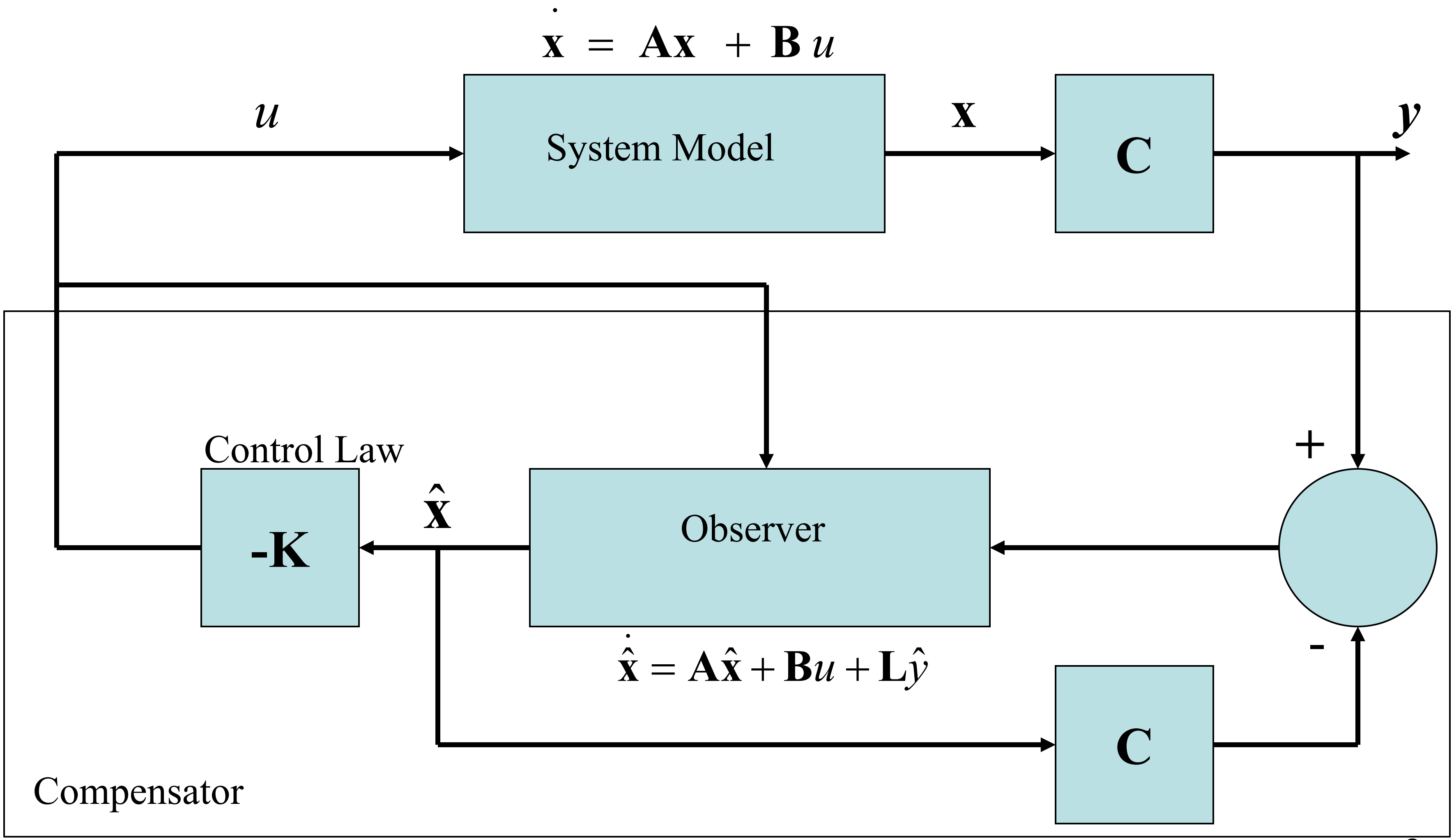
**Chapter 11**

# Compensator Design

When the full state is not available for feedback, we utilize an observer. The observer design process is described and the applicability of Ackermann's formula is established. The state variable compensator is obtained by connecting the full-state feedback law to the observer.

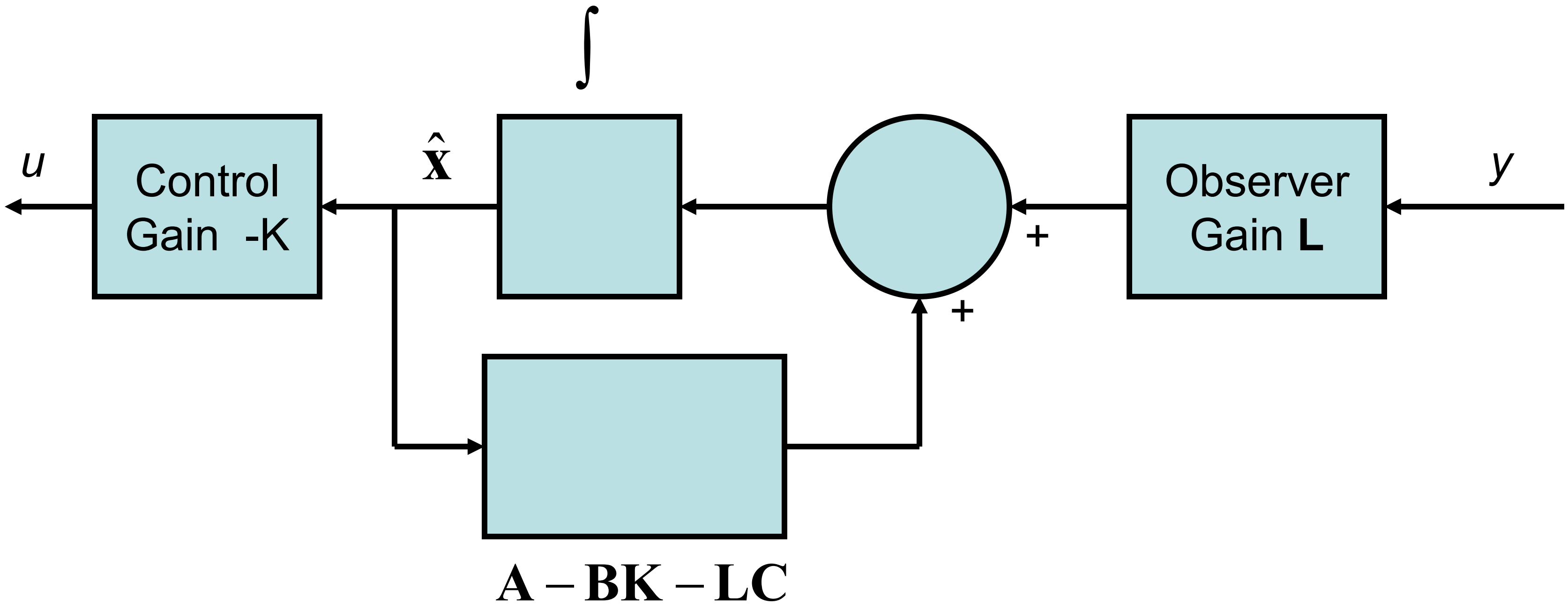
We consider optimal control system design and then describe the use of internal model design to achieve prescribed steady-state response to selected input commands.

# State Variable Compensator Employing Full-State Feedback in Series with a Full State Observer



# Compensator Design

## Integrated Full-State Feedback and Observer



$u(t) = -\mathbf{K}\hat{\mathbf{x}}(t)$ Feedback Law

Employ state estimate $\hat{\mathbf{x}}(t)$ in the feedback control in place of $\mathrm{x}(t)$

$\mathrm{Det}(s\mathbf{I} - (\mathbf{A} - \mathbf{BK})) = 0$ From full-state feedback. We need to verify that when using the feedback control law we retain the stability

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}u + \mathbf{L}(y - \mathbf{C}\hat{\mathbf{x}}) \text{ (From the observer, see the previos lecture)}$$

$$\dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{BK} - \mathbf{LC})\hat{\mathbf{x}} + \mathbf{L}y$$

$$u = -\mathbf{K}\hat{\mathbf{x}}$$

$$\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \mathbf{Ax} + \mathbf{Bu} - \mathbf{A}\hat{\mathbf{x}} - \mathbf{Bu} - \mathbf{Ly} + \mathbf{LC}\hat{\mathbf{x}} \text{ (time derivative of estimation error)}$$

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{LC})\mathbf{e}$$

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u; \ \mathrm{y} = \mathbf{Cx}$$

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u = \mathbf{Ax} - \mathbf{BK}\hat{\mathbf{x}}; \qquad \hat{\mathbf{x}} = \mathbf{x} - \mathbf{e}$$

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{BKe}$$

$$\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{pmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A - LC} \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{e} \end{pmatrix}$$

The goal is to verify that, with the value of $u(t)$ we retain the stability of the closed-loop system and the observer. The characteristic equation associated with the previous equation is

$$\mathbf{\Delta(s)} = \mathrm{Det}(\mathbf{sI} - (\mathbf{A} - \mathbf{BK}))\,\mathrm{Det}(\mathbf{sI} - (\mathbf{A} - \mathbf{LC}))$$

So if the roots of both parts of the above equation (first related to full-state feedback law and second to the design of the observer, then the overall system is stable.

**Design Procedure**

1. Determine $\mathbf{K}$ such that $\mathrm{Det}(\mathbf{sI} - (\mathbf{A} - \mathbf{BK})) = 0$ has roots in the left half plane and place the roots properly to meet the design criteria.

2. Determine $\mathbf{L}$ such that $\mathrm{Det}(\mathbf{sI} - (\mathbf{A} - \mathbf{LC})) = 0$ has roots in the left half plane and place the roots to achieve observer performance.

3. Connect the observer to the full-state feedback law using $u(t) = -\mathbf{K}\hat{\mathbf{x}}(t)$

# The Performance of Feedback Control Systems

- Because control systems are dynamics, their performance is usually specified in terms of both the transient response which is the response that disappears with time and the steady-state response which exists a long time following any input signal initiation.

- Any physical system inherently suffers steady-state error in response to certain types of inputs. A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input.

- Whether a given system will exhibit steady-state error for a given type of input depends on the type of open-loop transfer function of the system.

# The System Performance

- Modern control theory assumes that the systems engineer can specify quantitatively the required system performance. Then the performance index can be calculated or measured and used to evaluate the system's performance. A quantitative measure of the performance of a system is necessary for the operation of modern adaptive control systems and the design of optimum systems.

- Whether the aim is to improve the design of a system or to design a control system, a performance index must be chosen and measured.

- A performance index is a quantitative measure of the performance of a system and is chosen so that emphasis is given to the important system specifications.

# Optimal Control Systems

- A system is considered an optimum control system when the system parameters are adjusted so that the index reaches an **extreme** value, commonly a **minimum** value.

- A performance index, to be useful, must be a number that is always positive or zero. Then the best system is defined as the system that minimizes this index.

- A suitable performance index is the integral of the square of the error, ISE. The time $T$ is chosen so that the integral approaches a steady-state value. You may choose $T$ as the settling time $T_s$

$$\text{ISE} = \int_0^T e^2(t)\,dt$$

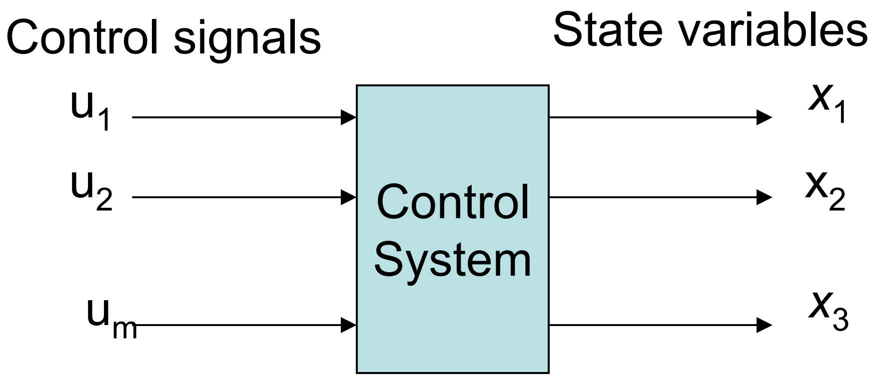
# The Performance of a Control System in Terms of State Variables

- The performance of a control system may be represented by integral performance measures [Section 5.9]. The design of a system must be based on minimizing a performance index such as the integral of the squared error (ISE). Systems that are adjusted to provide a minimum performance index are called **optimal control systems**.
- The performance of a control system, written in terms of the state variables of a system, can be expressed in general as

$$J = \int_0^{t_f} g(\mathrm{x}, \mathrm{u}, t)\,dt$$

- Where **x** equals the state vector, **u** equals the control vector, and $t_f$ equals the final time.
- We are interested in minimizing the error of the system; therefore when the desired state vector is represented as $x_d = 0$, we are able to consider the error as identically equal to the value of the state vector. That is, we desire the system to be at equilibrium, $x = x_d = 0$, and any deviation from equilibrium is considered an error.

**Design of Optimal Systems using State Variable Feedback and Error-Squared Performance Indices:** Consider the following control system in terms of x and u




The system can be represented by the vector differential equation : $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$

Select a feedback controller so that $\mathbf{u}$ is some function of the measured state variable x and therefore

$\mathbf{u} = \mathbf{k(x)}$ $[u_1 = -k_1 x_1;\ u_2 = -k_2 x_2;\ u_m = -k_m x_m]$. We may choose the control vector as

$u_1 = -k_1(x_1 + x_2);\ u_2 = -k_2(x_2 + x_3);..$

$$\begin{bmatrix} u_1 \\ u_2 \\ \cdots \\ u_m \end{bmatrix} = \begin{bmatrix} k_{11} & .. & k_{1n} \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \\ k_{m1} & .. & k_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$ Limit the feedbak function to a linear function so that u = Kx

$\dot{\mathbf{x}} = \mathbf{Ax} - \mathbf{BKx} = \mathbf{Hx}$; $\mathbf{H}$ is the $n \times n$ matrix resulting from the addition of the above elements.

# Now Return to the Error-Squared Performance Index

$$J = \int_0^{t_f} \left[x_1(t)\right]^2 dt \text{ [Single state variable]}$$

$$J = \int_0^{t_f} \left(x_1^2 + x_2^2\right) dt \text{ [Two state variables]. Utilize the matrix operation}$$

Performance index in terms of an integral of the sum of the state variables squared, use matrix operation

$$\mathrm{x}^{\mathrm{T}}\mathrm{x} = \left[x_1, x_2, x_3, ..., x_n\right] \begin{bmatrix} x_1 \\ x_2 \\ ... \\ xn \end{bmatrix} = \left(x_1^2 + x_2^2 + x_2^3 + ... + x_n^2\right); \ \mathrm{X}^{\mathrm{T}} \text{ is the transpose of the x matrix}$$

$$J = \int_0^{t_f} \left(\mathrm{x}^{\mathrm{T}}\mathrm{x}\right) dt; \frac{d}{dt}\left(\mathrm{x}^{\mathrm{T}}\mathrm{P}\mathrm{x}\right) = \text{-}\mathrm{x}^{\mathrm{T}}\mathrm{x} = \dot{\mathrm{x}}^{\mathrm{T}}\mathrm{P}\mathrm{x} + \mathrm{x}^{\mathrm{T}}\mathrm{P}\dot{\mathrm{x}} \text{ [A symmetric P will be used : } pij = pji \text{]}$$

To minimize the performance index $J$, we consider two equations

The design steps are: first to determine the matrix **P** that satisfies the second equation when **H** is known. Second minimize $J$ by determining the minimum of the first equation by adjusting one or more unspecified system parameters.

$$\frac{d}{dt}\left(\mathrm{x}^{\mathrm{T}}\mathrm{P}\mathrm{x}\right) = \dot{\mathrm{x}}^{\mathrm{T}}\mathrm{P}\mathrm{x} + \mathrm{x}^{\mathrm{T}}\mathrm{P}\dot{\mathrm{x}} = (\mathrm{H}\mathrm{x})^{\mathrm{T}}\mathrm{P}\mathrm{x} + \mathrm{x}^{\mathrm{T}}(\mathrm{H}\mathrm{x})$$

$$= \mathrm{x}^{\mathrm{T}}(\mathrm{H}^{\mathrm{T}}\mathrm{P} + \mathrm{P}\mathrm{H})\mathrm{x} = -\mathrm{x}^{\mathrm{T}}\mathrm{x}$$

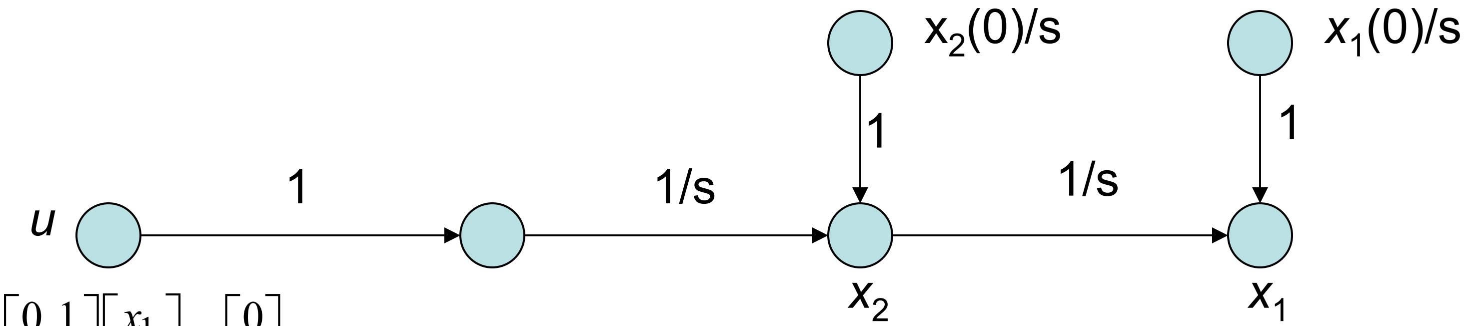
$$J = \int_0^\infty -\frac{d}{dx}(\mathrm{x}^{\mathrm{T}}\mathrm{P}\mathrm{x})\,dt = \mathrm{x}^{\mathrm{T}}(0)\mathrm{P}\mathrm{x}(0)$$

$$J = \int_0^\infty \mathbf{x}^{\mathrm{T}}\mathbf{x}\,dt = \mathrm{x}^{\mathrm{T}}(0)\,\mathbf{P}\mathbf{x}(0) \qquad \text{(Equation 1)}$$

$$\mathbf{H}^{\mathbf{T}}\mathbf{P} + \mathbf{P}\mathbf{H} = -\mathbf{I} \qquad \text{(Equation 2)}$$

The Design Criteria : Determine the matrix P (Equation 2) and minimize the value of $J$ by determining the minimum of Equation 1

# State Variable Feedback: State Variables $x_1$ and $x_2$




$$\frac{d}{dx}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

**Choose** feedback control system so that $u(t) = -k_1 x_1 - k_2 x_2$; sign negative to provide negative feedback

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -k_1 x_1 - k_2 x_2$$

In matrix form we have $\dot{\mathbf{x}} = \mathbf{H}\mathbf{x} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}\mathbf{x}.$

Let $k_1 = 1$ and find $k_2$ so that the performance index is minimized

$$\mathbf{H^T P} + \mathbf{PH} = \mathbf{-I}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & -k_2 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -k_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$-p_{12}-p_{12}=-1$$

$$p_{11}-k_2 p_{12}-p_{22}=0$$

$$p_{12}-k_2 p_{22}+p_{12}-k_2 p_{22}=-1$$

$$p_{12}=\frac{1}{2};\ p_{22}=\frac{1}{k_2};\ p_{11}=\frac{k_2^2+2}{2k_2}$$

$J=\mathbf{x}^{\mathrm{T}}(0)\mathbf{P}\mathbf{x}(0)$ (The integral performance index)

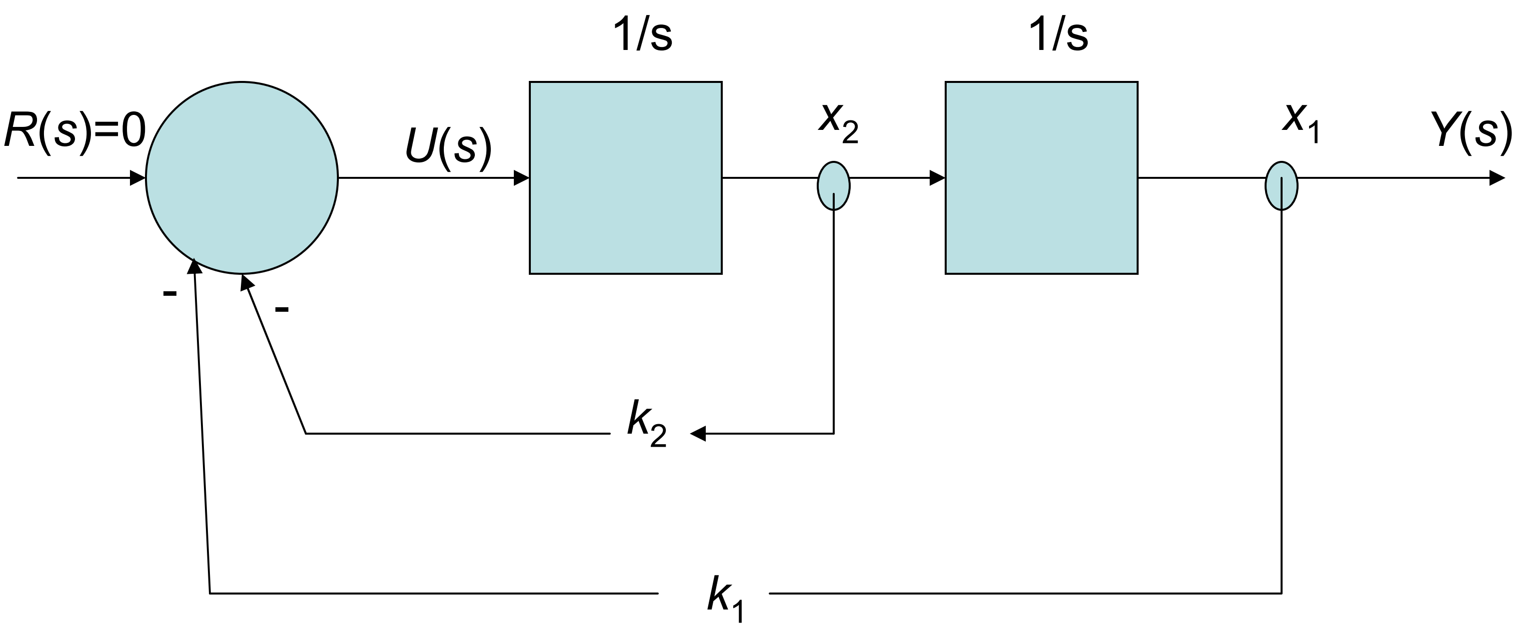
$\mathrm{x}^{\mathrm{T}}(0)=\begin{bmatrix}1 & 1\end{bmatrix}$ Assume each state is displaced one unit from equilibrium

$$J=\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}p_{11} & p_{12}\\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}=p_{11}+2p_{12}+p_{22}$$

$$J=\frac{k_2^2+2}{2k_2}+1+\frac{1}{k_2}=\frac{k_2^2+2k_2+4}{2k_2}$$

$\dfrac{dj}{dk_2}=\dfrac{2k_2(2k_2+2)-2(k_2^2+2k_2+4)}{(2k_2)^2}$; $k_2=2$ when $J$ is minimum

$J_{\min}=3$; $\mathbf{H}=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}$; $\mathrm{Det}\begin{bmatrix}\boldsymbol{\lambda}\mathbf{I}-\mathbf{H}\end{bmatrix}=\mathrm{Det}\begin{bmatrix}\lambda & -1\\ 1 & \lambda+1\end{bmatrix}=\lambda^2+2\lambda+1$



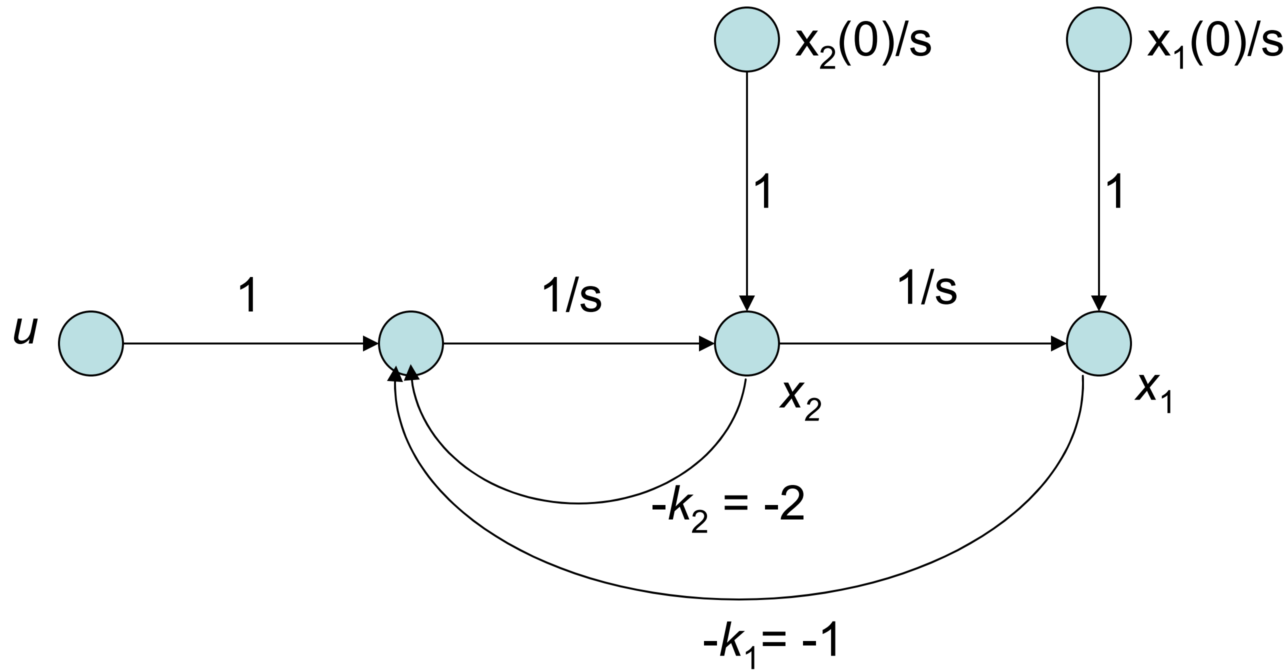

This is a second order system. The characteristic equation is of the form

$s^2 + 2\xi\omega ns + \omega_n^2$. Accordingly $\xi = 1.0$ (optimum system)

The sensitivity of an optimal system is

$$S_k^{opt} = \frac{\Delta J / J}{\Delta k / k}; \text{Assume } k = k_2, S_{k_2}^{opt} = \frac{\Delta J / J}{\Delta k_2 / k_2} = \frac{0.08/3}{0.5/2} = 0.107$$

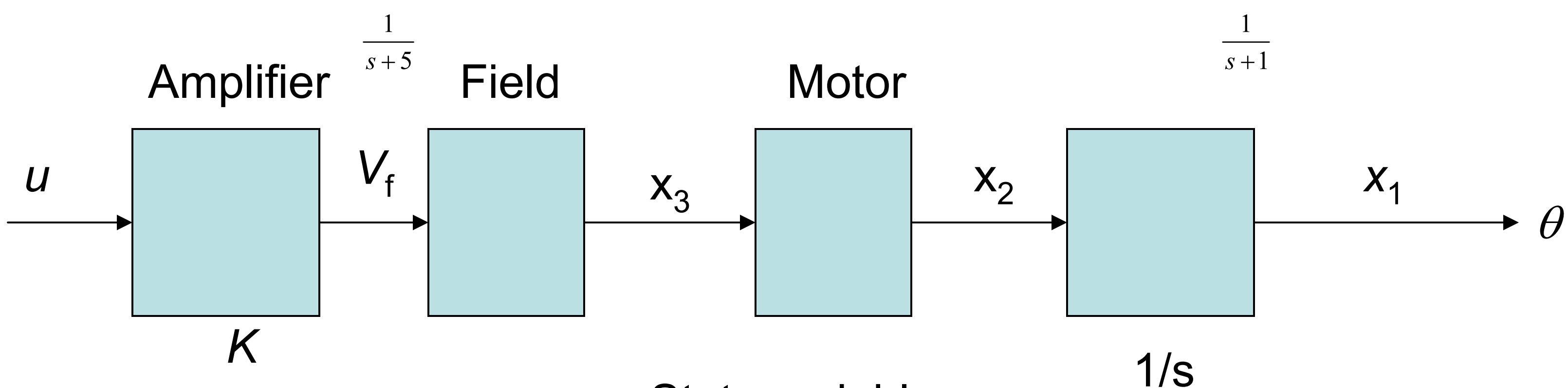
# Compensated Control System



# Design Example: Automatic Test System

A automatic test system uses a DC motor to move a set of test probes as shown in Figure 11.23 in the textbook. The system uses a DC motor with an encoded disk to measure position and velocity. The parameters of the system are shown with *K* representing the required power amplifier.

$$G(s) = \frac{K}{s(s+b/J)(s+R_f/L_f)} \text{(Section 2.5)} \qquad b/J = 1; R_f/L_f = 5$$




State variables

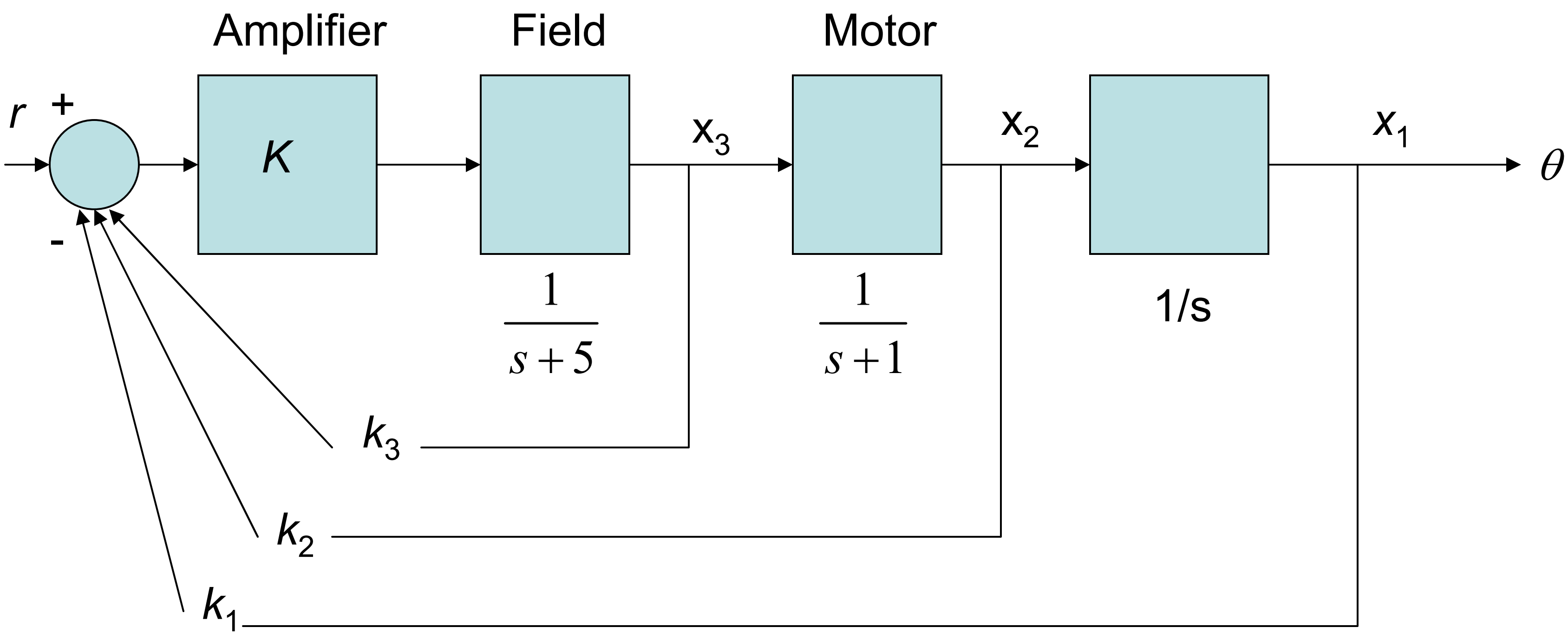
$$x_1 = \theta; \quad x_2 = d\theta/dt; \quad x_3 = I_f$$

$$G(s) = \frac{K}{s(s+1)(s+5)}; H(s) = K_3\left[s^2 + \left(\frac{K_3+K_2}{K_3}\right)s + \frac{1}{K_3}\right]$$

# State Variable Feedback

The goal is to select the gains so that the response to a step command has a settling time (with a 2%criterion) of less than 2 seconds and an overshoot of less than 4%

$$u = \left[-k_1 - k_2 - k_3\right]x + r = -k_1 x_1 - k_2 x_2 - k_3 x_3 = r$$



$$1+G(s)H(s)=1+\frac{kk_3(s^2+as+b)}{s(s+1)(s+5)}=0$$

$$a=\frac{k_2+k_3}{k_3};b=\frac{1}{k_3}$$

If we set $a=8$ and $b=20$ we will get zeros at s = -4 $\pm$ j2 in order to pull the locus to the left in the $s$ - plane.

$$8=\frac{k_2+k_3}{k_3};20=\frac{1}{k_3}$$

$k_1=1;k_2=0.35;k_3=0.05$

When $kk_3=12;\xi=0.76;k=240$

The roots are $s=-10.62;s=-3.69\pm j3.00$

To achieve an accurate output position, we let $k_1$ = 1 and determine $k$, $k_2$ and $k_3$. The aim is to find the characteristic equation

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ -k & -kk_2 & -(5 + k_3 k) \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} r$$

$$\mathrm{Det} \begin{bmatrix} s & -1 & 0 \\ 0 & \mathrm{s} + 1 & -1 \\ -k & kk_2 & \mathrm{s} + (5 + k_3 k) \end{bmatrix}$$

$$s^3 + 6s^2 + 5s + k_3 k s^2 + k_3 k s + kk_2 s + k = 0$$

$$1+\frac{kk_3(s^2+as+b)}{s(s+1)(s+5)}=0;\, a=\frac{k_2+k_3}{k_3};\, b=\frac{1}{k_3}$$

$$\text{Let } k_1=1;\;\; a=8;\;\; b=20$$

$$\text{We place zeros at }\;\; s=-4\pm j2$$

$$\frac{k_2+k_3}{k_3}=8 \qquad\qquad \frac{1}{k_3}=20$$

$$k_1=1; k_2=0.35; k_3=0.05; k=240$$

$$\text{The roots at } k=240$$

$$s=\text{-}10.62 \qquad\quad \text{and } s=\text{-}\,3.69\pm j3.00$$

## P11.1:

$$\dot{x} = x + u$$

$$u = -kx$$

$$\dot{x} = x - kx = (1-k)x$$

$x(t) = e^{(1-k)t}x(0)$ System is stable if $k \rangle 1$.

Calculate the value of $J$ assuming $k \rangle 1$ yields

$$J = \int_0^\infty e^{2(1-k)t} x^2(0) dt = \frac{1}{k-1}$$

If $k = \infty$, them $J$ is minimum. Physically, this is not realizable

Select $k = 31$, then the value of the performance index $J$ is $1/30$

**P11.3:** An unstable robot system is described below by the vector differential equation. Design gain *k* so that the performance index is minimized.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t); \quad u(t) = -k\left(x_1 + x_2\right) = -k[1\ 1]\mathrm{x}$$

Then, with feedback applied, the system is

$$\dot{\mathbf{x}} = \mathbf{Hx} = \begin{bmatrix} (1-k) & -k \\ -(1+k) & (2-k) \end{bmatrix}\mathbf{x}$$

$$\mathbf{H}^{\mathbf{T}}\mathbf{P} + \mathbf{PH} = \mathbf{-I}$$

$$2p_{11}(1-k) - 2p_{12}(k+1) = -1; p_{12}(3-2k) - p_{11}k - p_{22}(k+1) = 0; -2kp_{12} + 2p_{22}(2-k) = -1$$

$$p_{11} = \frac{-\left(2k^2 - 6k + 7\right)}{4(4k^2 - 8k + 3)}; \ p_{12} = \frac{2k^2 - 2k - 1}{4(4k^2 - 8k + 3)}; \ p_{22} = \frac{-(2k^2 - 6k + 3)}{4(4k^2 - 8k + 3)}$$

The performance index is computed to be : $J = \mathbf{x}^{\mathrm{T}}(0)\,\mathbf{Px}(0) = p_{11} + 2p_{12} + p_{22} = \dfrac{1}{2k-1}$

When $\mathrm{x}(0) = [1 \quad 1]^{\mathrm{T}}$. As $k$ goes to $\infty$, $J$ goes to 0. The system is unstable without feedback

Select $k = 10$, $J = \dfrac{1}{19}$; $\dot{\mathrm{x}} = \begin{bmatrix} -9 & -10 \\ -11 & -8 \end{bmatrix}\mathrm{x} = \mathrm{Ax}$

The closed - loop system roots are determined by Det [SI - A] $= s^2 + 17s - 38 = 0$

where $s_1 = -19$ and $s_2 = 2$; The system is unstable.