Chapter 3: State variable models

1. definition of state variable models;
2. how to calculate the output from the state variable model;
3. from transfer function to state variable model;
4. from state variable representation to transfer function;

1. state variable model

In last DGD, we reviewed the Laplace transform and transfer function models in Chapter 2, how ato obtain the transfer function of a system from the differential equations, and also some basic concepts, like characteristic equation, damping ratio, natural frequency.

In chapter 3, we will learn an alternative method of system modeling using time-domain method, called state variable modeling.

Scheme:

1) describe the physical systems by a set of nth order ordinary differential equations;
2) using a set of variables (may be non-unique), called state variables, we can obtain a set of first order differential equations. These first order equations are grouped using a compact matrix notation, known as the state variable model;

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

3) the system response and analysis can be computed directly from the state variable model.

2. how to calculate the output from the state variable model:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

Laplace transform: $sX(s) - x(0) = aX(s) + bU(s)$

$$X(s) = \frac{x(0)}{s-a} + \frac{b}{s-a}U(s),$$

then

$$x(t) = e^{at}x(0) + \int_{0^-}^{\infty} e^{a(t-\tau)} bu(\tau)d\tau$$

in general,

$$\begin{cases} X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \exp(At)\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \int_0^{\infty} \exp(A(t-\tau))Bu(\tau)d\tau \\ y(t) = C\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + Du(t). \end{cases}$$

$X(s) = \begin{bmatrix} x_1(s) \\ x_2(s) \\ \vdots \\ x_n(s) \end{bmatrix} = [sI - A]^{-1}x(0) + [sI - A]^{-1}BU(s)$, we define $\Phi(s) = [sI - A]^{-1}$, called state transition matrix. $\phi(t) = L^{-1}(\Phi(s))$

$$x(t) = \phi(t)x(0) + \int_0^t \phi(t-\tau)bu(\tau)d\tau$$

output $y(t) = Cx(t)$

example 1:

$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt}ky(t) = u(t)$$

here the input is $r(t)$, output is $y(t)$.

Possible state variables: in most of case they are non-unique

Principle: the number of state variable chosen to represent the system should be as small as possible.

Method: start from the variable, then first order derivative, second order, …, n-1th order.

Set $x_1(t) = y(t)$

$$x_2(t) = \frac{dy(t)}{dt}$$

then: $M\dot{x}_2(t) + bx_2(t) + kx_1(t) = u(t)$

$$\dot{x}_1(t) = x_2(t)$$

$$\begin{cases} \dot{x} = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{M} & -\dfrac{b}{M} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \dfrac{1}{M} u(t) \\ \\ y(t) = x1(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + [0]u(t) \end{cases}$$

example 2:

DP11.4 A high performance helicopter has a model as

$$\frac{d^2\theta}{dt^2} = -\sigma_1 \frac{d\theta}{dt} - \alpha_1 \frac{dx}{dt} + n\delta$$

$$\frac{d^2x}{dt^2} = g\theta - \alpha_2 \frac{d\theta}{dt} - \sigma_2 \frac{dx}{dt} + g\delta \ .$$

The goal is to control the pitch angle θ by adjust the rotor angle δ, where *x* is the translation in the horizontal direction.

Given: σ_1 =0.415 α_1 =1.43 n=6.27

σ_2 =0.0198 α_2 =0.0111 g=9.8

the state variable representation of the system;

As stated in the problem, the input of the system is δ, the output of the system is θ. To be clear in notation, we use *z* instead *x* for the translation in the horizontal direction. Then the given system is $\frac{d^2\theta}{dt^2} = -\sigma_1 \frac{d\theta}{dt} - \alpha_1 \frac{dz}{dt} + n\delta$

$$\frac{d^2 z}{dt^2} = g\theta - \alpha_2 \frac{d\theta}{dt} - \sigma_2 \frac{dz}{dt} + g\delta$$

a) Set the state variable as $\mathrm{x} = \begin{bmatrix} \theta \\ \dot{\theta} \\ \dot{z} \end{bmatrix}$, $\dot{\mathrm{x}} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \ddot{z} \end{bmatrix}$, $y = \theta = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathrm{x}$.

$$\begin{cases} \dot{\theta} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\mathrm{x} \\ \ddot{\theta} = -\sigma_1\dot{\theta} - \alpha_1\dot{z} + n\delta = \begin{bmatrix} 0 & -\sigma_1 & -\alpha_1 \end{bmatrix}\mathrm{x} + \mathrm{n}\delta; \\ \ddot{z} = g\theta - \alpha_2\dot{\theta} - \sigma_2\dot{z} + g\delta = \begin{bmatrix} g & -\alpha_2 & -\sigma_2 \end{bmatrix}\mathrm{x} + \mathrm{g}\delta \end{cases}$$

so the state space representation is

$$\begin{cases} \dot{\mathrm{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\sigma_1 & -\alpha_1 \\ g & -\alpha_2 & -\sigma_2 \end{bmatrix}\mathrm{x} + \begin{bmatrix} 0 \\ n \\ g \end{bmatrix}\delta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.415 & -0.0111 \\ 9.8 & -1.43 & -0.0198 \end{bmatrix}\mathrm{x} + \begin{bmatrix} 0 \\ 6.27 \\ 9.8 \end{bmatrix}\delta \\ y = \theta = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathrm{x} \end{cases}$$

3. from transfer function to state variable representation:
using the signal flow graph

4. from state variable model to transfer function:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

$$G(s)=\frac{Y(s)}{U(s)}=C_{1\times n}\cdot\Phi(s)_{n\times n}\cdot B_{n\times 1}$$

example 3:

b) Find the transfer function of the system in example 2:

From Chapter 3 Eq 3.71, we knew that transfer function of a system is

$$G(s)=\frac{Y(s)}{X(s)}=C[sI-A]^{-1}B.$$

$$[sI-A]^{-1}=\begin{bmatrix} s & -1 & 0 \\ 0 & s+\sigma_1 & \alpha_1 \\ -g & \alpha_2 & s+\sigma_2 \end{bmatrix}^{-1}$$

$$=\frac{\begin{bmatrix} (s+\sigma_1)(s+\sigma_2)-\alpha_1\alpha_2 & -\alpha_1 g & (s+\sigma_1)g \\ s+\sigma_2 & s(s+\sigma_2) & -(s\alpha_2-g) \\ -\alpha_1 & 0 & s(s+\sigma_1) \end{bmatrix}^T}{\Delta}$$

$$G(s)=C[sI-A]^{-1}B=\begin{bmatrix}1 & 0 & 0\end{bmatrix}[sI-A]^{-1}\begin{bmatrix}0 \\ n \\ g\end{bmatrix}=\frac{(s+\sigma_2)n-\alpha_1 g}{\Delta}$$, where

$$\Delta=\det[sI-A]=s(s+\sigma_1)(s+\sigma_2)+g\alpha_1-\alpha_1\alpha_2 s=s^3+(\sigma_1+\sigma_2)s^2+(\sigma_1\sigma_2-a_1a_2)s+a_1g$$

i.e. $G(s)=\dfrac{6.27s+0.0154}{s^3+0.435s^2-0.0077s+0.109}$