ELG 4151

Linear Systems

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My agenda for this tutorial session

- I will introduce the Laplace Transforms as a useful tool for you to tackle linear systems analysis.
- I will give examples on how to derive the transfer function of different models.
- I will talk about solving the system equation to obtain the system response and I will give examples regarding that.
- I will talk about performing system analysis using the state transition matrix and I will give examples on that.

Laplace Transforms

- We transform the system (model), which is identified by a differential equation (transfer function), from time domain to frequency domain.

$$\frac{d}{dt} = D = s \qquad\qquad s = \sigma + j\omega$$

- We normally assume zero initial conditions at t=0. If any of the initial conditions are non-zero, then they must be added.

What is Laplace Transform

$$f(s) = \int_0^{\infty} f(t)e^{-st}dt$$

where,

$f(t)$ = the function in terms of time t

$f(s)$ = the function in terms of the Laplace s

Example

For $f(t) = 5$,

$$f(s) = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty} 5e^{-st}dt = \frac{5}{s}e^{-st}\Big|_0^{\infty} = \left[\frac{5}{s}e^{-s\infty}\right] - \left[\frac{5e^{-s0}}{s}\right] = \frac{5}{s}$$

- How can we carry out system (model) analysis using Laplace Transforms ?
  1. We convert the system transfer function (differential equation) to the s-domain using Laplace Transform by replacing ‘*d/dt*’ or ’*D*’ with ’*s*’.
  2. We convert the input function to the s-domain using the transform tables.
  3. We combine algebraically the input and the transfer function to find out an output function.
  4. We Use partial fractions to reduce the output function to simpler components.
  5. We convert the output equation from the s-domain back to the time-domain to obtain the response using Inverse Laplace Transforms according to the tables.

Laplace Transforms Properties

TIME DOMAIN	FREQUENCY DOMAIN
$f(t)$	$f(s)$
$Kf(t)$	$KL[f(t)]$
$f_1(t) + f_2(t) - f_3(t) + \ldots$	$f_1(s) + f_2(s) - f_3(s) + \ldots$
$\frac{df(t)}{dt}$	$sL[f(t)] - f(0^-)$
$\frac{d^2 f(t)}{dt^2}$	$s^2 L[f(t)] - sf(0^-) - \frac{df(0^-)}{dt}$
$\frac{d^n f(t)}{dt^n}$	$s^n L[f(t)] - s^{n-1} f(0^-) - s^{n-2} \frac{df(0^-)}{dt} - \ldots - \frac{d^{n-1} f(0^-)}{dt^n}$
$\int_0^t f(t)dt$	$\frac{L[f(t)]}{s}$
$f(t-a)u(t-a),\ a > 0$	$e^{-as} L[f(t)]$
$e^{-at} f(t)$	$f(s-a)$
$f(at),\ a > 0$	$\frac{1}{a} f\left(\frac{s}{a}\right)$
$tf(t)$	$\frac{-df(s)}{ds}$
$t^n f(t)$	$(-1)^n \frac{d^n f(s)}{ds^n}$
$\frac{f(t)}{t}$	$\int_s^\infty f(u)du$

- Q: What is the Laplace Transform for the convolution function.

$$L[F(s)*G(s)] = F(s).G(s)$$

$$F(s)*G(s) = \int_{-\infty}^{+\infty} f(t-\tau)g(\tau)d\tau$$

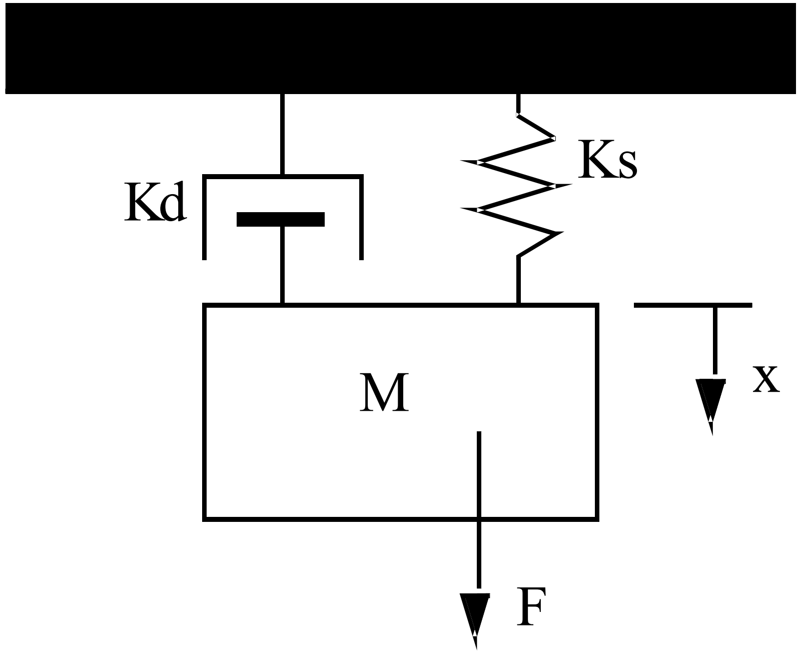
Laplace Transforms Table

TIME DOMAIN		FREQUENCY DOMAIN
$\delta(t)$	unit impulse	1
A	step	$\frac{A}{s}$
t	ramp	$\frac{1}{s^2}$
t^2		$\frac{2}{s^3}$
$t^n,\ n > 0$		$\frac{n!}{s^{n+1}}$
e^{-at}	exponential decay	$\frac{1}{s+a}$
$\sin(\omega t)$		$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$		$\frac{s}{s^2+\omega^2}$
te^{-at}		$\frac{1}{(s+a)^2}$
t^2e^{-at}		$\frac{2!}{(s+a)^3}$

TIME DOMAIN	FREQUENCY DOMAIN
$e^{-at}\sin(\omega t)$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\dfrac{s+a}{(s+a)^2+\omega^2}$
$e^{-at}\sin(\omega t)$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\left[B\cos\omega t+\left(\dfrac{C-aB}{\omega}\right)\sin\omega t\right]$	$\dfrac{Bs+C}{(s+a)^2+\omega^2}$
$2\lvert A\rvert e^{-\alpha t}\cos(\beta t+\theta)$	$\dfrac{A}{s+\alpha-\beta j}+\dfrac{A^{\text{complex conjugate}}}{s+\alpha+\beta j}$
$2t\lvert A\rvert e^{-\alpha t}\cos(\beta t+\theta)$	$\dfrac{A}{(s+\alpha-\beta j)^2}+\dfrac{A^{\text{complex conjugate}}}{(s+\alpha+\beta j)^2}$
$\dfrac{(c-a)e^{-at}-(c-b)e^{-bt}}{b-a}$	$\dfrac{s+c}{(s+a)(s+b)}$
$\dfrac{e^{-at}-e^{-bt}}{b-a}$	$\dfrac{1}{(s+a)(s+b)}$

Example

For this mechanical system obtain the transfer function in s-domain



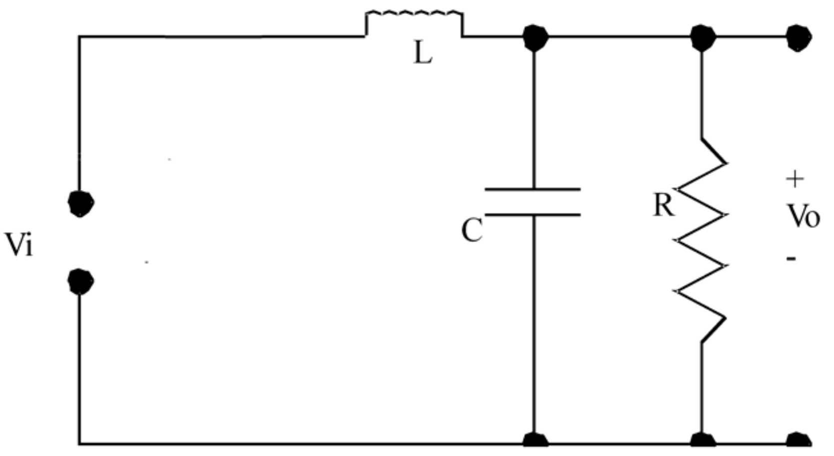
$$F = MD^2x + K_dDx + K_sx$$

$$\frac{F(t)}{x(t)} = MD^2 + K_dD + K_s$$

$$L\left[\frac{F(t)}{x(t)}\right] = \frac{F(s)}{x(s)} = Ms^2 + K_ds + K_s$$

Example

For this electrical circuit obtain the transfer function in s-domain



$$V_o = \frac{Vi\left(\dfrac{1}{DC + \dfrac{1}{R}}\right)}{DL + \left(\dfrac{1}{DC + \dfrac{1}{R}}\right)} = \frac{V_i\left(\dfrac{R}{1 + DCR}\right)}{DLR + \left(\dfrac{R}{1 + DCR}\right)} = V_i\left(\frac{R}{D^2R^2LC + DLR + R}\right)$$

$$L\left[\frac{V_o(t)}{V_i(t)}\right] = \frac{V_o(s)}{V_i(s)} = \left(\frac{R}{s^2R^2LC + sLR + R}\right)$$

Device	Time domain	s-domain	Impedance
Resistor	$V(t) = RI(t)$	$V(s) = RI(s)$	$Z = R$
Capacitor	$V(t) = \frac{1}{C}\int I(t)dt$	$V(s) = \left(\frac{1}{C}\right)\frac{I(s)}{s}$	$Z = \frac{1}{sC}$
Inductor	$V(t) = L\frac{d}{dt}I(t)$	$V(s) = LsI(s)$	$Z = Ls$

Impedances of electrical components

- **Have more examples on how to obtain the transfer function in the s-domain for the given systems.**

- **Now back to our simple mechanical system to obtain its output response to a step input of magnitude 1000 N.**

Given,

$$\frac{x(s)}{F(s)} = \frac{1}{Ms^2 + K_d s + K_s}$$

$$F(s) = \frac{A}{s}$$

Therefore,

$$x(s) = \left(\frac{x(s)}{F(s)}\right)F(s) = \left(\frac{1}{Ms^2 + K_d s + K_s}\right)\frac{A}{s}$$

Assume,

$$K_d = 3000\frac{Ns}{m}$$

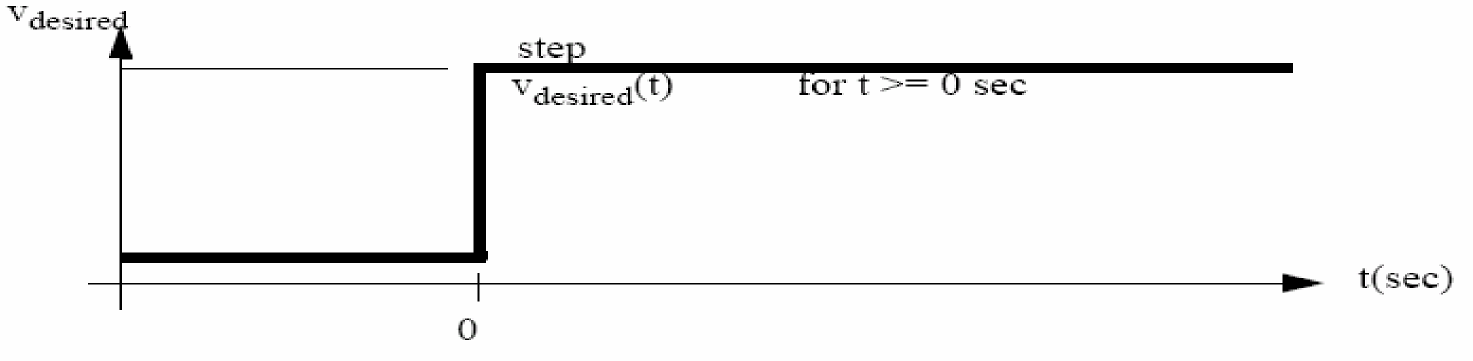
$$K_s = 2000\frac{N}{m}$$

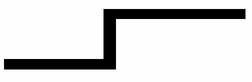
$$M = 1000kg$$

$$A = 1000N$$

$$\therefore x(s) = \frac{1}{(s^2 + 3s + 2)s}$$

Types of inputs (driving force)



	Input type	Time function	Laplace function
	STEP	$f(t) = Au(t)$	$f(s) = \frac{A}{s}$
	RAMP	$f(t) = Atu(t)$	$f(s) = \frac{A}{s^2}$
	SINUSOID	$f(t) = A\sin(\omega t)u(t)$	$f(s) = \frac{A\omega^2}{s^2 + \omega^2}$

- Performing partial fraction simplification

$$x(s) = \frac{1}{(s^2 + 3s + 2)s} = \frac{1}{(s+1)(s+2)s} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$A = \lim_{s \to 0}\left[s\left(\frac{1}{(s+1)(s+2)s}\right)\right] = \frac{1}{2}$$

$$B = \lim_{s \to -1}\left[(s+1)\left(\frac{1}{(s+1)(s+2)s}\right)\right] = -1$$

$$C = \lim_{s \to -2}\left[(s+2)\left(\frac{1}{(s+1)(s+2)s}\right)\right] = \frac{1}{2}$$

$$x(s) = \frac{1}{(s^2 + 3s + 2)s} = \frac{0.5}{s} + \frac{-1}{s+1} + \frac{0.5}{s+2}$$

- Now we proceed with the Inverse Laplace Transforms to obtain the system time response

$$x(t) = L^{-1}[x(s)] = L^{-1}\left[\frac{0.5}{s} + \frac{-1}{s+1} + \frac{0.5}{s+2}\right]$$

$$x(t) = L^{-1}\left[\frac{0.5}{s}\right] + L^{-1}\left[\frac{-1}{s+1}\right] + L^{-1}\left[\frac{0.5}{s+2}\right]$$

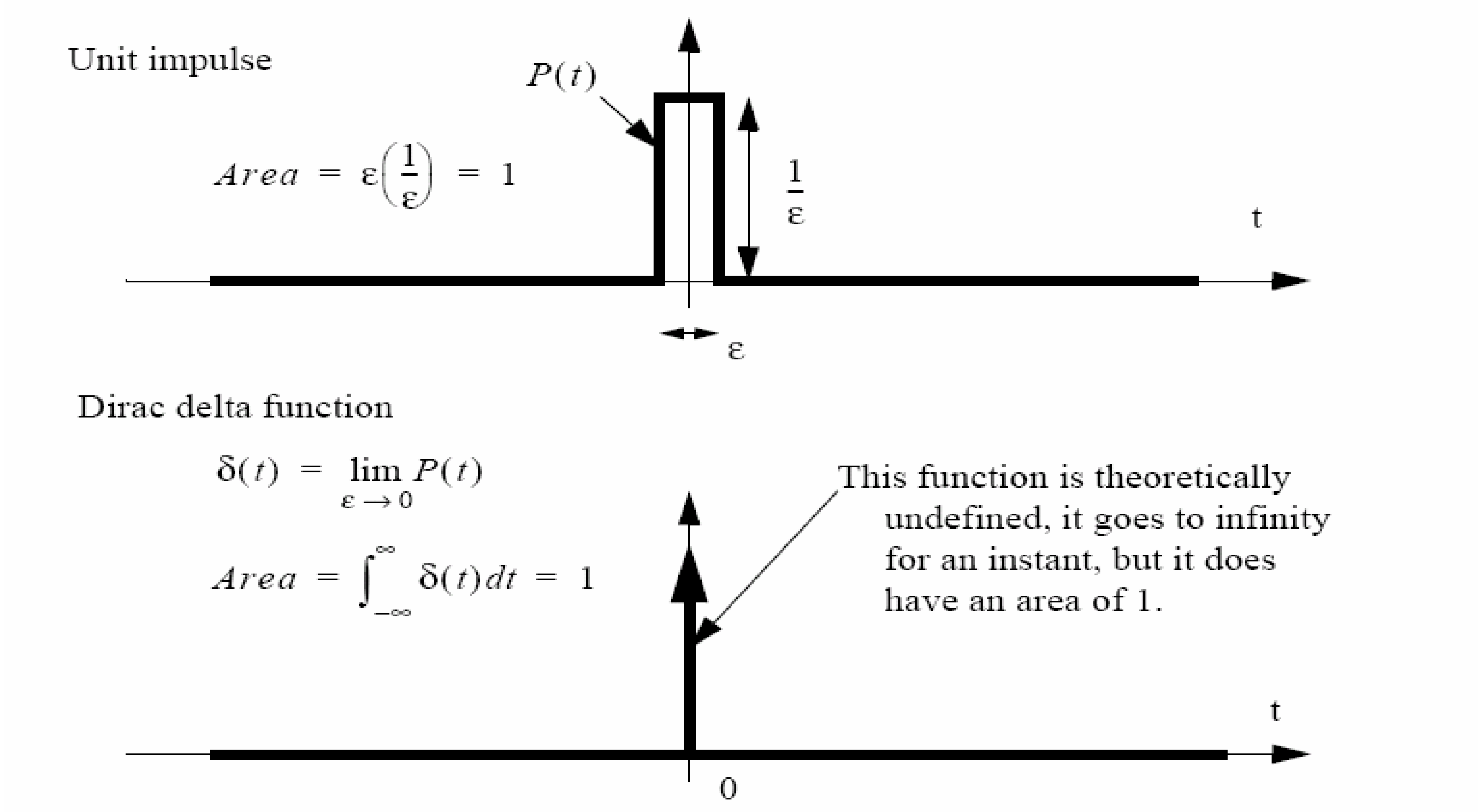
$$x(t) = [0.5] + [(-1)e^{-t}] + [(0.5)e^{-2t}]$$

$$x(t) = 0.5 - e^{-t} + 0.5e^{-2t}$$

- Try to think about the case where the driving force is an impulse input. So what will be the impulse response ?

$$F(s) = L[\delta(t)] = 1$$

$$x(s) = \frac{1}{(s^2 + 3s + 2)} = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$


Unit impulse
$P(t)$
$Area = \varepsilon\left(\frac{1}{\varepsilon}\right) = 1$
$\frac{1}{\varepsilon}$
t
ε
Dirac delta function
$\delta(t) = \lim_{\varepsilon \to 0} P(t)$
$Area = \int_{-\infty}^{\infty} \delta(t)dt = 1$
This function is theoretically undefined, it goes to infinity for an instant, but it does have an area of 1.
t
0

- What's about the partial fractions simplification for the repeated roots.

Example

$$x(s) = \frac{1}{s^2(s+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}$$

$$C = \lim_{s \to -1}\left[(s+1)\left(\frac{1}{s^2(s+1)}\right)\right] = 1$$

$$A = \lim_{s \to 0}\left[s^2\left(\frac{1}{s^2(s+1)}\right)\right] = \lim_{s \to 0}\left[\frac{1}{s+1}\right] = 1$$

$$B = \lim_{s \to 0}\left[\frac{d}{ds}\left[s^2\left(\frac{1}{s^2(s+1)}\right)\right]\right] = \lim_{s \to 0}\left[\frac{d}{ds}\left(\frac{1}{s+1}\right)\right] = \lim_{s \to 0}\left[-(s+1)^{-2}\right] = -1$$

Have another example

$$F(s) = \frac{5}{s^2(s+1)^3}$$

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

$$A = \lim_{s \to 0}\left[\left(\frac{5}{s^2(s+1)^3}\right)s^2\right] = \lim_{s \to 0}\left[\frac{5}{(s+1)^3}\right] = 5$$

$$B = \lim_{s \to 0}\left[\frac{d}{ds}\left(\frac{5}{s^2(s+1)^3}\right)s^2\right] = \lim_{s \to 0}\left[\frac{d}{ds}\left(\frac{5}{(s+1)^3}\right)\right] = \lim_{s \to 0}\left[\frac{5(-3)}{(s+1)^4}\right] = -15$$

$$C = \lim_{s \to -1}\left[\left(\frac{5}{s^2(s+1)^3}\right)(s+1)^3\right] = \lim_{s \to -1}\left[\frac{5}{s^2}\right] = 5$$

$$D = \lim_{s \to -1}\left[\frac{1}{1!}\frac{d}{ds}\left(\frac{5}{s^2(s+1)^3}\right)(s+1)^3\right] = \lim_{s \to -1}\left[\frac{1}{1!}\frac{d}{ds}\frac{5}{s^2}\right] = \lim_{s \to -1}\left[\frac{1}{1!}\frac{-2(5)}{s^3}\right] = 10$$

$$E = \lim_{s \to -1}\left[\frac{1}{2!}\frac{d}{ds}^2\left(\frac{5}{s^2(s+1)^3}\right)(s+1)^3\right] = \lim_{s \to -1}\left[\frac{1}{2!}\frac{d}{ds}^2\frac{5}{s^2}\right] = \lim_{s \to -1}\left[\frac{1}{2!}\frac{30}{s^4}\right] = 15$$

$$\frac{5}{s^2(s+1)^3} = \frac{5}{s^2} + \frac{-15}{s} + \frac{5}{(s+1)^3} + \frac{10}{(s+1)^2} + \frac{15}{(s+1)}$$

Initial and Final Value Theorems

$$x(s) = \frac{1}{(s^2+3s+2)s}$$

$$x(t\to\infty) = \lim_{s\to 0}[sx(s)]$$

Final value theorem

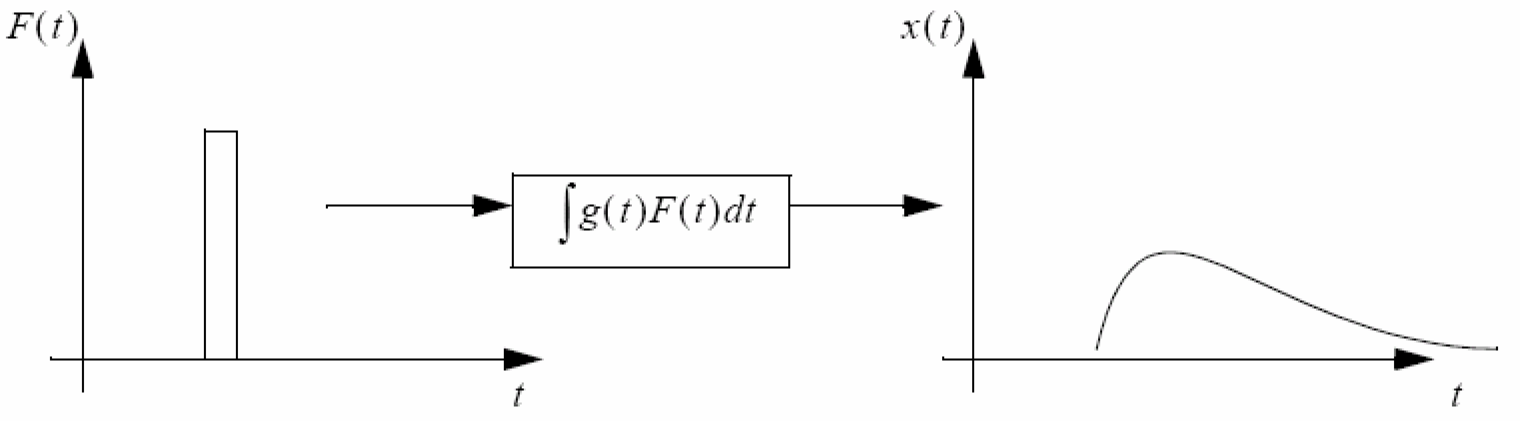
$$\therefore x(t\to\infty) = \lim_{s\to 0}\left[\frac{1s}{(s^2+3s+2)s}\right] = \lim_{s\to 0}\left[\frac{1}{s^2+3s+2}\right] = \frac{1}{(0)^2+3(0)+2} = \frac{1}{2}$$

$$x(t\to 0) = \lim_{s\to\infty}[sx(s)]$$

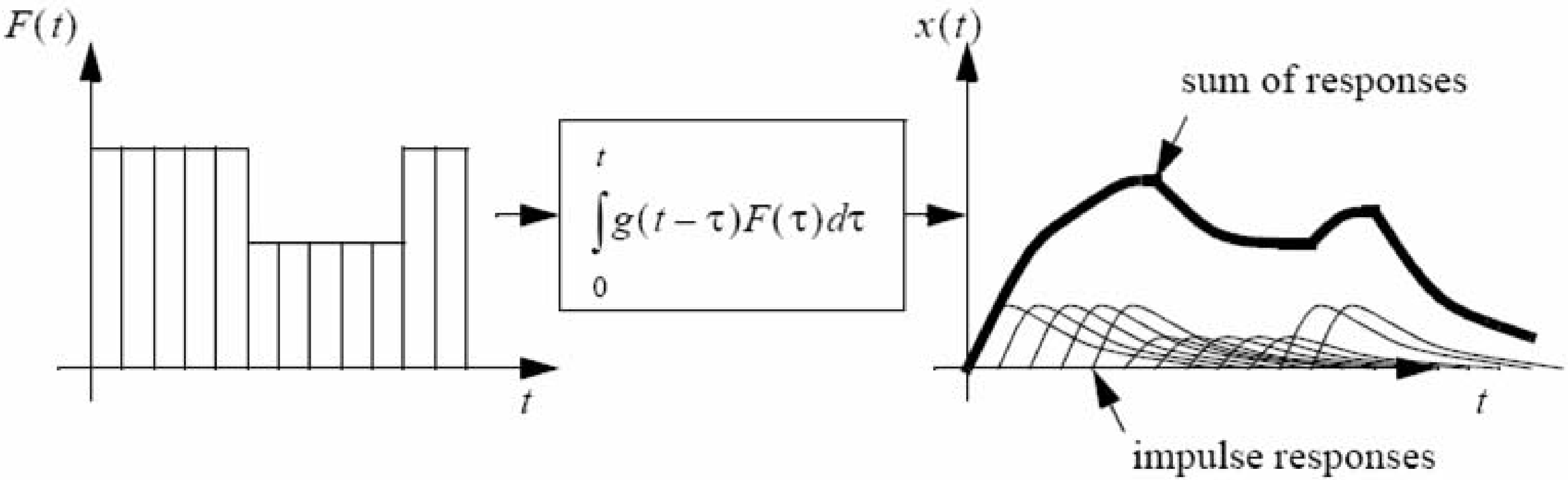
Initial value theorem

$$\therefore x(t\to 0) = \lim_{s\to\infty}\left[\frac{1(s)}{(s^2+3s+2)s}\right] = \frac{1}{((\infty)^2+3(\infty)+2)} = \frac{1}{\infty} = 0$$

- Why Laplace Transform is that powerful tool ?
    - Solving the Convolution Integral Problem



Response of the system to a single pulse



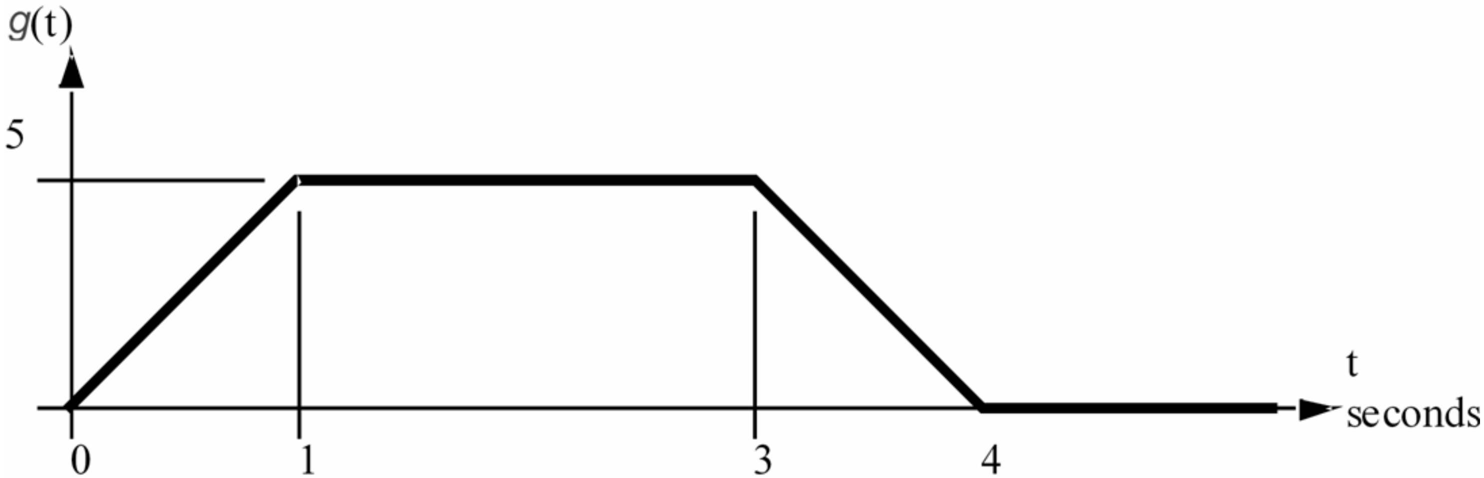
The convolution integral

$$x(t) = \int_0^t g(t-\tau)F(\tau)d\tau$$

A set of pulses for a system gives summed responses to give the output

Example

*L[y(t)=g(t)*u(t)]=y(s)=g(s).u(s)*



$$y(t) = 5tu(t) + 5(t-1)u(t-1) + 5(t-3)u(t-3) - 5(t-4)u(t-4)$$

$$y(s) = \frac{5}{s^2} + \frac{5e^{-s}}{s^2} + \frac{5e^{-3s}}{s^2} - \frac{5e^{-4s}}{s^2}$$

Solving the System Equation (Response)

Example

$$\dot{X} + 0.5X = 2\delta(t)$$

The homogeneous solution can be found.

$$\dot{X} + 0.5X = 0 \quad , \quad X_h = e^{At} \quad \dot{X}_h = Ae^{At}$$

$$A + 0.5 = 0$$

$$X_h = Ce^{-0.5t}$$

The particular solution is found.

$$\dot{X} + 0.5X = 2\delta(t) \quad , \quad X_p = A \quad \dot{X}_p = 0$$

$$0 + 0.5A = 2(0)$$

$$X_p = A = 0$$

The initial condition caused by the impulse function found, assuming a zero initial condition.

$$\left(\frac{1}{dt}\right)X_0 + 0.5(0) = 2\left(\frac{1}{dt}\right)$$

$$X_0 = 2$$

The initial condition caused by the impulse function found, assuming a zero initial condition.

$$X(t) = Ce^{-0.5t}$$

$$X(0) = 2 = Ce^{0}$$

$$X(t) = 2e^{-0.5t}$$

- Solving for system response using Laplace Transforms

$$(s+0.5)X(s) = 2$$

$$X(s) = \frac{2}{s+0.5}$$

$$X(t) = L^{-1}[X(s)] = 2e^{-0.5t}$$

System Analysis Based on State Transition Matrix

State equations as functions of time

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

In the s-domain

$$sX - X_0 = AX + BU$$

$$X(sI - A) = BU + X_0$$

$$X = (sI - A)^{-1}BU + (sI - A)^{-1}X_0$$

$$Y = CX + DU$$

$$Y = C((sI - A)^{-1}BU + (sI - A)^{-1}X_0) + DU$$

$$Y = (C(sI - A)^{-1}B + D)U + C(sI - A)^{-1}X_0$$

Assuming the system starts at rest,

$$Y = (C(sI - A)^{-1}B + D)U$$

$$\frac{Y}{U} = (C(sI - A)^{-1}B + D) \qquad \text{(the transfer function)}$$

State Transition Matrix

The transfer function can be said to be equivalant to the determinants of the matrix form.

$$\frac{\begin{vmatrix} (sI-A) & -B \\ C & D \end{vmatrix}}{|sI-A|} = \frac{(sI-A)D-(-B)C}{sI-A} = (sI-A)D+BC(sI-A)$$

$$G = C(sI-A)^{-1}B+D = \frac{\begin{vmatrix} (sI-A) & -B \\ C & D \end{vmatrix}}{|sI-A|} = \frac{poles}{zeros}$$

$|sI-A|$ = characteristic equation = homogeneous

Solving the Model Equations in Time Domain

$$\dot{x} = Ax + Bu$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y = Cx + Du$$

$$y(t) = \underbrace{Ce^{At}x(0)}_{\text{initial response}} + \underbrace{\int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau}_{\text{impulse response}} + Du(t)$$

Solving in s-domain (Open form)

The homogeneous equation can be written in the s-domain, and then converted to time.

$$X_h = |sI - A| X_0$$

$$x_h(t) = L^{-1}[|sI - A| X_0]$$

$$x_h(t) = L^{-1}\left[\left[\frac{I}{s} + \frac{A}{s} + \frac{A^2}{s^2} + \frac{A^3}{s^3} + \ldots\right] X_0\right]$$

$$x_h(t) = e^{At} x_0$$

$$e^{At} = \text{transition matrix}$$

aside: This expansion is a McLaurin (Taylor) series.

$$e^{At} = I + At + \left(\frac{1}{2!}\right) A^2 t^2 + \left(\frac{1}{3!}\right) A^3 t^3 + \ldots$$

- Solving for the Closed Form of the State Transition Matrix

$$e^{At} = L^{-1}[(sI - A)^{-1}]$$

Example

$$F = M\ddot{x}$$

$$\dot{x} = v$$

$$\dot{v} = \frac{F}{M}$$

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} F \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

This can be used to find the inverse matrix,

$$(sI - A)^{-1} = \left(s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}}{s^2 - 0} = \begin{bmatrix} \frac{s}{s^2} & \frac{1}{s^2} \\ \frac{0}{s^2} & \frac{s}{s^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}$$

- The forced/particular solution

The function of time can be found assuming an initial position of 10 and velocity of 5.

$$e^{At} = L^{-1}[(sI - A)^{-1}] = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$x_h(t) = \begin{bmatrix} x \\ v \end{bmatrix} = e^{At}\begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 + 5t \\ 5 \end{bmatrix}$$

- Another Example

Back to our motor model!

Your Questions

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