Chapter 13: Design of a digital controller

**Method 1**: a. design the controller \( G_c(s) \) in \( s \) domain;

b. convert the \( G_c(s) \) to \( z \) domain \( D(z) \).

**Method 2**: a. convert the given \( s \) domain plant \( G(s) \) into \( z \) domain;

b. design the digital controller in \( z \) domain directly.

This week we explain the first method.

**Step b: convert \( s \) domain \( G_c(s) \) to \( z \) domain \( D(z) \):**

**Section 13.11**, if \( G_c(s) = k_i + \frac{k_2}{s} + k_3s = \frac{X(s)}{U(s)} \), then

\[
U(s) = (k_i + \frac{k_2}{s} + k_3s)X(s),
\]

if \( U(s) = X(s), U(z) = X(z) \);

if \( U(s) = \frac{1}{s}X(s) \), then \( u(t) = \int_0^t x(\tau)d\tau \),

in discrete notation, \( u(k) = u(kT) = u((k-1)T) + T x(kT) \rightarrow U(z) = z^{-1}U(z) + TX(z) \)

i.e. \( U(z) = \frac{T}{1 - z^{-1}} X(z) \)

if \( U(s) = sX(s) \), then \( u(t) = \frac{dx(t)}{dt} \), in discrete notation, \( u(kT) = \frac{x(kT) - x((k-1)T)}{T} \),
\[
U(z) = \frac{X(z) - z^{-1}X(z)}{T} = 1 - z^{-1}X(z)
\]

so, \(D(z) = Z(Gc(s)) = (k_1 + k_2 \frac{Tz}{z-1} + k_3 \frac{z-1}{Tz}X(s)\)

**section 13.8:** if \(Gc(s) = k \frac{s+a}{s+b}\), corresponding \(D(z) = \frac{z-A}{z-B}\), with

\[
A = e^{-aT}, \quad B = e^{-bT}, \quad C = k \frac{a}{b} \frac{1-B}{1-A}
\]

**Step a: design the controller \(Gc(s)\) in \(s\) domain**

\(Gc(s)\) can be designed using the method explained in Chapter 10,11, and 12.

Review the design of a phase lead controller in Chapter 10 and explain the example 13.5 in details.

1. review the concept of phase margin (Chapter 7):

**Definition:** The phase margin is the amount of phase shift of the \(GH(jw)\) at unity magnitude that will result in a marginally stable system with intersection of the \(-1+j0\) point on the Nyquist diagram.

It means phase margin is the phase shift needed to 180 or \(-180\) at the frequency where the magnitude of \(GH(jw)\) is one, i.e. 0 dB, or it is 180 or \(-180\) minus the phase 180 at the frequency where the magnitude of \(GH(jw)\) is 0dB. Using the Bode plot to explain the phase margin is shown in the following figure.

For example: \(GH(s) = \frac{1740}{s(0.25s + 1)}\)
pm = 2.7466 = (-177.3) - 180. Usually, we define phase margin within (-180 180).

The preferred closed-loop phase margin is greater than 45, we need phase compensation about at least 45 - 2 = 43.

Another example from section 9.4

\[ GH(s) = \frac{1}{s(s + 1)(0.2s + 1)} = \frac{1}{0.2s^3 + 1.2s^2 + s} \]
2. review the phase-lead design using phase margin (Section 10.3 and 10.4):

\[ G_c(s) = \frac{k s - a}{s - b}, \text{ i.e., } G_c(j\omega) = \frac{k j\omega + z}{j\omega + p} = k \frac{\frac{j\omega}{z} + 1}{\frac{j\omega}{p} + 1} = k \frac{1 + j\omega \alpha \tau}{\alpha + j\omega \tau}, \text{ where} \]

\[ a = \frac{p}{z}, \quad \tau = \frac{1}{p}. \]

The magnitude is \[ |G_c(j\omega)| = \frac{k}{a} \sqrt{\frac{1 + (\omega \alpha \tau)^2}{1 + (\omega \tau)^2}}, \quad (\text{Eq 1}) \]

The phase is \[ \angle G_c(j\omega) = \tan^{-1}(\omega \alpha \tau) - \tan^{-1}(\omega\tau) \quad (\text{Eq 2}) \]

Or we can write the phase in another formula,

\[ G_c(j\omega) = k \frac{1 + j\omega \alpha \tau}{\alpha + j\omega \tau} = \frac{k (1 + j\omega \alpha \tau)(1 - j\omega \tau)}{\alpha (1 - (\omega \tau)^2)} = \frac{k 1 + \omega^2 \alpha^2 \tau^2 + j(\omega \alpha \tau - \omega \tau)}{a 1 - (\omega \tau)^2} \]
The phase is \( \angle G_c(j\omega) = \tan^{-1} \frac{\omega \alpha \tau - \omega \tau}{1 + \omega^2 \alpha^2 \tau^2} \) \hspace{1cm} (Eq 3)

The bode diagram of the general phase-lead compensator is shown in textbook figure 10.3, or reference to the figure in the last page of this notes. The maximum value of the phase lead occurs at a frequency \( \omega_m \), and \( \omega_m = \sqrt{z p} = \frac{1}{\tau \sqrt{\alpha}} \).

It should also be noted that at \( \omega_m \), the \( G_c(s) \) add additional gain about

\[
20\log_{10} \alpha / 2 = 10\log_{10} \alpha .
\]

At frequency \( \omega_m \), the corresponding maximum phase is

\[
\Phi_m = \angle G_c(j\omega) \bigg|_{\omega = \omega_m} = \tan^{-1} \frac{\omega \alpha \tau - \omega \tau}{1 + \omega^2 \alpha^2 \tau^2} = \frac{\alpha - 1}{2\sqrt{\alpha}} , \text{ or it is equivalent to } \sin \Phi m = \frac{\alpha - 1}{\alpha + 1} \) \hspace{1cm} (Eq 4).

So if we can derive the requested phase compensation \( \Phi_m \), then the can be calculated from (Eq 4).

Design step is expressed in Section 10.4:

1. plot the Bode diagram of the uncompensated G(s), find the phase margin of G(s);
2. calculate the maximum phase lead \( \Phi_m \),
3. using Eq 4 calculated \( \alpha \);
4. calculate \( 10\log_{10} \alpha \) and determine the frequency where the uncompensated magnitude curve is equal to \( -10\log_{10} \alpha \). Because the compensation network provide a gain of \( 10\log_{10} \alpha \) at \( \omega_m \). This frequency is the new 0 dB crossover frequency and \( \omega_m \) simultaneously.
5. using \( \omega_m \) and \( \alpha \) to calculate \( z \) and \( p \) using \( a = \frac{p}{z} \), \( \tau = \frac{1}{p} \). \( \omega_m = \sqrt{z p} \), i.e.

\[
p = \omega_m \sqrt{\alpha} \text{ and } z = p / a .
\]
6. $k$ is calculated to yield $|GcG| = 1$

Example 13.5: $GH(s) = \frac{1740}{s(0.25s + 1)}$

As shown previously, the phase margin of the uncompensated system is 2.7, to achieve the closed-loop phase margin greater than 45, we need additional phase lead $\Phi_m > 45 - 2.7$.

![Graph showing phase and magnitude plots with additional gain and phase compensation](image)

...the preferred closed-loop phase margin is greater than 45, we need phase compensation about at least $45 - 2 = 43$. But considering that $Gc(s) = \frac{s - a}{s - b}$ adds additional gain about $10\log_{10} \alpha$ to the system, to be safely, we need larger $\Phi_m$ as illustrated in above figure. So we chose $\alpha = 46$.

$\alpha = \frac{1 + \sin \Phi_m}{1 - \sin \Phi_m} = 6.25$. At the end, we need verify the performance, if it does not achieve required performance, we need to pick a larger $\Phi_m$. 


$10 \log_{10} \alpha = 7.9588 \text{ dB}$

$|G_p(j \omega)| = \frac{1740}{\sqrt{(0.25 \omega^2)^2 + \omega^2}} = 10^{-8/20} = 0.4$, then we get $\omega = 125$

so $z = 50$, $p = 312$.

$G_c(s) = k \frac{s + 50}{s + 312}$,

$G_c G_p(s) = k \frac{s + 50}{s + 312} \frac{1740}{s(0.25s + 1)}$,

magnitude is $|k \frac{j \omega + 50}{j \omega + 312} \frac{1740}{-0.25 \omega^2 + j \omega}| = k \frac{\sqrt{\omega^2 + 2500}}{\sqrt{\omega^2 + 312^2}} \frac{1740}{\sqrt{(0.25 \omega^2)^2 + \omega^2}} = 1$

i.e. $k \frac{\sqrt{\omega^2 + 2500}}{\sqrt{\omega^2 + 312^2}} = 0.4$, $k = \frac{1}{2.5 \cdot 0.44} = 5.67$