Assignment 1 solution: about Chapter 11, State variable model

Solution:

E11.5

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

$$y = x_1(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

the controllability matrix is $Pc = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & -1+2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$,

$\det(Pc) = 1 \cdot 1 - (-1) \cdot (-1) = 0$, the system is uncontrollable.

The observability matrix is $Q = \begin{bmatrix} C \\ CA \end{bmatrix}. = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\det(Q) = 1 \neq 0$. the system is observable.

E11.6

the controllability matrix is $Pc = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$, $\det(Pc) = -1 \neq 0$, the system is controllable.

The observability matrix is $Q = \begin{bmatrix} C \\ CA \end{bmatrix}. = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\det(Q) = 1 \neq 0$. the system is observable.

P11.12:

A dc motor has a transfer function $G(s) = \dfrac{10}{s^2(s+1)(s^2+2s+2)}$. Determine whether this system is controllable and observable.

Solution:

$$G(s) = \frac{10}{s^2(s+1)(s^2+2s+2)} = \frac{10}{s^5+3s^4+4s^3+2s^2} = \frac{10s^{-5}}{1+3s^{-1}+4s^{-2}+2s^{-3}}$$

the state variable model of the system is $\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -4 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$

$$y = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \end{bmatrix} x$$

the controllable matrix is $Pc = \begin{bmatrix} B & AB & A^2B & A^3B & A^4B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 1 & -3 & 5 & -5 \\ 1 & -3 & 5 & -5 & 1 \end{bmatrix}$,

$\det(Pc) = -1 \neq 0$

the system is controllable;

the observability matrix is $Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \end{bmatrix} . = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix}, \det(Q) \neq 0$

the system is observable

P11.13:

A feedback system has a plant transfer function $\frac{Y(s)}{R(s)} = G(s) = \frac{K}{s(s+70)}$. It is desired that the velocity error constant K_v be 35 and the overshoot to a step input be approximately 4% so that ξ is $\frac{1}{\sqrt{2}}$. The setting time (2% criterion) desired is 0.11 second. Design an appropriate state variable feedback system.

Solution:

$$G(s) = \frac{K}{s(s+70)} = \frac{K}{s^2 + 70s}$$

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 0 & -70 \end{bmatrix} x + \begin{bmatrix} 0 \\ K \end{bmatrix} u$$

let $u = -Kx = \begin{bmatrix} -k_1 & -k_2 \end{bmatrix} x$,

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 0 & -70 \end{bmatrix} x + \begin{bmatrix} 0 \\ K \end{bmatrix} \begin{bmatrix} -k_1 & -k_2 \end{bmatrix} x = \begin{bmatrix} 0 & 1 \\ -Kk_1 & -70 - Kk_2 \end{bmatrix} x$$

given velocity error constant K_v be 35. According to chapter 5.7,

$$Kv = \lim_{s \to 0} sG(s) = \lim_{s \to 0} s \cdot \frac{K}{s(s+70)} = \frac{K}{70} = 35, \quad K = 2450.$$

$Ts = 4\tau = \dfrac{4}{\xi\omega_n} = 0.11,\ \ P.O. = 100e^{-\xi\pi/\sqrt{1-\xi^2}}$ =4, given ξ is $\dfrac{1}{\sqrt{2}}$, then $\omega_n = 51.42$.

the characteristic equation is

$$\Delta = \det[sI - H] = s^2 + 2\xi\omega_n s + \omega_n^2 = s2 + 72.73s + 2644.63 = \det\begin{bmatrix} s & -1 \\ Kk_1 & s + 70 + Kk_2 \end{bmatrix}$$

$= s(s + 70 + Kk_2) + Kk_1 = s2 + (70 + Kk_2)s + Kk_1$,

so $k_1 = \dfrac{2644.63}{2450} = 1.08$, $k_2 = \dfrac{72.73 - 70}{2450} = 0.0011$.

AP11.1

The closed-loop transfer function of the feedback system is

$T(s) = \dfrac{4K}{s^3 + (2kk_3 + 5)s^2 + (4Kk_2 + 2Kk_3 + 4)s + 4Kk_1}$ select the feedback gain to keep the steady-state error equal to zero, so we need

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{s}(1 - T(s)) = \lim_{s\to 0}(1 - T(s)) = 1 - \frac{4K}{4Kk_1} = 0$$

so $k_1 = 1$.

$$T(s) = \frac{4K}{s^3 + (2Kk_3 + 5)s^2 + (4Kk_2 + 2Kk_3 + 4)s + 4K} = \frac{4K}{(s + \frac{1}{\tau})(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

$$= \frac{4K}{s^3 + (2\xi\omega_n + \frac{1}{\tau})s^2 + (2\xi\omega_n\frac{1}{\tau} + \omega_n^2)s + \omega_n^2\frac{1}{\tau}}$$

To keep the overshoot rate P.O less than 3%, P.O=$100e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$ <3, i.e. $\zeta > 0.7448$, pick $\zeta = 0.75$,

Assume $Ts = \dfrac{4}{\zeta\omega_n} < 1.5$, choose $\omega_n > 5$. $1/\tau > 10\xi\omega_n = 37.5$, Choose $1/\tau = 40$

then $(s + \frac{1}{\tau})(s^2 + 2\xi\omega_n s + \omega_n^2) = (s + 40)(s^2 + 7.5s + 25) = s^3 + 47.5s^2 + 325s + 1000$

$= s^3 + (2Kk_3 + 5)s^2 + (4Kk_2 + 2Kk_3 + 4)s + 4K$

so K=250, k_3=0.085, k_2=0.2785

AP11.2

A system has a plant $G(s) = \dfrac{3s^2 + 4s - 2}{s^3 + 3s^2 + 7s + 5}$. Add state variable feedback so that the closed-loop poles are s=-4,-4, and –5.

1. convert the transfer function to state space representation.

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -7 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = Cx = \begin{bmatrix} -2 & 4 & 3 \end{bmatrix} x$$

2. add the state variable feedback $u = -Kx = \begin{bmatrix} -k_1 & -k_2 & -k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\dot{x} = Ax + Bu = Ax + B(-K)x = (A - BK)x = Hx = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5-k_1 & -7-k_2 & -3-k_3 \end{bmatrix} x$$

3. given the closed-loop poles are –4, -4, -5. It means the characteristic equation of the closed loop transfer function is (s+4)(s+4)(s+5)= $s^3 + 13s^2 + 56s + 80$ and the characteristic equation= $\Delta = \det[sI - H] = \det \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 5+k_1 & 7+k_2 & s+3+k_3 \end{bmatrix}$,

$= s^2(s + 3 + k_3) + (5 + k_1) - s(-1)(7 + k_2) = s^3 + (3 + k_3)s^2 + (7 + k_2)s + (5 + k_1)$. So

$3 + k_3 = 13$,

$7 + k_2 = 56$,

$5 + k_1 = 80$

Then we get *K*= $\begin{bmatrix} 10 & 49 & 75 \end{bmatrix}$.

AP11.3:

A system has a matrix differential equation $\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$.

What values for b1 and b2 are required so that the system is controllable?

To be controllable, the determinant of the controllability matrix Pc should not be zero.

$$Pc = [B \quad AB] = \begin{bmatrix} b_1 & b_1 \\ b_2 & 2b_2 \end{bmatrix}, \quad \det(Pc) = f(b_1, b_2) = 2b_1b_2 - b_1b_2 = b_1b_2 \neq 0$$

so the system is controllable when $b_1b_2 \neq 0, i.e. b_1 \neq 0 \ and \ b_2 \neq 0$

AP11.5:

A system and it state variable feedback is define in figure AP11.5. determine the parameters K2, k3 to keep the poles of the closed-loop system between –3 and –6, also select Kp so that the steady-state error for a step input is equal to zero.

Hints:

1. find the closed-loop transfer function from block diagram directly, which includes the feedback parameters;
2. according to the steady error from chapter 4.5, steady error $e_{ss} = \lim_{s\to 0} sE(s)$.
3. to keep the roots of the characteristic equation are three real root and lying between –3 to –6, the characteristic equation=(s-s1)(s-s2)(s-s3), then we can choose the appropriate k2,k3.

Solution:

From the block diagram, the closed-loop transfer function is

$$T(s) = \frac{2Kp}{s^3 + (9 + 2k_3)s^2 + (26 + 2K_2 + 10K_3)s + (26 + 6K_2 + 12K_3)}$$ with given k1=1.

the roots of the characteristic equation is real and within (-6 -3).

The standard format of the third order characteristic equation is $\Delta = (s + \tau)(s^2 + 2\xi\omega_n s + \omega_n^2)$.

To have real roots, $\xi = 1$, $\Delta = (s + \frac{1}{\tau})(s + \omega_n)^2 = (s - s_1)(s^2 - 2s_2 s + s_2^2) =$

$$s^3 - (s_1 + 2s_2)s^2 + (2s_1s_2 + s_2^2)s - s_1 s_2^2$$

Pick the roots $s_{1,2,3} = -5, -4, -4$

Then $K_3 = 2, \ K_2 = 5$

So $e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{s}(1 - T(s)) = \lim_{s\to 0}(1 - T(s)) = 1 - \frac{2Kp}{26 + 6K_2 + 12K_3} = 0$

$$Kp = 13 + 3K_2 + 6K_3 = 40$$

DP11.4:

A high performance helicopter has a model as

$$\frac{d^2\theta}{dt^2} = -\sigma_1 \frac{d\theta}{dt} - \alpha_1 \frac{dx}{dt} + n\delta$$

$$\frac{d^2x}{dt^2} = g\theta - \alpha_2 \frac{d\theta}{dt} - \sigma_2 \frac{dx}{dt} + g\delta .$$

The goal is to control the pitch angle θ by adjust the rotor angle δ, where *x* is the translation in the horizontal direction.

Given: σ_1 =0.415 α_1 =1.43 n=6.27 σ_2 =0.0198 α_2 =0.0111 g=9.8

As stated in the problem, the input of the system is δ, the output of the system is θ. To be clear in notation, we use *z* instead *x* for the translation in the horizontal direction. Then the given system is

$$\frac{d^2\theta}{dt^2} = -\sigma_1 \frac{d\theta}{dt} - \alpha_1 \frac{dz}{dt} + n\delta \qquad \frac{d^2z}{dt^2} = g\theta - \alpha_2 \frac{d\theta}{dt} - \sigma_2 \frac{dz}{dt} + g\delta$$

a) Set the state variable as $\mathrm{x} = \begin{bmatrix} \theta \\ \dot{\theta} \\ \dot{z} \end{bmatrix}$, $\dot{\mathrm{x}} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \ddot{z} \end{bmatrix}$, $y = \theta = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathrm{x}$.

Try to express the elements in $\dot{\mathrm{x}}$, $(\dot{\theta}, \ddot{\theta}, \ddot{z})$ by x, i.e. $(\theta, \dot{\theta}, \dot{z})$.

$$\begin{cases} \dot{\theta} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}\mathrm{x} \\ \ddot{\theta} = -\sigma_1\dot{\theta} - \alpha_1\dot{z} + n\delta = \begin{bmatrix} 0 & -\sigma_1 & -\alpha_1 \end{bmatrix}\mathrm{x} + \mathrm{n}\delta ; \\ \ddot{z} = g\theta - \alpha_2\dot{\theta} - \sigma_2\dot{z} + g\delta = \begin{bmatrix} g & -\alpha_2 & -\sigma_2 \end{bmatrix}\mathrm{x} + \mathrm{g}\delta \end{cases}$$

so the state space representation is

$$\begin{cases} \dot{\mathrm{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\sigma_1 & -\alpha_1 \\ g & -\alpha_2 & -\sigma_2 \end{bmatrix}\mathrm{x} + \begin{bmatrix} 0 \\ n \\ g \end{bmatrix}\delta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.415 & -0.0111 \\ 9.8 & -1.43 & -0.0198 \end{bmatrix}\mathrm{x} + \begin{bmatrix} 0 \\ 6.27 \\ 9.8 \end{bmatrix}\delta \\ y = \theta = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathrm{x} \end{cases}$$

b) From Chapter 3 Eq 3.71, we knew that transfer function of a system is

$$G(s) = \frac{Y(s)}{X(s)} = C[sI - A]^{-1}B .$$

$$[sI-A]^{-1} = \begin{bmatrix} s & -1 & 0 \\ 0 & s+\sigma_1 & \alpha_1 \\ -g & \alpha_2 & s+\sigma_2 \end{bmatrix}^{-1}$$

$$= \frac{\begin{bmatrix} (s+\sigma_1)(s+\sigma_2)-\alpha_1\alpha_2 & -\alpha_1 g & (s+\sigma_1)g \\ s+\sigma_2 & s(s+\sigma_2) & -(s\alpha_2-g) \\ -\alpha_1 & -s\alpha_1 & s(s+\sigma_1) \end{bmatrix}^T}{\Delta}$$

$$G(s) = C[sI-A]^{-1}B = \begin{bmatrix}1 & 0 & 0\end{bmatrix}[sI-A]^{-1}\begin{bmatrix}0 \\ n \\ g\end{bmatrix} = \frac{(s+\sigma_2)n-\alpha_1 g}{\Delta} = \frac{ns+(\sigma_2 n-\alpha g)}{\Delta}$$

, where

$$\Delta = \det[sI-A] = s(s+\sigma_1)(s+\sigma_2)+g\alpha_1-\alpha_1\alpha_2 s = s^3+(\sigma_1+\sigma_2)s^2+(\sigma_1\sigma_2-a_1a_2)s+a_1g$$

i.e. $G(s) = \dfrac{6.27s+0.0154}{s^3+0.435s^2-0.0077s+0.109}$

b) design the state variable feedback.

Let the feedback as $\delta = \begin{bmatrix}-k_1 & -k_2 & -k_3\end{bmatrix}\mathrm{x}$, then the closed-loop system has

$$\dot{\mathrm{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\sigma_1 & -\alpha_1 \\ g & -\alpha_2 & -\sigma_2 \end{bmatrix}\mathrm{x} + \begin{bmatrix}0 \\ n \\ g\end{bmatrix}\delta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\sigma_1 & -\alpha_1 \\ g & -\alpha_2 & -\sigma_2 \end{bmatrix}\mathrm{x} + \begin{bmatrix}0 \\ n \\ g\end{bmatrix}\begin{bmatrix}-k_1 & -k_2 & -k_3\end{bmatrix}\mathrm{x} + \begin{bmatrix}0 \\ n \\ g\end{bmatrix}r$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -nk_1 & -\sigma_1-nk_2 & -\alpha_1-nk_3 \\ g-gk_1 & -\alpha_2-gk_2 & -\sigma_2-gk_3 \end{bmatrix}\mathrm{x} + \begin{bmatrix}0 \\ n \\ g\end{bmatrix}r$$

$$H = \begin{bmatrix} 0 & 1 & 0 \\ -nk_1 & -\sigma_1-nk_2 & -\alpha_1-nk_3 \\ g-gk_1 & -\alpha_2-gk_2 & -\sigma_2-gk_3 \end{bmatrix}$$

$$\det(sI-H) = \det(\begin{bmatrix} s & -1 & 0 \\ nk_1 & s+\sigma_1+nk_2 & \alpha_1+nk_3 \\ -g+gk_1 & \alpha_2+gk_2 & s+\sigma_2+gk_3 \end{bmatrix})$$

$$= s(s+\sigma_1+nk_2)(s+\sigma_2+gk_3)-(a_1+nk_3)(gk_1-g)+nk_1(s+\sigma_2+gk_3)(gk_2+a_2)$$

$$= s^3+(gk_3+nk_2+\sigma_1+\sigma_2)s^2+(\sigma_1\sigma_2+n\sigma_2k_2+g\sigma_1k_3+nk_1-a_1gk_2-a_1a_2)s+$$

$$a_1g-a_1gk_1+gnk_3+\sigma_2nk_1 \qquad \text{(Eq 1)}$$

This is a third order system, the general form of the characteristic equation is

$$(s^2+2\zeta\omega_n s+\omega_n{}^2)(\gamma s+1)$$

To achieve the design specification, we need *Ts*<1.5 P.O<20, i.e.

Ts= $\frac{4}{\zeta\omega_\mathrm{n}}<1.5$, i.e. $\zeta\omega_n>4/1.5=2.6667$, pick $\zeta\omega_n=3$;

P.O= $100e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$ <20, i.e. $\zeta>0.4558$, pick $\zeta=0.46$,

then $\omega_n>6.52$, pick $\omega_n=6.7$.

According to the explanation in section 5.4, the real part of the third root should greater than 10 times of the real part of the dominant roots (the roots of the second order system).

So $|\frac{1}{\gamma}|\geq 10\,|\zeta\omega_n|=30$

Then $(s^2+2\zeta\omega_n s+\omega_n{}^2)(s+1/\gamma)=(s^2+6s+45)(s+30)=s^3+36s^2+225s+1350$

Comparing coefficients in the characteristic equation Eq (1), yields

$$\begin{bmatrix} 0 & n & g \\ n & n\sigma_2-a_1g & -a_2n+g\sigma_1 \\ -a_1g+\sigma_2n & 0 & gn \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}=\begin{bmatrix} 36-\sigma_1-\sigma_2 \\ 225+a_1+a_2 \\ 1350-a_1g \end{bmatrix}$$

so K=[53.1116 -28.6441 21.9555].

(the results may be a little bit different because different ζ and ω_n were picked up. It is OK as long as the performance requirements are meet.)