

Problem E13.3

A system has a response $y(kT) = kT$ for $k \geq 0$. Find $Y(z)$ for this response.

Solution:

$$Y(z) = \sum_{k=0}^{\infty} y(kT)z^{-k} = \sum_{k=0}^{\infty} kT z^{-k} \quad (I)$$

On the other hand we can find the geometric series as follows

$$\sum_{k=0}^{\infty} z^{-k} = \frac{1}{1-z^{-1}} \quad (II)$$

In order to find the series in (I), take derivatives of both sides of (II) with respect to z , then

$$\sum_{k=0}^{\infty} (-k)z^{-k} = -\frac{z^{-2}}{(1-z^{-1})^2} \quad \Rightarrow \quad \sum_{k=0}^{\infty} kT z^{-k} = \frac{Tz^{-2}}{(1-z^{-1})^2}$$

Now $Y(z)$ can be found

$$Y(z) = \frac{Tz^{-2}}{(1-z^{-1})^2} = \frac{Tz}{(z-1)^2}$$

Problem E13.9

(a) Determine $y(kT)$ for $k = 0$ to 3 when

$$Y(z) = \frac{z+1}{z^2-1}$$

(b) Determine the closed form solution for $y(kT)$ as a function of k .

Solution:

Part (a)

We have

$$Y(z) = \sum_{k=0}^{\infty} y(kT)z^{-k} = \frac{1}{z-1}$$

Using long division,

$$Y(z) = \frac{1}{z-1} = z^{-1} + z^{-2} + z^{-3} + \dots$$

Using the expansion for $Y(z)$ we have

k	0	1	2	3	...
$y(kT)$	0	1	1	1	...

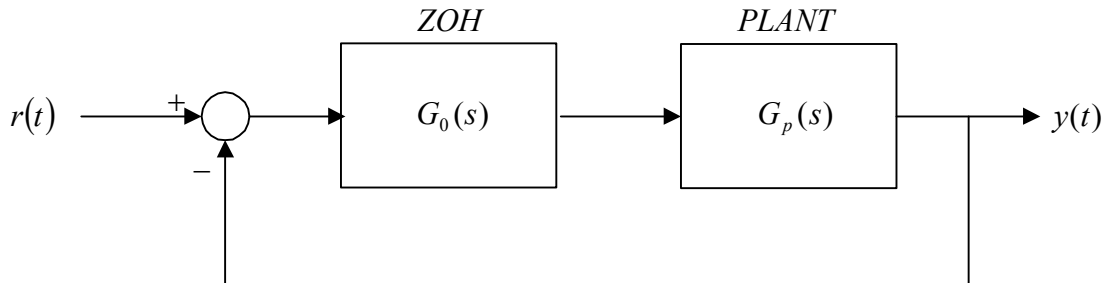
Part (b)

$$y(kT) = 1 - \delta(k) = \begin{cases} 1 & ; \text{ if } k \neq 0 \\ 0 & ; \text{ if } k = 0 \end{cases}$$

Problem P13.4

A closed loop system has a hold circuit and process as shown in Fig.13.18. Determine $G(z)$ when $T = 1$ and

$$G_p(z) = \frac{2}{s+2}$$



Solution:

The transfer function of the Zero-Order-Hold circuit is $G_0(s) = \left(\frac{1 - e^{-sT}}{s} \right)$, then

$$G(s) = G_0(s)G_p(s) = \left(\frac{1 - e^{-sT}}{s} \right) \left(\frac{2}{s+2} \right) = (1 - e^{-sT}) \left(\frac{2}{s(s+2)} \right) = (1 - e^{-sT}) \left(\frac{1}{s} - \frac{1}{s+2} \right)$$

$G(z)$ in Z-domain is the following,

$$G(z) = (1 - z^{-1}) \left(\frac{z}{z-1} - \frac{z}{z - e^{-2T}} \right) = 1 - \frac{z-1}{z - e^{-2T}}$$

$$G(z) = \frac{1 - e^{-2T}}{z - e^{-2T}}$$

for $T = 1$, we have $G(z) = \frac{0.8647}{z - 0.1353}$

Problem P13.5

For the system in P13.4, let $r(t)$ be a unit step input and calculate the response of the system by synthetic division.

Solution:

$$Y(z) = T(z)R(z) \text{ where } R(z) = \frac{z}{z-1} \text{ and } T(z) = \frac{G(z)}{1+G(z)} = \frac{0.8647}{z+0.7294}.$$

So the Z-transform of the output is found as the following

$$Y(z) = \left(\frac{0.8647}{z+0.7294} \right) \left(\frac{z}{z-1} \right) = \frac{0.8647z}{z^2 - 0.2706z - 0.7294}$$

Using long division we get

$$Y(z) = 0.8647z^{-1} + 0.2339z^{-2} + 0.694z^{-3} + 0.3585z^{-4} + \dots$$

Now $y(kT)$ in time domain is found as the following

$$y(0) = 0, \quad y(T) = 0.8647, \quad y(2T) = 0.2339, \quad y(3T) = 0.694, \quad y(4T) = 0.3585, \dots$$

Also in MATLAB the inverse Z-transform of $Y(z)$ can be found as follows

```
>> num=[0.8647 0];  
>> den=[1 -0.2706 -0.7294];  
>> sys=tf(num,den,1);  
>> [y,k]=impz(sys);  
>> y
```

y =

```
0  
0.8647  
0.2340  
0.6940  
0.3585  
0.6032
```

Problem P13.6

For the output of the system in Problem 13.4, find the initial and final values of the output directly from $Y(z)$.

Solution:

Using Table 13.1, we have

$$Y_{ss} = \lim_{z \rightarrow 1} (z-1)Y(z) = \lim_{z \rightarrow 1} (z-1) \left(\frac{z}{z-1} \right) \left(\frac{0.8647}{z+0.7294} \right)$$

$$Y_{ss} = 0.5$$

Also using the initial value theorem for Z-transform we have

$$Y_0 = \lim_{z \rightarrow \infty} Y(z) = \lim_{z \rightarrow \infty} \left(\frac{z}{z-1} \right) \left(\frac{0.8647}{z+0.7294} \right)$$

$$Y_0 = 0$$

This initial value is the same value found for $y(0)$ in problem P13.5

Problem P13.12

The transfer function of a plant and a zer-order hold in Fig.13.18 is

$$G(z) = \frac{K(z+0.5)}{z(z-1)}$$

- (a) Plot the root locus
- (b) Determine the range of gain K for a stable system

Solution:

Part (a)

We follow root locus procedure step by step as explained in Table 13.5, page 765 for root locus in the z-plane.

- 1) The root locus lies on a section of real axis to the left of an odd number of poles and zeros. In this example, there are two sections on the real axis, one between $z=0$ and $z=1$, and the other part is $z \leq -\frac{1}{2}$.

2) Breakaway points

The characteristics equation is $1 + K \frac{z+0.5}{z(z-1)} = 0$, from which K is found as a function of z.

$$K = -\frac{z(z-1)}{z+0.5}$$

$$\frac{dK}{dz} = \frac{d}{dz} \left(-\frac{z^2 - z}{z+0.5} \right) = \frac{-z^2 - z + 0.5}{(z+0.5)^2} = 0 \Rightarrow z = \frac{-1 \pm \sqrt{3}}{2} = \begin{cases} -1.366 \\ 0.366 \end{cases}$$

3) Equations for the different parts of root locus

Since the model of the system is simple (second order), we are able to find exact equations describing different branches of the root locus. We start by characteristic equation

$$q(z) = z(z-1) + K(z+0.5) = z^2 + (K-1)z + 0.5K = 0$$

Substitute $z = \sigma + j\omega$ in the above equation

$$\left((\sigma^2 - \omega^2) + (K-1)\sigma + 0.5K \right) + j(2\sigma\omega + (K-1)\omega) = 0$$

From this we get two equations

$$\begin{cases} (\sigma^2 - \omega^2) + (K-1)\sigma + 0.5K = 0 \\ 2\sigma\omega + (K-1)\omega = 0 \end{cases} \quad \text{(III)}$$

From the second equation we get two possible cases:

1) If $\omega = 0$ then using the first equation in (III), we have

$$\sigma^2 + (K-1)\sigma + 0.5K = 0$$

This equation corresponds to the section on the real axis.

2) If $2\sigma + K - 1 = 0$ then we can find K as $K = -2\sigma + 1$. Substitute this in the first equation in (III) to get

$$-\sigma^2 - \omega^2 + 0.5 - \sigma = 0 \Rightarrow (\sigma + 0.5)^2 + \omega^2 = \frac{3}{4}$$

This is a circle centered at $(0.5, 0)$ with radius $\frac{\sqrt{3}}{2}$. So finally the equations of different sections of root locus are found as follows

$$\begin{cases} \sigma^2 + (K - 1)\sigma + 0.5K = 0 & (\text{Section on real axis}) \\ (\sigma + 0.5)^2 + \omega^2 = \frac{3}{4} & (\text{Circle centered at } (0.5, 0) \text{ with radius } \sqrt{3}/2) \end{cases}$$

Also root locus can be found using MATLAB

```
>> sys=tf([1 0.5],[1 -1 0],1);
>> sys
```

Transfer function:

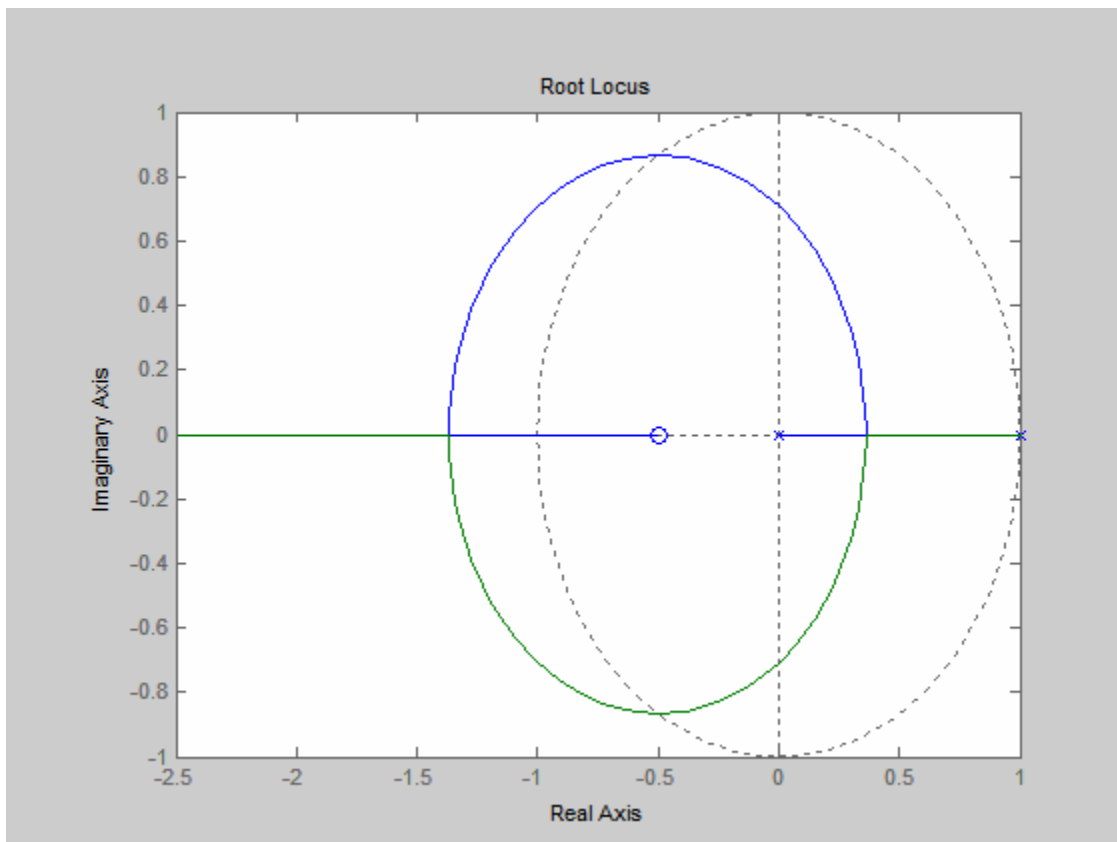
$z + 0.5$

$z^2 - z$

Sampling time: 1

```
>> rlocus(sys)
```

The result is given in the following figure, which shows that our equations are valid.



Part (b)

From characteristic equation we can find the conditions for stability,

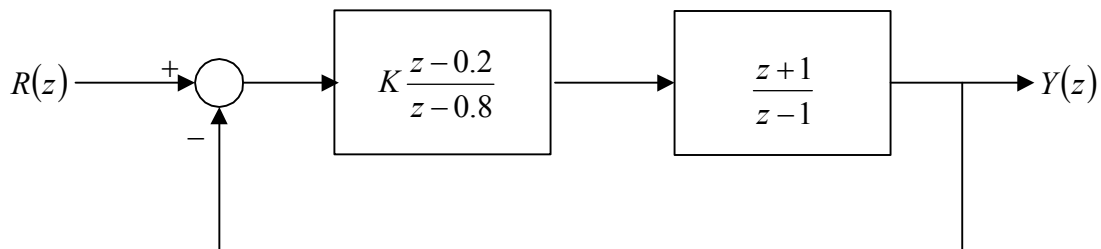
$$q(z) = z(z-1) + K(z+0.5) = z^2 + (K-1)z + 0.5K = 0$$

$$\text{Stability Conditions are } \begin{cases} |q(0)| < 1 \\ q(1) > 0 \\ q(-1) > 0 \end{cases} \Rightarrow \begin{cases} |0.5K| < 1 \\ 1.5K > 0 \\ 2 - 0.5K > 0 \end{cases} \Rightarrow \begin{cases} |K| < 2 \\ K > 0 \\ K < 4 \end{cases} \quad (\text{IV})$$

Combining all three inequalities in (IV) we get $0 < K < 2$.

Problem MP13.5

Consider the feedback system in Fig.MP13.5. Obtain the root locus and determine the range of K for stability.



Solution:

- 1) The root locus lies on a section of real axis to the left of an odd number of poles and zeros. In this example, there are two sections on the real axis, one between $z=0.8$ and $z=1$, and the other part is between $z=0.2$ and $z=-1$.
- 2) Breakaway points

The characteristics equation is $1 + K \left(\frac{z-0.2}{z-0.8} \right) \left(\frac{z+1}{z-1} \right) = 0$, from which K is found as a function of z.

$$K = -\frac{(z-0.8)(z-1)}{(z-0.2)(z+1)}$$

$$\frac{dK}{dz} = \frac{d}{dz} \left(-\frac{(z-0.8)(z-1)}{(z-0.2)(z+1)} \right) = \frac{-2.6z^2 + 2z + 0.28}{(z^2 + 0.8z - 0.2)^2} = 0 \Rightarrow z = \frac{-1 \pm 1.3145}{2} = \begin{cases} -0.121 \\ 0.89 \end{cases}$$

In this problem, we don't intend to find the exact equations of different branches of root locus, instead we look into intersection with imaginary axis.

- 3) Intersection with $j\omega$ -axis. This can be found by applying Routh-Hurwitz criterion and finding the auxiliary equation.

$$q(z) = (z-0.8)(z-1) + K(z-0.2)(z+1) = 0$$

$$q(z) = (1+K)z^2 + (0.8K-1.8)z + 0.8-0.2K = 0$$

Routh table is the following

z^2	$1+K$	$0.8-0.2K$
z^1	$0.8K-1.8$	0
z^0	a	

$$a = \frac{-1}{0.8K-1.8} \begin{vmatrix} 1+K & 0.8-0.2K \\ 0.8K-1.8 & 0 \end{vmatrix} = 0.8-0.2K$$

Auxiliary equation corresponds to the row preceding an all-zero row. If we let second row equal to zero then $0.8K-1.8=0$ and $K=2.25$. Substitute this value in the first row to obtain auxiliary equation,

$$(1+K)z^2 + (0.8-0.2K) = 0 \text{ with } K=2.25 \Rightarrow 3.25z^2 + 0.35 = 0 \Rightarrow z \cong \pm j0.328$$

Using the information obtained in parts (1), (2) and (3), we are able to sketch root locus.

Also in MATLAB we have

```
>> sys1=tf([1 -0.2],[1 -0.8],1);  
>> sys2=tf([1 1],[1 -1],1);  
>> sys1
```

Transfer function:

$$\frac{z - 0.2}{z - 0.8}$$

Sampling time: 1

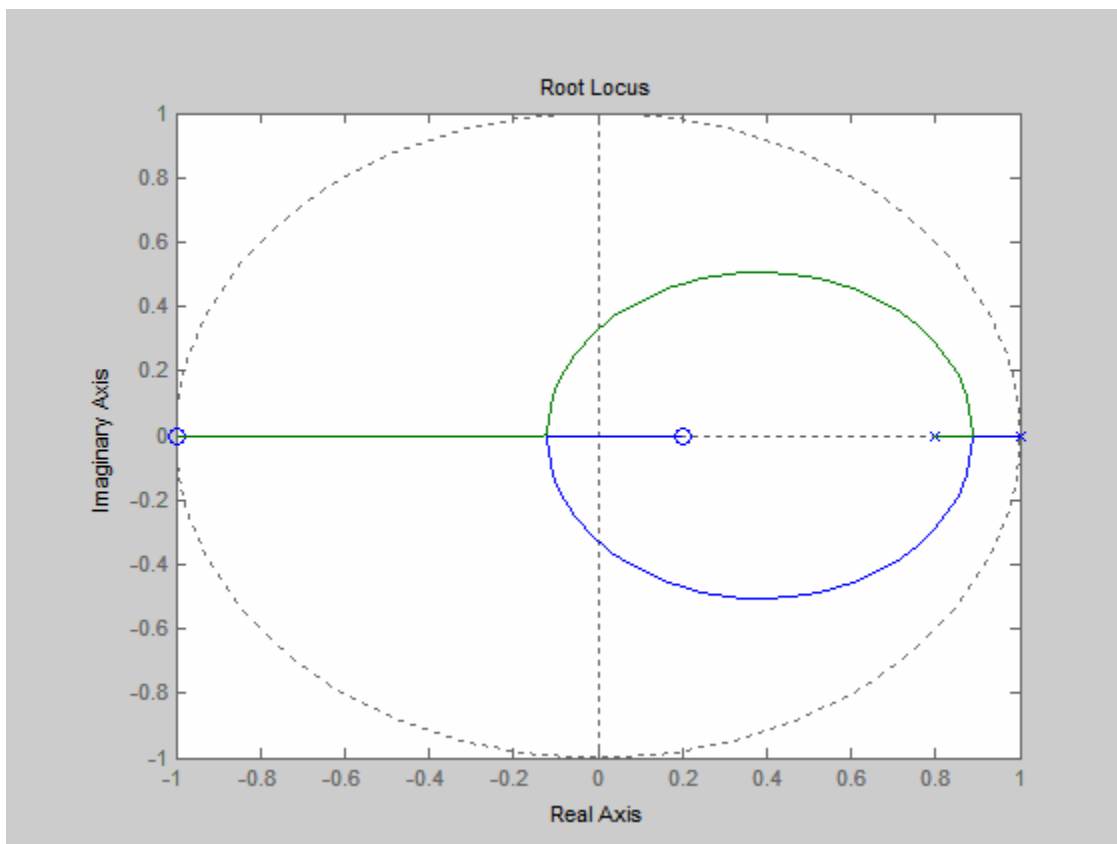
```
>> sys2
```

Transfer function:

$$\frac{z + 1}{z - 1}$$

Sampling time: 1

```
>> sys3=series(sys1,sys2);  
>> rlocus(sys3)
```



From the root locus, it is obvious that for $K > 0$, root locus is inside the unit circle, therefore for $K > 0$ system is stable. Also we can check this fact using characteristics equation,

$$q(z) = (1 + K)z^2 + (0.8K - 1.8)z + 0.8 - 0.2K = 0$$

$$q(z) = z^2 + \left(\frac{0.8K - 1.8}{K + 1}\right)z + \left(\frac{0.8 - 0.2K}{K + 1}\right) = 0$$

Stability conditions are

$$\begin{cases} |q(0)| < 1 \\ q(1) > 0 \\ q(-1) > 0 \end{cases} \Rightarrow \begin{cases} \left| \frac{0.8 - 0.2K}{K + 1} \right| < 1 \\ 1 + \frac{0.8K - 1.8}{K + 1} + \frac{0.8 - 0.2K}{K + 1} > 0 \\ 1 + \frac{-0.8K + 1.8}{K + 1} + \frac{0.8 - 0.2K}{K + 1} > 0 \end{cases} \Rightarrow \begin{cases} \left| \frac{0.8 - 0.2K}{K + 1} \right| < 1 \\ \frac{1.6K}{K + 1} > 0 \\ \frac{2.6}{K + 1} > 0 \end{cases} \Rightarrow$$

$$\begin{cases} \left| \frac{0.8 - 0.2K}{K + 1} \right| < 1 \\ \frac{K}{K + 1} > 0 \\ K + 1 > 0 \end{cases} \Rightarrow \begin{cases} |0.8 - 0.2K| < K + 1 \\ K > 0 \\ K + 1 > 0 \end{cases} \Rightarrow \begin{cases} -0.8K < 1.8 \text{ and } 1.2K > -0.2 \\ K > 0 \\ K + 1 > 0 \end{cases}$$

$\Rightarrow K > 0$ (*Stability Condition*)