Chapter 5
The Performance of Feedback Control Systems

Test Input Signals
Performance of a Second-Order System
Effects of Third Pole and a Zero on the Second-Order System Response
Estimation of the Damping Ratio
The Steady-State Error of Feedback Control Systems
Performance Indices
The ability to adjust the transient and steady-state response of a feedback control system is a beneficial outcome of the design of control systems.

Input signals such as step and ramp are used to test the response of the control system.

In this chapter, common time-domain specifications are introduced:
- Transient Response and Steady-State Response
- Percent overshoot
- Settling time
- Time to peak
- Time to Rise
- Steady-State Tracking Error.

The concept of a performance index that represents a system's performance by a single number (or index) will be considered.
Test Input Signals
Table 2.3 is used to obtain the Laplace transform

<table>
<thead>
<tr>
<th>Signal</th>
<th>$r(t)$</th>
<th>$R(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step</td>
<td>$A, t \to 0$</td>
<td>$A/s$</td>
</tr>
<tr>
<td>Ramp</td>
<td>$At, t \to 0$</td>
<td>$A/s^2$</td>
</tr>
<tr>
<td>Parabolic</td>
<td>$At^2, t \to 0$</td>
<td>$2A/s^3$</td>
</tr>
</tbody>
</table>

See Table 5.1
Performance of a Second-Order System

\[ G(s) = \frac{K}{s(s + p)} \]

\[ Y(s) = \frac{G(s)}{1 + G(s)} \quad R(s) = \frac{K}{s^2 + ps + K} \]

\[ Y(s) = \frac{\omega_n^2}{s(s^2 + 2\xi \omega_n s + \omega_n^2)} \text{ [Use generalized notation in Section 2.4]} \]

\[ y(t) = 1 - \frac{1}{\beta} e^{-\xi \omega_n t} \sin(\omega_n \beta t + \theta) \text{ [See Laplace Transform from Table 2.3]} \]

\[ \beta = \sqrt{1 - \xi^2}, \quad \theta = \cos^{-1} \xi \]

\[ Y(s) = \frac{\omega_n^2}{s^2 + 2\xi \omega_n s + \omega_n^2} \] For a unit impulse \( R(s) = 1 \)

\[ y(t) = \frac{\omega_n}{\beta} e^{-\xi \omega_n t} \sin \omega_n \beta t \]
Standard Performance Measures

Rise Time $T_r$ and Peak Time, $T_p$

Percent overshoot P. O. $= \frac{M_{p_i} - f_v}{f_v} \times 100\%$

$M_{p_i}$ is the peak value of the time response,
$f_v$ is the final value of the response

Settling time, $T_s = \frac{4}{\xi \omega_n}$
Peak Time, $T_p = \frac{\pi}{\omega_n \sqrt{1 - \xi^2}}$

Peak response, $M_{p_i} = 1 + e^{-\xi \pi / \sqrt{1 - \xi^2}}$; P. O. $= 100e^{-\xi \pi / \sqrt{1 - \xi^2}}$

$\xi$: Damping ratio
Step Response of a Control System

The diagram illustrates the step response of a control system, showing key performance metrics:

- **Peak time** \( T_p \)
- **Settling time** \( T_s \)
- **Rise time** \( T_r \)
- **Overshoot**

The graph also highlights the steady-state error \( e_{ss} \).
The Steady-State Error of Feedback Control Systems

\[ H(s) = \frac{K_2}{(\alpha s + 1)}; \lim_{s \to 0} H(s) = K_2 \]

\[ E(s) = R(s) - Y(s) = [1 - T(s)]R(s) \]

\[ T(s) = \frac{K_1G(s)}{1 + K_1G(s)}; E(s) = \frac{1}{1 + K_1G(s)}R(s) \]

\[ e_{ss} = \lim_{s \to 0} sE(s) = \frac{1}{1 + K_1G(0)} \]
### Table 5.5 Summary of Steady-State Errors

*K<sub>p</sub>*: position error constant; *K<sub>v</sub>*: Velocity error constant

<table>
<thead>
<tr>
<th>Number of Integrations in <em>G(s)</em>, Type Number</th>
<th>Step <em>R(s)=A/s</em></th>
<th>Input Ramp</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td><em>e&lt;sub&gt;ss&lt;/sub&gt;=1/(1+K&lt;sub&gt;p&lt;/sub&gt;)</em></td>
<td>Infinite</td>
</tr>
<tr>
<td>1</td>
<td><em>e&lt;sub&gt;ss&lt;/sub&gt;=0</em></td>
<td><em>A/K&lt;sub&gt;v&lt;/sub&gt;</em></td>
</tr>
<tr>
<td>2</td>
<td><em>e&lt;sub&gt;ss&lt;/sub&gt;=0</em></td>
<td>0</td>
</tr>
</tbody>
</table>
E5.1: In order to get $e_{ss} = 0$; When the input is a step we require one integration (type 1 system). For a ramp input we require type 2 system.

See Table 5.5
Performance Indices

A performance index is a quantitative measure of the performance of a system and is chosen so that emphasis is given to the important system specifications.

A system is considered an optimum control system when the system parameters are adjusted so that index reaches an extremum value, commonly a minimum value.

A performance index, to be useful, must be a number that is always positive or zero

\[
\begin{align*}
\text{ISE} &= \int_0^T e^2(t) \, dt \quad \text{[Integral of the square of the error]} \\
\text{IAE} &= \int_0^T |e(t)| \, dt \quad \text{[Integral of the absolute magnitude of the error]} \\
\text{ITAE} &= \int_0^T t |e(t)| \, dt \quad \text{[Integral of time multiplied by absolute error]}
\end{align*}
\]
The Optimum Coefficients of $T(s)$ Based on the ITAE Criterion for a Step Input

$$s + \omega_n$$
$$s^2 + 1.4\omega_n s + \omega_n^2$$
$$s^3 + 1.75\omega_n s^2 + 2.15\omega_n^2 s + \omega_n^3$$

The coefficients that will minimize the ITAE performance criterion for a step input have been determined for the general closed-loop transfer function

$$T(s) = \frac{Y(s)}{R(s)} = \frac{b_0}{s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0} \quad (5.47)$$
E5.2:

\[ T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = \frac{100}{(s + 2)(s + 5) + 100} = \frac{100}{s^2 + 2\xi\omega_n s + \omega_n^2} \]

\[ e_{ss} = \frac{A}{1 + K_p} (R(s) = A/s); \quad K_p = \lim_{s \to 0} G(s) = \frac{100}{10} = 10; \quad e_{ss} = \frac{A}{11} \]

The closed-loop system is a second order system with natural frequency \( \omega_n = \sqrt{110} \)

The damping ratio \( \xi = \frac{7}{2\sqrt{110}} = 0.334 \)

\[ P.O. = 0.909 \left( 100 e^{-\pi\xi/\sqrt{1-\xi^2}} \right) = 29\% \text{ (Equation 5.16)} \]
\[ T(s) = \frac{G(s)}{1 + G(s)} = \frac{K}{s(s + 14) + K} = \frac{K}{s^2 + 14s + K} \]

Refer to Table 5.6 to find the optimum coefficients \( s^2 + 1.4\omega_n s + \omega_n^2 \); \( s^2 + 14s + K \). We will get

\[ \omega_n = 10; \quad K = \omega_n^2 = 100; \quad \xi = \frac{14}{2\omega_n} = 0.7 \]

Use Figure 5.8 to find P.O. \( \approx 5\% \)
E5.8:

\[ G(s) = \frac{K}{s(s + \sqrt{2K})}; \quad T(s) = \frac{K}{s^2 + \sqrt{2K}s + K} \]

\[ \xi = \frac{\sqrt{2}}{2}; \quad \omega_n = \sqrt{K}; \quad \text{P.O.} = 100 \ e^{-\pi \xi \sqrt{1-\xi^2}} = 4.3\%; \quad T_s = \frac{4}{\xi \omega_n} = \frac{8}{\sqrt{2K}} \]

The settling time is less than 1 second whenever \( K > 32 \)
The system is a type 1 (Table 5.5). The error constants are

\[ G(s) = \frac{10(s + 4)}{s(s + 1)(s + 3)(s + 8)} \]

\[ K_p = \infty \text{ and } K_v = 1.67 \text{ (Equation 5.29)} \]

The steady-state error for a step input is 0.

The steady-state error for a ramp is 0.6A (Equation 5.29).

A is the amplitude of the ramp input.
The tracking error is given by: \( E(s) = [1 - T(s)]R(s) \) (Read Section 5.8)

The steady-state tracking error with \( R(s) = 1/s \) is

\[
e_{ss} = \lim_{s \to 0} s \left[1 - T(s)\right] R(s) = \lim_{s \to 0} \left[1 - T(s)\right] = 1 - T(0)
\]

The closed-loop transfer function is

\[
T(s) = \frac{K (s + 0.1)}{s(s + 0.1)(s + 2) + K (s + 3)}; \quad T(0) = 0.033
\]

\[
e_{ss} = 1 - 0.033 = 0.967
\]

b) Use \( G_P(s) = 30 \)

\[
\lim_{s \to 0} s \left[1 - T(s) G_P(s)\right] R(s) = 1 - \lim_{s \to 0} T(s) G_P(s) = 1 - 30T(0) = 0
\]