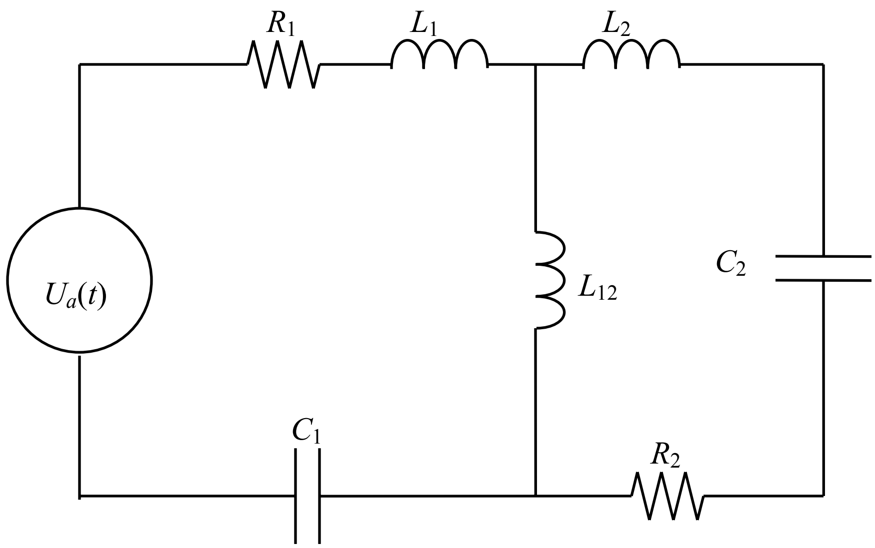
Applications of Lagrange Equations

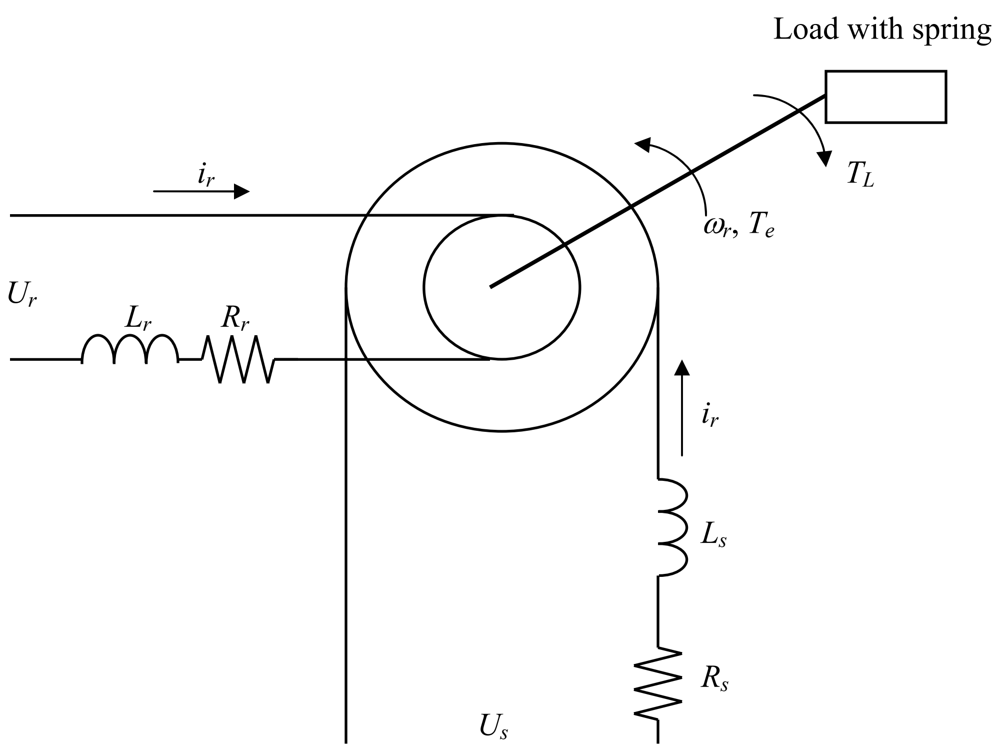
Case Study 1: Electric Circuit

Using the Lagrange equations of motion, develop the mathematical models for the circuit shown in Figure 1.Simulate the results by SIMULINK. The circuitry parameters are: L_1 = 0.01 H, L_2 = 0.005 H, L_{12} = 0.0025 H, C_1 = 0.02 F, C_2 = 0.1 F, R_1 = 10 Ω, R_2 = 5 Ω and U_a = 100 sin (200 t) V.



Case Study 2: Servomechanism

Using the Lagrange equations of motion for the directly driven servo-system. Consider a servomechanism actuated by a motor with two independently excited stator and rotor.

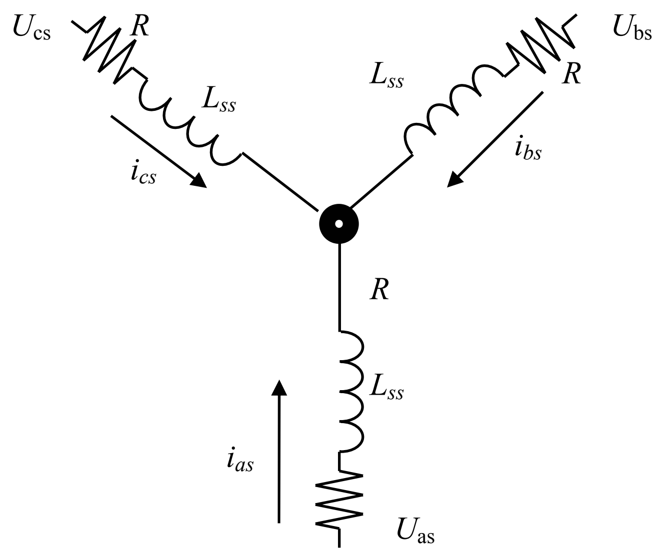


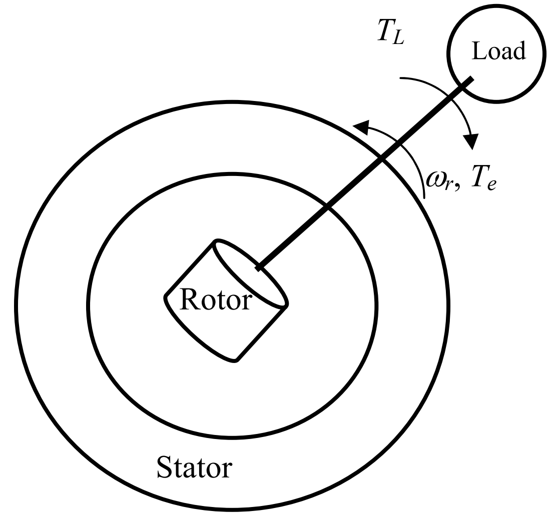
Data:

Using the block diagram of permanent-magnet synchronous motors, as illustrated in the Figure above, develop the SIMULINK diagram. Simulate the servo-system with the following parameters: $R_s = R_r = 0.5\ \Omega$, $L_s = L_r = 0.001$ H, $L_M = 0.0009$ H, $B_m = 0.000015$ N-m-s-rad^{-1}, and $J = 0.000017$ kg-m^2. $U_s = 100 \sin(200\,t)$ V; $U_r = 50 \sin(200\,t)$ V.

Case Study 3: A Three-Phase Permanent Magnet Synchronous Motor

Apply Lagrange equations of motion to study the dynamics of the following permanent-magnet synchronous motor. Simulate this system.





Solution:

$$q_1 = \frac{i_{as}}{s}; \dot{q}_1 = i_{as}$$
$$q_2 = \frac{i_{bs}}{s}; \dot{q}_2 = i_{bs}$$
$$q_3 = \frac{i_{cs}}{s}; \dot{q}_3 = i_{cs}$$
$$q_4 = \theta_r$$
$$\dot{q}_4 = \omega_r$$
$$Q_1 = U_{as}$$
$$Q_2 = U_{bs}$$
$$Q_3 = U_{cs}$$
$$Q_4 = -T_L$$

The resulting Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_1}\right)-\frac{\partial K_e}{\partial q_1}+\frac{\partial P}{\partial \dot{q}_1}+\frac{\partial V}{\partial q_1}=Q_1$$
$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_2}\right)-\frac{\partial K_e}{\partial q_2}+\frac{\partial P}{\partial \dot{q}_2}+\frac{\partial V}{\partial q_2}=Q_2$$
$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_3}\right)-\frac{\partial K_e}{\partial q_3}+\frac{\partial P}{\partial \dot{q}_3}+\frac{\partial V}{\partial q_3}=Q_3$$
$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_4}\right)-\frac{\partial K_e}{\partial q_4}+\frac{\partial P}{\partial \dot{q}_4}+\frac{\partial V}{\partial q_4}=Q_4$$

The total kinetic energy includes kinetic energies of electrical and mechanical systems; in particular

$$K_e = K_{ee} + K_m = \frac{1}{2}L_{asas}\dot{q}_1^2 + \frac{1}{2}\left(L_{asbs}+L_{bsas}\right)\dot{q}_1\dot{q}_2 + \frac{1}{2}\left(L_{ascs}+L_{csas}\right)\dot{q}_1\dot{q}_3 + \frac{1}{2}L_{bsbs}\dot{q}_2^2 +$$
$$\frac{1}{2}\left(L_{bscs}+L_{csbs}\right)\dot{q}_2\dot{q}_3 + \frac{1}{2}L_{\csc s}\dot{q}_3^2 + \psi_m\dot{q}_1\sin q_4 + \psi_m\dot{q}_2\sin(q_4-\frac{2}{3}\pi) + \psi_m\dot{q}_3\sin\left(q_4+\frac{2}{3}\pi\right) + \frac{1}{2}J\dot{q}_4^2$$

The self- and mutual inductances are defined by their subscripts, and the flux established by the permanent magnet is denoted by Ψ_m.

$$\frac{\partial K_e}{\partial q_1}=0;\frac{\partial K_e}{\partial \dot{q}_1}=L_{asas}\dot{q}_1+\frac{1}{2}\left(L_{asbs}+L_{bsas}\right)\dot{q}_2+\frac{1}{2}\left(L_{ascs}+L_{csas}\right)\dot{q}_3+\psi_m\sin q_4$$
$$\frac{\partial K_e}{\partial q_2}=0;\frac{\partial K_e}{\partial \dot{q}_2}=\frac{1}{2}\left(L_{asbs}+L_{bsas}\right)\dot{q}_1+L_{bsbs}\dot{q}_2+\frac{1}{2}\left(L_{bscs}+L_{csbs}\right)\dot{q}_3+\psi_m\sin\left(q_4-\frac{2}{3}\pi\right)$$
$$\frac{\partial K_e}{\partial q_3}=0;\frac{\partial K_e}{\partial \dot{q}_3}=\frac{1}{2}\left(L_{ascs}+L_{csas}\right)\dot{q}_1+\frac{1}{2}\left(L_{ascs}+L_{csbs}\right)\dot{q}_2+L_{\csc s}\dot{q}_3+\psi_m\sin\left(q_4+\frac{2}{3}\pi\right)$$
$$\frac{\partial K_e}{\partial q_4}=\psi_m\dot{q}_1\cos q_4+\psi_m\dot{q}_2\cos\left(q_4-\frac{2}{3}\pi\right)+\psi_m\dot{q}_3\cos\left(q_4+\frac{2}{3}\pi\right);\frac{\partial K_e}{\partial \dot{q}_4}=J\dot{q}_4$$

Since there is no spring in the mechanical system, the potential energy $V = 0$

The dissipated energy should be found as a sum of the heat energy dissipated by the electrical system and the heat energy dissipated by the mechanical system

$$P=\frac{1}{2}\left(R_s\dot{q}_1^2+R_s\dot{q}_2^2+R_s\dot{q}_3^2+B_m\dot{q}_4^2\right)$$

We obtain

$$\frac{\partial P}{\partial \dot{q}_1}=R_s\dot{q}_1;\frac{\partial P}{\partial \dot{q}_2}=R_s\dot{q}_2;\frac{\partial P}{\partial \dot{q}_3}=R_s\dot{q}_3;\frac{\partial P}{\partial \dot{q}_4}=B_m\dot{q}_4$$

The Lagrange equations, which are expressed in terms of each independent coordinate, lead to four differential equations

$$L_{asas}\frac{di_{as}}{dt}+\frac{1}{2}\left(L_{asbs}+L_{bsas}\right)\frac{di_{bs}}{dt}+\frac{1}{2}\left(L_{ascs}+L_{csas}\right)\frac{di_{cs}}{dt}+\psi_m\omega_r\cos\theta_r+r_s i_{as}=U_{as}$$

$$\frac{1}{2}\left(L_{asbs}+L_{bsas}\right)\frac{di_{as}}{dt}+L_{bsbs}\frac{di_{bs}}{dt}+\frac{1}{2}\left(L_{bscs}+L_{csbs}\right)\frac{di_{cs}}{dt}+\psi_m\omega_r\cos(\theta_r-\frac{2}{3}\pi)+R_s i_{bs}=U_{bs}$$

$$\frac{1}{2}\left(L_{ascs}+L_{csas}\right)\frac{di_{as}}{dt}+\frac{1}{2}\left(L_{bscs}+L_{csbs}\right)\frac{di_{bs}}{dt}+L_{\csc s}\frac{di_{cs}}{dt}+\psi_m\omega_r\cos(\theta_r+\frac{2}{3}\pi)+R_s i_{cs}=U_{cs}$$

$$J\frac{d^2\theta_r}{dt^2}-\psi_m i_{as}\cos\theta_r-\psi_m i_{bs}\cos\left(\theta_r-\frac{2}{3}\pi\right)-\psi_m i_{cs}\cos\left(\theta_r+\frac{2}{3}\pi\right)+B_m\frac{d\theta_r}{dt}=-T_L$$

Notes:

$$U_{as}(t)=\sqrt{2}\,U_M\cos\theta_{\mathrm{r}}$$

$$U_{bs}(t)=\sqrt{2}\,U_M\cos(\theta_{\mathrm{r}}-\frac{2}{3}\pi)$$

$$U_{bs}(t)=\sqrt{2}\,U_M\cos(\theta_{\mathrm{r}}+\frac{2}{3}\pi)$$

Data:

Using the block diagram of permanent-magnet synchronous motors, as illustrated the Figure above, develop the SIMULINK diagram. Simulate a three-phase, two pole permanent magnet synchronous motor with the following parameters: R_s = 0.5 Ω, L_{ss} = 0.001 H, L_{ls} = 0.001 H, L_m = 0.0009 H, ψ_m = 0.069 V-s-rad^{-1} (N-m-A-1), B_m = 0.000015 N-m-s-rad^{-1}, and J = 0.000017 kg-m^2. Perform the transient analysis by supplying a balanced three-phase voltages set; U_M = 40 V.

Note:

The mutual inductances between sinusoidally distributed stator windings L_{asbs}, L_{ascs}, L_{bsas}, L_{bscs}, and L_{csbs} are periodic functions of θ_r and have the average values (DC components). Assuming the magnetic field is uniform, and making use of the fact that the magnetic axes are displaced by (2/3) π, one concludes that the DC component of L_{asbs}, L_{ascs}, L_{bsas}, L_{bscs}, and L_{csbs}.

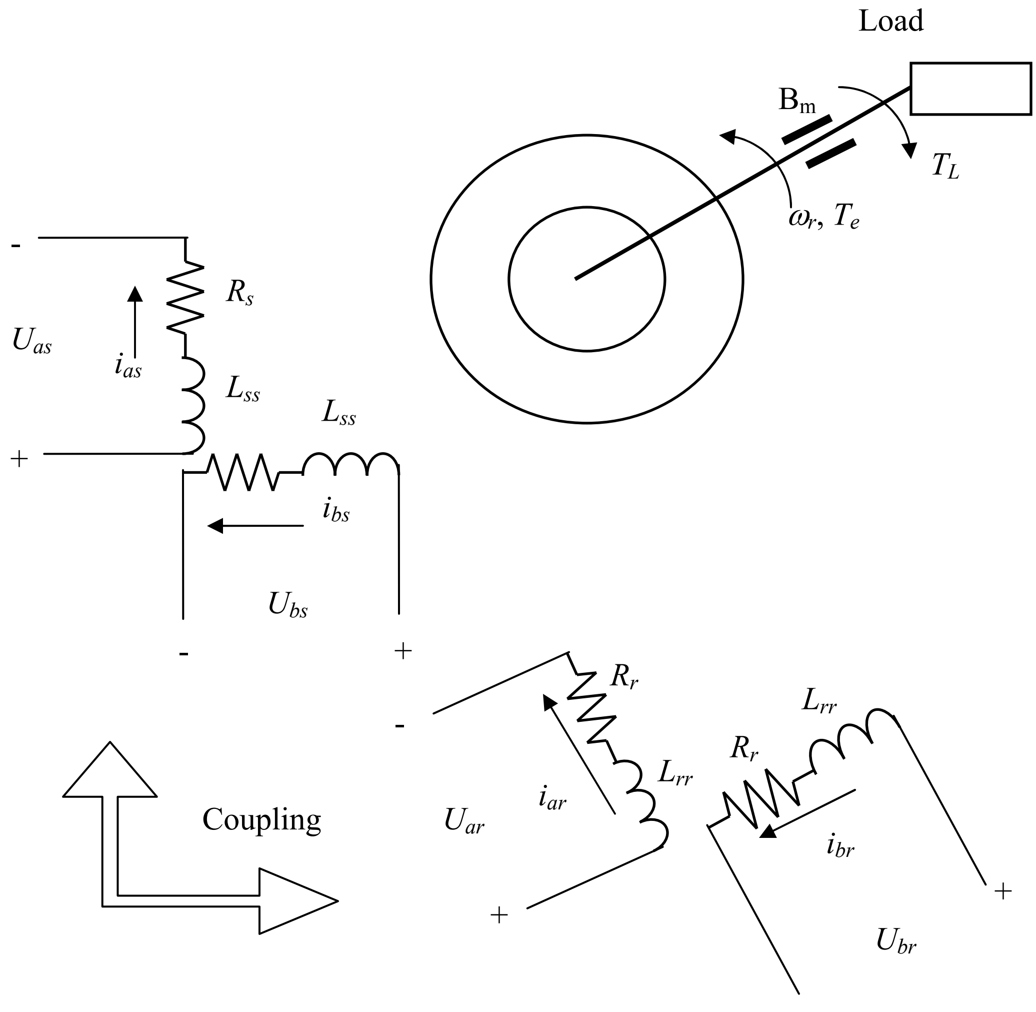
$$L_{asbs}=L_{ascs}=L_{bsas}=L_{bscs}=L_{csbs}=L_m\cos(\frac{2}{3}\pi)=-\frac{1}{2}L_m$$

Symbols:

L_{ss}	Self-inductance of the stator windings
L_{ls}	Stator leakage inductance
L_{lr}	Rotor leakage inductance
B_m	Viscous friction coefficient
J	Equivalent moment of inertia
ψ_m	Magnetic of the flux linkage established by the permanent-magnet.

Case Study 4: A Two-Phase Induction Motor

Find the mathematical model using the Lagrange equations of motion for a two-phase induction motor motor.



$$q_1 = \frac{i_{as}}{s}; q_2 = \frac{i_{bs}}{s}; q_3 = \frac{i_{ar}^{'}}{s}; q_4 = \frac{i_{br}^{'}}{s}; q_5 = \theta_r$$

$$Q_1 = U_{as}; Q_2 = U_{bs}; Q_3 = U_{ar}^{'}; Q_4 = U_{br}^{'}, Q_5 = -T_L$$

We may write the following equations:

$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_1}\right)-\frac{\partial K_e}{\partial q_1}+\frac{\partial P}{\partial \dot{q}_1}+\frac{\partial V}{\partial q_1}=Q_1$$
$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_2}\right)-\frac{\partial K_e}{\partial q_2}+\frac{\partial P}{\partial \dot{q}_2}+\frac{\partial V}{\partial q_2}=Q_2$$
$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_3}\right)-\frac{\partial K_e}{\partial q_3}+\frac{\partial P}{\partial \dot{q}_3}+\frac{\partial V}{\partial q_3}=Q_3$$
$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_4}\right)-\frac{\partial K_e}{\partial q_4}+\frac{\partial P}{\partial \dot{q}_4}+\frac{\partial V}{\partial q_4}=Q_4$$
$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_5}\right)-\frac{\partial K_e}{\partial q_5}+\frac{\partial P}{\partial \dot{q}_5}+\frac{\partial V}{\partial q_5}=Q_5$$

The expressions for the total kinetic, potential, and dissipated energies are given by:

$$Ke=\frac{1}{2}L_{ss}\dot{q}_1^2+L_{ms}\dot{q}_1\dot{q}_3\cos q_5-L_{ms}\dot{q}_1\dot{q}_4\sin q_5+\frac{1}{2}L_{ss}\dot{q}_2^2+L_{ms}\dot{q}_2\dot{q}_3\sin q_5$$
$$+\frac{1}{2}L_{rr}^{'}\dot{q}_3^2+\frac{1}{2}L_{rr}^{'}\dot{q}_4^2+\frac{1}{2}J\dot{q}_5^2$$
$$V=0$$
$$P=\frac{1}{2}\left(R_s\dot{q}_1^2+R_s\dot{q}_2^2+R_r^{'}\dot{q}_3^2+R_r^{'}\dot{q}_4^2+B_m\dot{q}_5^2\right)$$

$$L_{ms}=\frac{N_s}{N_r}L_{sr}$$
$$L_{sr}=\frac{N_sN_r}{\Re_m}$$
$$L_{ss}=\frac{N_s^2}{\Re_m}$$
$$Lrr=\frac{N_r^2}{\Re_m}$$
$$L_{asar}=L_{aras}=L_{sr}\cos\theta r$$
$$L_{asbr}=L_{bras}=-L_{sr}\sin\theta r$$
$$L_{bsar}=L_{arbs}=L_{sr}\sin\theta r$$
$$L_{bsbr}=L_{brbs}=L_{sr}\cos\theta r$$

$$\frac{\partial K_e}{\partial q_1} = 0; \frac{\partial K_e}{\partial \dot{q}_1} = L_{ss}\dot{q}_1 + L_{ms}\dot{q}_3\cos q_5 - L_{ms}\dot{q}_4\sin q_5$$
$$\frac{\partial K_e}{\partial q_2} = 0; \frac{\partial K_e}{\partial \dot{q}_2} = L_{ss}\dot{q}_2 + L_{ms}\dot{q}_3\cos q_5 + L_{ms}\dot{q}_4\cos q_5$$
$$\frac{\partial K_e}{\partial q_3} = 0; \frac{\partial K_e}{\partial \dot{q}_3} = L_{rr}^{'}\dot{q}_3 + L_{ms}\dot{q}_3\cos q_5 + L_{ms}\dot{q}_4\sin q_5$$
$$\frac{\partial K_e}{\partial q_4} = 0; \frac{\partial K_e}{\partial \dot{q}_4} = L_{rr}^{'}\dot{q}_4 - L_{ms}\dot{q}_1\cos q_5 + L_{ms}\dot{q}_2\cos q_5$$
$$\frac{\partial K_e}{\partial q_5} = -L_{ms}\dot{q}_1\dot{q}_3\sin q_5 - L_{ms}\dot{q}_1\dot{q}_4\cos q_5 + L_{ms}\dot{q}_2\dot{q}_3\cos q_5 - L_{ms}\dot{q}_2\dot{q}_4\sin q_5$$
$$= L_{ms}\left[(\dot{q}_1\dot{q}_3 + \dot{q}_2\dot{q}_4)\sin q_5 + (\dot{q}_1\dot{q}_4 - \dot{q}_2\dot{q}_3)\cos q_5\right]$$
$$\frac{\partial K_e}{\partial q_5} = J\dot{q}_5$$
$$\frac{\partial V}{\partial q_1} = 0; \frac{\partial V}{\partial q_2} = 0; \frac{\partial V}{\partial q_3} = 0; \frac{\partial V}{\partial q_4} = 0; \frac{\partial V}{\partial q_5} = 0$$
$$\frac{\partial P}{\partial \dot{q}_1} = R_s\dot{q}_1; \frac{\partial P}{\partial \dot{q}_2} = R_s\dot{q}_2; \frac{\partial P}{\partial \dot{q}_3} = R_r^{'}\dot{q}_1; \frac{\partial P}{\partial \dot{q}_4} = R_r^{'}\dot{q}_4; \frac{\partial P}{\partial \dot{q}_5} = B_m\dot{q}_5$$

In terms of original values, we have

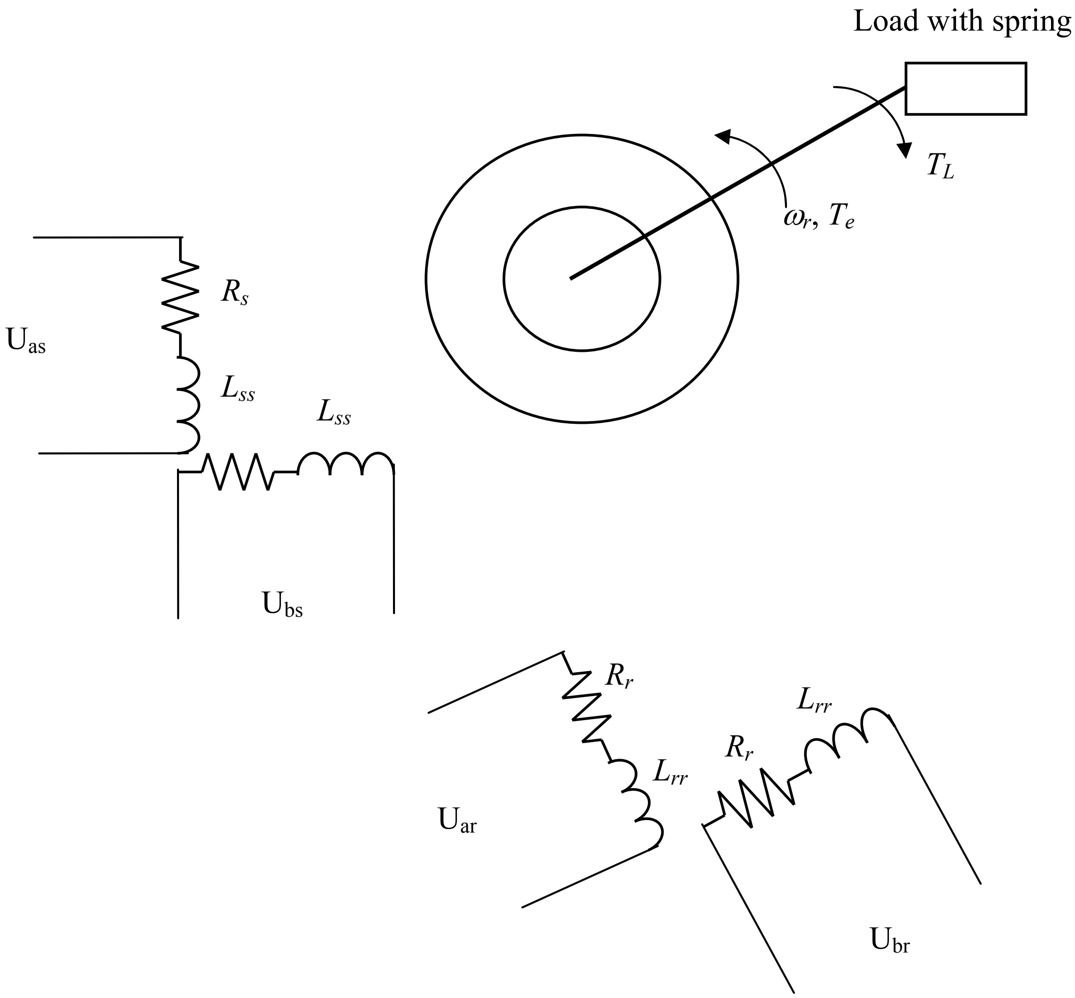
$$L_{ss}\frac{di_{as}}{dt} + L_{ms}\frac{d\left(i_{ar}^{'}\cos\theta_r\right)}{dt} - L_{ms}\frac{d\left(i_{ar}^{'}\sin\theta_r\right)}{dt} + R_s i_{as} = U_{as}$$
$$L_{ss}\frac{di_{bs}}{dt} + L_{ms}\frac{d\left(i_{ar}^{'}\sin\theta_r\right)}{dt} + L_{ms}\frac{d\left(i_{bs}\sin\theta_r\right)}{dt} + +R_s i_{as} = U_{as}$$
$$L_{ss}\frac{d\left(i_{as}\cos\theta_r\right)}{dt} - L_{ms}\frac{d\left(i_{ar}^{'}\sin\theta_r\right)}{dt} + L_{rr}^{'}\frac{di_{ar}^{'}}{dt} + R_r^{'}i_{ar}^{'} = U_{ar}^{'}$$
$$-L_{ss}\frac{d\left(i_{as}\sin\theta_r\right)}{dt} + L_{ms}\frac{d\left(i_{bs}\cos\theta_r\right)}{dt} + L_{rr}^{'}\frac{di_{br}^{'}}{dt} + R_r^{'}i_{br}^{'} = U_{br}^{'}$$
$$J\frac{d^2\theta_r}{dt^2} + L_{ms}\left[\left(i_{as}i_{ar}^{'} + i_{bs}i_{br}^{'}\right)\sin\theta_r + \left(i_{as}i_{br}^{'} - i_{bs}i_{ar}^{'}\right)\cos\theta_r\right] + B_m\frac{d\theta_r}{dt} = -T_l$$

Data:

Using the block diagram as illustrated in the Figure above, develop the SIMULINK diagram. Simulate the servo-system with the following parameters: $R_s = R_r = 0.5\ \Omega$, $L_{ss} = L_{rr} = 0.001$ H, $L_{ms} = 0.0009$ H, $B_m = 0.000015$ N-m-s-rad^{-1}, and $J = 0.000017$ kg-m^2. $U_{as} = 100 \sin(200\ t)$ V; $U_{bs} = 50 \sin(200\ t)$ V, $U_{ar} = 100 \sin(200\ t)$ V; $U_{br} = 50 \sin(200\ t)$ V.

Case Study 5: A Two-Phase Induction Generator

Find the mathematical model using the Lagrange equations of motion for a two-phase induction generator.



$$q_1 = -\frac{i_{as}}{s}; q_2 = -\frac{i_{bs}}{s}; q_3 = \frac{i'_{ar}}{s}; q_4 = \frac{i'_{br}}{s}; q_5 = \theta_r$$
$$Q_1 = U_{as}; Q_2 = U_{bs}; Q_3 = U'_{ar}; Q_4 = U'_{br}, Q_5 = -T_{pm}$$

We may write the following equations:

$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_1}\right)-\frac{\partial K_e}{\partial q_1}+\frac{\partial P}{\partial \dot{q}_1}+\frac{\partial V}{\partial q_1}=Q_1$$
$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_2}\right)-\frac{\partial K_e}{\partial q_2}+\frac{\partial P}{\partial \dot{q}_2}+\frac{\partial V}{\partial q_2}=Q_2$$
$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_3}\right)-\frac{\partial K_e}{\partial q_3}+\frac{\partial P}{\partial \dot{q}_3}+\frac{\partial V}{\partial q_3}=Q_3$$
$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_4}\right)-\frac{\partial K_e}{\partial q_4}+\frac{\partial P}{\partial \dot{q}_4}+\frac{\partial V}{\partial q_4}=Q_4$$
$$\frac{d}{dt}\left(\frac{\partial K_e}{\partial \dot{q}_5}\right)-\frac{\partial K_e}{\partial q_5}+\frac{\partial P}{\partial \dot{q}_5}+\frac{\partial V}{\partial q_5}=Q_5$$

The expressions for the total kinetic, potential, and dissipated energies are given by:

$$K_e=\frac{1}{2}L_{ss}\dot{q}_1^2+L_{ms}\dot{q}_1\dot{q}_2\cos q_5-L_{ms}\dot{q}_1\dot{q}_4\sin q_5+\frac{1}{2}L_{ss}\dot{q}_2^2+L_{ms}\dot{q}_2\dot{q}_3\sin q_5$$
$$+\frac{1}{2}L_{rr}^{'}\dot{q}_3^2+\frac{1}{2}L_{rr}^{'}\dot{q}_4^2+\frac{1}{2}J\dot{q}_5^2$$
$$V=0$$
$$P=\frac{1}{2}\left(R_s\dot{q}_1^2+R_s\dot{q}_2^2+R_r^{'}\dot{q}_3^2+R_r^{'}\dot{q}_4^2+B_m\dot{q}_5^2\right)$$

$$\frac{\partial K_e}{\partial q_1} = 0; \frac{\partial K_e}{\partial \dot{q}_1} = L_{ss}\dot{q}_1 + L_{ms}\dot{q}_3 \cos q_5 - L_{ms}\dot{q}_4 \sin q_5$$

$$\frac{\partial K_e}{\partial q_2} = 0; \frac{\partial K_e}{\partial \dot{q}_2} = L_{ss}\dot{q}_2 + L_{ms}\dot{q}_3 \cos q_5 + L_{ms}\dot{q}_4 \cos q_5$$

$$\frac{\partial K_e}{\partial q_3} = 0; \frac{\partial K_e}{\partial \dot{q}_3} = L_{rr}^{'}\dot{q}_3 + L_{ms}\dot{q}_3 \cos q_5 + L_{ms}\dot{q}_4 \sin q_5$$

$$\frac{\partial K_e}{\partial q_4} = 0; \frac{\partial K_e}{\partial \dot{q}_4} = L_{rr}^{'}\dot{q}_4 - L_{ms}\dot{q}_1 \cos q_5 + L_{ms}\dot{q}_2 \cos q_5$$

$$\frac{\partial K_e}{\partial q_5} = -L_{ms}\dot{q}_1\dot{q}_3 \sin q_5 - L_{ms}\dot{q}_1\dot{q}_4 \cos q_5 + L_{ms}\dot{q}_2\dot{q}_3 \cos q_5 - L_{ms}\dot{q}_2\dot{q}_4 \sin q_5$$

$$= -L_{ms}\left[(\dot{q}_1\dot{q}_3 + \dot{q}_2\dot{q}_4)\sin q_5 + (\dot{q}_1\dot{q}_4 - \dot{q}_2\dot{q}_3)\cos q_5\right]$$

$$\frac{\partial K_e}{\partial q_5} = J\dot{q}_5$$

$$\frac{\partial V}{\partial q_1} = 0; \frac{\partial V}{\partial q_2} = 0; \frac{\partial V}{\partial q_3} = 0; \frac{\partial V}{\partial q_4} = 0; \frac{\partial V}{\partial q_5} = 0$$

$$\frac{\partial P}{\partial \dot{q}_1} = R_s\dot{q}_1; \frac{\partial P}{\partial \dot{q}_2} = R_s\dot{q}_2; \frac{\partial P}{\partial \dot{q}_3} = R_r^{'}\dot{q}_1; \frac{\partial P}{\partial \dot{q}_4} = R_r^{'}\dot{q}_4; \frac{\partial P}{\partial \dot{q}_5} = B_m\dot{q}_5$$

In terms of original values, we have

$$-L_{ss}\frac{di_{as}}{dt} + L_{ms}\frac{d\left(i_{ar}^{'}\cos\theta_r\right)}{dt} - L_{ms}\frac{d\left(i_{ar}^{'}\sin\theta_r\right)}{dt} + R_s i_{as} = U_{as}$$

$$-L_{ss}\frac{di_{bs}}{dt} + L_{ms}\frac{d\left(i_{ar}^{'}\sin\theta_r\right)}{dt} + L_{ms}\frac{d\left(i_{bs}\sin\theta_r\right)}{dt} + +R_s i_{as} = U_{as}$$

$$-L_{ss}\frac{d\left(i_{as}\cos\theta_r\right)}{dt} - L_{ms}\frac{d\left(i_{ar}^{'}\sin\theta_r\right)}{dt} + L_{rr}^{'}\frac{di_{ar}^{'}}{dt} + R_r^{'}i_{ar}^{'} = U_{ar}^{'}$$

$$L_{ss}\frac{d\left(i_{as}\sin\theta_r\right)}{dt} + L_{ms}\frac{d\left(i_{bs}\cos\theta_r\right)}{dt} + L_{rr}^{'}\frac{di_{br}^{'}}{dt} + R_r^{'}i_{br}^{'} = U_{br}^{'}$$

$$J\frac{d^2\theta_r}{dt^2} - L_{ms}\left[\left(i_{as}i_{ar}^{'} + i_{bs}i_{br}^{'}\right)\sin\theta_r + \left(i_{as}i_{br}^{'} - i_{bs}i_{ar}^{'}\right)\cos\theta_r\right] + B_m\frac{d\theta_r}{dt} = T_{pm}$$