ELG4139: Op Amp-based Active Filters

• **Advantages:**
  – Reduced size and weight, and therefore parasitics.
  – Increased reliability and improved performance.
  – Simpler design than for passive filters and can realize a wider range of functions as well as providing voltage gain.
  – In large quantities, the cost of an IC is less than its passive counterpart.

• **Disadvantages:**
  – Limited bandwidth of active devices limits the highest attainable pole frequency and therefore applications above 100 kHz (passive RLC filters can be used up to 500 MHz).
  – The achievable quality factor is also limited.
  – Require power supplies (unlike passive filters).
  – Increased sensitivity to variations in circuit parameters caused by environmental changes compared to passive filters.

• For applications, particularly in voice and data communications, the economic and performance advantages of active RC filters far outweigh their disadvantages.
First-Order Low-Pass Filter

\[ H(f) = \frac{V_o}{V_i} = -\frac{Z_f}{Z_i} \]

\[ \frac{1}{Z_f} = \frac{1}{R_f} + \frac{1}{j2\pi f C_f} \]

\[ Z_f = \frac{R_f}{1 + j2\pi f R_f C_f} \]

\[ H(f) = -\frac{Z_f}{Z_i} = -\left(\frac{R_f}{R_i}\right) \frac{1}{1 + j2\pi f R_f C_f} \]

\[ f_B = \frac{1}{2\pi R_f C_f} \]

A low-pass filter with a dc gain of \(-R_f/R_i\)
$v_o(t) = -\frac{1}{RC} \int_0^t v_{in}(t) \, dt$
First-Order High-Pass Filter

\[ H(f) = \frac{v_o}{v_i} = -\frac{Z_f}{Z_i} \]

\[ Z_i = R_i + \frac{1}{j2\pi f C_i} \quad Z_f = R_f \]

\[ H(f) = -\frac{Z_f}{Z_i} = -\frac{R_f}{R_i + \frac{1}{j2\pi f C_i}} \]

\[ = -\frac{j2\pi f R_f C_i}{1 + j2\pi f R_i C_i} = -\left(\frac{R_i}{R_f}\right)\frac{j2\pi f R_f C_i}{1 + j2\pi f R_i C_i} \]

\[ = -\left(\frac{R_f}{R_i}\right)\left[\frac{j(f / f_B)}{1 + j(f / f_B)}\right] \]

\[ f_B = \frac{1}{2\pi R_i C_i} \]

A high-pass filter with a high frequency gain of \(-R_f/R_i\)
Higher Order Filters

\[ H(f) = H_1(f)H_2(f)\ldots H_n(f) \]

\[ = (-1)^n \left( \frac{R_f}{R_i} \right)^n \left[ \frac{1}{1 + j(f / f_B)} \right]^n \]
Single-Pole Active Filter Designs

**High Pass**

\[ \frac{v_{out}}{v_{in}} = \frac{1}{sCR} + 1 = \frac{1}{1 + sRC} \]

\[ = \frac{sRC}{RC(s + 1/RC)} = \frac{s}{s + 1/RC} \]

**Low Pass**

\[ \frac{v_{out}}{v_{in}} = \frac{1}{s + 1/RC} \]
Two-Pole (Sallen-Key) Filters

Low Pass Filter

High Pass Filter
Three-Pole Low-Pass Filter
Two-Stage Band-Pass Filter

Stage 1
Two-pole low-pass

Stage 2
Two-pole high-pass

\[ BW = f_2 - f_1 \]
\[ Q = \frac{f_0}{BW} \]
Multiple-Feedback Band-Pass Filter

\[ v_{\text{in}} \rightarrow R_1 \rightarrow C_1 \rightarrow - \rightarrow + \rightarrow - \rightarrow R_f \rightarrow C_2 \rightarrow +V \rightarrow -V \rightarrow V_{\text{out}} \]
Transfer function $H(j\omega)$

$$H(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)}$$

$$H = \text{Re}(H) + j\text{Im}(H)$$

$$|H| = \sqrt{\text{Re}(H)^2 + \text{Im}(H)^2}$$

$$\angle H = \tan^{-1}\left(\frac{\text{Im}(H)}{\text{Re}(H)}\right)$$

$$\angle H = 180^\circ + \tan^{-1}\left(\frac{\text{Im}(H)}{\text{Re}(H)}\right)$$

For $\text{Re}(H) > 0$

For $\text{Re}(H) < 0$
Frequency Transfer Function of Filters

\[ H(j\omega) \]

(I) Low - Pass Filter

\[ |H(j\omega)| = 1 \quad f < f_o \]
\[ |H(j\omega)| = 0 \quad f > f_o \]

(II) High - Pass Filter

\[ |H(j\omega)| = 0 \quad f < f_o \]
\[ |H(j\omega)| = 1 \quad f > f_o \]

(III) Band - Pass Filter

\[ |H(j\omega)| = 1 \quad f_L < f < f_H \]
\[ |H(j\omega)| = 0 \quad f < f_L \text{ and } f > f_H \]

(IV) Band - Stop (Notch) Filter

\[ |H(j\omega)| = 0 \quad f_L < f < f_H \]
\[ |H(j\omega)| = 1 \quad f < f_L \text{ and } f > f_H \]

(V) All - Pass (or phase - shift) Filter

\[ |H(j\omega)| = 1 \quad \text{for all } f \]

has a specific phase response
Bode Plot

To understand Bode plots, you need to use Laplace transforms!

![Bode Plot Diagram]

The transfer function of the circuit is:

\[
A_v = \frac{V_o(s)}{V_in(s)} = \frac{1/sC}{R + 1/sC} = \frac{1}{sRC + 1}
\]

\[
A_v(f) = \frac{1}{j\omega RC + 1} = \frac{1}{1 + j2\pi RCf} = \frac{1}{1 + j\left(\frac{f}{f_b}\right)}
\]

where \(f_c\) is called the break frequency, or corner frequency, and is given by:

\[
f_c = \frac{1}{2\pi RC}
\]
Bode Plot (Single Pole)

\[ H(j\omega) = \frac{1}{1 + j\omega CR} = \frac{1}{1 + j\left(\frac{\omega}{\omega_o}\right)} \]

\[ |H(j\omega)| = \sqrt{1 + \left(\frac{\omega}{\omega_o}\right)^2} \]

\[ |H(j\omega)|_{dB} = 20\log_{10}|H(j\omega)| = 20\log_{10}\left(\frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_o}\right)^2}}\right) \]

For \( \omega >> \omega_o \)

\[ |H(j\omega)|_{dB} \approx -20\log_{10}\left(\frac{\omega}{\omega_o}\right) \]
\[ |H(j\omega)| \approx -20 \log_{10} \left( \frac{\omega}{\omega_o} \right) \]

For octave apart,
\[ \frac{\omega}{\omega_o} = \frac{2}{1} \quad |H(j\omega)| \approx -6dB \]

For decade apart,
\[ \frac{\omega}{\omega_o} = \frac{10}{1} \quad |H(j\omega)| \approx -20dB \]

slope
-6dB/octave
-20dB/decade
Bode Plot (Two-Pole)

\[ |H(j\omega)| = 20\log_{10} \left\{ \frac{1}{\sqrt{1 + \left( \frac{\omega}{\omega_{o1}} \right)^2}} \sqrt{1 + \left( \frac{\omega}{\omega_{o2}} \right)^2} \right\} \]
Corner Frequency

• The significance of the break frequency is that it represents the frequency where

\[ A_v(f) = 070.7 \angle -45^\circ \]

• This is where the output of the transfer function has an amplitude 3-\(dB\) below the input amplitude, and the output phase is shifted by \(-45^\circ\) relative to the input.

• Therefore, \(f_c\) is also known as the 3-\(dB\) frequency or the corner frequency.
Bode plots use a logarithmic scale for frequency, where a *decade* is defined as a range of frequencies where the highest and lowest frequencies differ by a factor of 10.
Magnitude of the Transfer Function in dB

\[ |A_v(f)| = \frac{1}{\sqrt{1 + \left(\frac{f}{f_b}\right)^2}} \]

\[ |A_v(f)|_{dB} = 20 \log 1 - 20 \log \sqrt{1 + \left(\frac{f}{f_b}\right)^2} \]

\[ = -20 \log \sqrt{1 + \left(\frac{f}{f_b}\right)^2} = -10 \log \left[1 + \left(\frac{f}{f_b}\right)^2\right] \]

\[ = -20 \log \left(\frac{f}{f_b}\right) \]

• See how the above expression changes with frequency:
  – at low frequencies \( f \ll f_b \), \( |A_v|_{dB} = 0 \text{ dB} \)
    • low frequency asymptote
  – at high frequencies \( f \gg f_b \),
    \[ |A_v(f)|_{dB} = -20 \log \frac{f}{f_b} \]
    • high frequency asymptote
Real Filters

• Butterworth Filters
  – Flat Pass-band.
  – $20n$ dB per decade roll-off.

• Chebyshev Filters
  – Pass-band ripple.
  – Sharper cut-off than Butterworth.

• Elliptic Filters
  – Pass-band and stop-band ripple.
  – Even sharper cut-off.

• Bessel Filters
  – Linear phase response – i.e. no signal distortion in pass-band.
Filter Response Characteristics
Butterworth Filters

The magnitude response of a Butterworth filter.

Magnitude response for Butterworth filters of various order with $\varepsilon = 1$. Note that as the order increases, the response approaches the ideal brickwall type transmission.
Chebyshev Filters

Sketches of the transmission characteristics of a representative even- and odd-order Chebyshev filters.
# First-Order Filter Functions

| Filter Type and $T(s)$ | $s$-Plane Singularities | Bode Plot for $|T|$ | Passive Realization | Op Amp-RC Realization |
|------------------------|--------------------------|---------------------|---------------------|-----------------------|
| (a) Low-Pass (LP)      | ![Low-Pass S-plane](image1) | ![Low-Pass Bode](image2) | ![Low-Pass Passive](image3) | ![Low-Pass Op Amp](image4) |
| $T(s) = \frac{a_0}{s + \omega_0}$ | $0$ at $\omega$ | $20 \log_{10} \frac{a_0}{\omega_0}$ dB | $CR = \frac{1}{\omega_0}$ | $CR_2 = \frac{1}{\omega_0}$ |
|                        | $0$ at $\omega_0$       | $-20$ dB/decade     | $dc$ gain = 1        | $dc$ gain = $-\frac{R_2}{R_1}$ |
| (b) High-Pass (HP)     | ![High-Pass S-plane](image5) | ![High-Pass Bode](image6) | ![High-Pass Passive](image7) | ![High-Pass Op Amp](image8) |
| $T(s) = \frac{a_1 s}{s + \omega_0}$ | $0$ at $\omega$ | $20 \log_{10} |a_1|$ dB | $CR = \frac{1}{\omega_0}$ | $CR_1 = \frac{1}{\omega_0}$ |
|                        | $0$ at $\omega_0$       | $+20$ dB/decade     | High-frequency gain = 1 | High-frequency gain = $-\frac{R_2}{R_1}$ |
| (c) General            | ![General S-plane](image9) | ![General Bode](image10) | ![General Passive](image11) | ![General Op Amp](image12) |
| $T(s) = \frac{a_1 s + a_0}{s + \omega_0}$ | $0$ at $\omega$ | $20 \log_{10} \frac{a_0}{\omega_0}$ dB | $(C_1 + C_2) \frac{R_1}{R_2} = \frac{1}{\omega_0}$ | $C_{2R_2} = \frac{1}{\omega_0}$ |
|                        | $0$ at $\omega_0$       | $-20$ dB/decade     | $C_1 R_1 = \frac{a_0}{a_1}$ | $C_1 R_1 = \frac{a_1}{a_0}$ |
|                        | $0$ at $\omega_0$       | $-20$ dB/decade     | $dc$ gain = $\frac{R_2}{R_1 + R_2}$ | $dc$ gain = $-\frac{R_2}{R_1}$ |
|                        | $0$ at $\omega_0$       | $-20$ dB/decade     | HF gain = $\frac{C_1}{C_1 + C_2}$ | HF gain = $-\frac{C_1}{C_2}$ |
First-Order Filter Functions

| $T(s)$ | Singularities | $|T|$ and $\phi$ | Passive Realization | Op Amp-RC Realization |
|--------|---------------|-----------------|---------------------|----------------------|
| $T(s) = -a_1 \frac{s - \omega_0}{s + \omega_0}$ | $a_1 > 0$ | $|T|$, dB
$20 \log |a_1|$ | $V_i$ $- V_0$ $+$ $R_1$ $R$ $C$ $R_1$ $R$ $C$ | $CR = 1/\omega_0$
Flat gain ($a_1$) = 0.5 |
| | | $\phi$ | $\omega_0$ | $CR = 1/\omega_0$
Flat gain ($a_1$) = 1 |
# Second-Order Filter Functions

| Filter Type and $T(s)$ | $s$-Plane Singularities | $|T|_s$ |
|------------------------|-------------------------|---------|
| (a) Low-Pass (LP)      | ![Diagram](low_pass.png) | ![Diagram](low_pass_freq.png) |
| $T(s) = \frac{a_0}{s^2 + s \frac{\omega_0}{Q} + \omega_0^2}$ | $\omega_0$ at $\sigma = 0$  | $|T| = \frac{|a_0|Q}{\omega_0^2 \sqrt{1 - \frac{1}{4Q^2}}}$ |
| dc gain = $\frac{a_0}{\omega_0}$ | ![Diagram](low_pass_freq.png) | |
| (b) High-Pass (HP)     | ![Diagram](high_pass.png) | ![Diagram](high_pass_freq.png) |
| $T(s) = \frac{a_2 s^2}{s^2 + s \frac{\omega_0}{Q} + \omega_0^2}$ | $\omega_0$ at $\sigma = 0$  | $|T| = \frac{|a_2|}{\sqrt{1 - \frac{1}{2Q^2} \omega_0^2}}$ |
| High-frequency gain = $a_2$ | ![Diagram](high_pass_freq.png) | |
| (c) Bandpass (BP)      | ![Diagram](bandpass.png) | ![Diagram](bandpass_freq.png) |
| $T(s) = \frac{a_1 s}{s^2 + s \frac{\omega_0}{Q} + \omega_0^2}$ | $\omega_0$ at $\sigma = 0$  | $T_{max} = \frac{(a_1 Q)}{\omega_0}$ |
| Center-frequency gain = $\frac{a_1 Q}{\omega_0}$ | ![Diagram](bandpass_freq.png) | $0.707 T_{max} = \frac{(a_1 Q/\sqrt{2\omega_0})}{\omega_0}$ |

\[ \omega_1, \omega_2 = \omega_0 \sqrt{1 + \frac{1}{4Q^2} + \frac{\omega_0^2}{2Q}} \]

\[ \omega_a, \omega_b = \omega_0^2 \]

\[ \omega_0 \sqrt{\frac{1}{Q^2} + \frac{\omega_0^2}{2Q}} \]

\[ \omega_1 \omega_2 = \omega_0^2 \]
### Second-Order Filter Functions

<table>
<thead>
<tr>
<th>Type</th>
<th>Transfer Function</th>
<th>Notch Frequency</th>
<th>High-Frequency Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>(d) Notch</strong></td>
<td>$T(s) = a_2 \frac{s^2 + \omega_n^2}{s^2 + s \frac{\omega_n}{Q} + \omega_0^2}$</td>
<td>$\omega_n$</td>
<td>$a_2$</td>
</tr>
<tr>
<td></td>
<td>dc gain = $\frac{\omega_0}{2Q}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>high-frequency gain = $a_2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(e) Low-Pass Notch (LPN)</strong></td>
<td>$T(s) = a_2 \frac{s^2 + \omega_n^2}{s^2 + s \frac{\omega_n}{Q} + \omega_0^2}$</td>
<td>$\omega_n$</td>
<td>$\frac{\omega_n^2}{\omega_0^2}$</td>
</tr>
<tr>
<td></td>
<td>$\omega_n \gg \omega_0$</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>dc gain = $a_2 \frac{\omega_n^2}{\omega_0^2}$</td>
<td>$\omega_0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>high-frequency gain = $a_2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>(f) High-Pass Notch (HPN)</strong></td>
<td>$T(s) = a_2 \frac{s^2 + \omega_n^2}{s^2 + s \frac{\omega_n}{Q} + \omega_0^2}$</td>
<td>$\omega_0$</td>
<td>$\frac{\omega_n^2}{\omega_0^2}$</td>
</tr>
<tr>
<td></td>
<td>$\omega_n \leq \omega_0$</td>
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</tr>
<tr>
<td></td>
<td>dc gain = $a_2 \frac{\omega_n^2}{\omega_0^2}$</td>
<td>$\omega_0$</td>
<td></td>
</tr>
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<td></td>
<td>high-frequency gain = $a_2$</td>
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<td></td>
</tr>
</tbody>
</table>
Second-Order Filter Functions

\[ T(s) = a_2 \frac{s^2 - s \frac{\omega_0}{Q} + \omega_0^2}{s^2 + s \frac{\omega_0}{Q} + \omega_0^2} \]

Flat gain = \( a_2 \)
Second-Order LCR Resonator

(a) General structure

(b) LP

(c) HP

(d) BP

(e) Notch at $\omega_0$

(f) General notch

(g) LPN ($\omega_n > \omega_0$)

(h) LPN as $s \rightarrow \infty$

(i) HPN ($\omega_n < \omega_0$)
The Antoniou inductance-simulation circuit. (b) Analysis of the circuit assuming ideal op amps. The order of the analysis steps is indicated by the circled numbers.
The Antoniou inductance-simulation circuit. Analysis of the circuit assuming ideal op amps. The order of the analysis steps is indicated by the circled numbers.
Realizations for the various second-order filter functions using the op amp-RC resonator of Fig. 11.21 (b). (a) LP; (b) HP; (c) BP, (d) notch at $\omega_0$.
The Second-Order Active Filter: Inductor Replacement

(e) LPN, $\omega_n \geq \omega_0$

(f) HPN, $\omega_n \leq \omega_0$

(g) All-pass
Derivation of an alternative two-integrator-loop biquad in which all op amps are used in a single-ended fashion. The resulting circuit in (b) is known as the Tow-Thomas biquad.
Low-Pass Active Filter Design

Design a fourth-order low-pass Butterworth filter having a frequency cut-off of 100 Hz

Choose \( C = 0.1 \mu F \)

\[
R = \frac{1}{2\pi C f_b} = \frac{1}{2\pi (0.1 \times 10^{-6}) (100 \text{Hz})} = \frac{1}{15.92 \text{k}\Omega}
\]

\( R_1 = R_2 = R_{11} = R_{12} = 15.8 \text{k}\Omega \)

\( C_1 = C_2 = C_{11} = C_{12} = 0.1 \mu F \)
Low-Pass Active Filter Design

$$20 \log \left| \frac{V_o}{V_{in}} \right| \text{ (dB)}$$

-80 dB/decade slope

$f \text{ (Hz)}$
Low-Pass Active Filter Design

Normalized gain

Second stage

First stage

Overall

$f (\text{Hz})$
Infinite-Gain Multiple-Feedback (IGMF) Negative Feedback Active Filter

\[ v_i^+ = 0 \quad v_i^- = 0 \quad \Rightarrow \quad V_o = -\frac{Z_5}{Z_3} V_x \quad \Rightarrow \quad V_x = -\frac{Z_3}{Z_5} V_o \quad (1) \]

By KCL at \( V_x \),

\[ \frac{V_i - V_x}{Z_1} = \frac{V_x}{Z_2} + \frac{V_x}{Z_3} + \frac{V_x - V_o}{Z_4} \quad (2) \]

Substitute (1) into (2) gives

\[ \frac{V_i}{Z_1} + \frac{Z_3}{Z_1 Z_5} V_o = -\frac{Z_3}{Z_5 Z_2} V_o - \frac{V_o}{Z_5} - \frac{Z_3}{Z_4 Z_5} V_o - \frac{V_o}{Z_4} \quad (3) \]
Rearranging equation (3), it gives,

\[ H = \frac{V_o}{V_i} = -\frac{1}{Z_1Z_3} \left( \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \frac{1}{Z_4} \right) + \frac{1}{Z_3Z_4} \]

Or in admittance form:

\[ H = \frac{V_o}{V_i} = -\frac{Y_1Y_3}{Y_5(Y_1 + Y_2 + Y_3 + Y_4) + Y_3Y_4} \]

<table>
<thead>
<tr>
<th>Filter</th>
<th>Value</th>
<th>( Z_1 )</th>
<th>( Z_2 )</th>
<th>( Z_3 )</th>
<th>( Z_4 )</th>
<th>( Z_5 )</th>
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<tbody>
<tr>
<td>LP</td>
<td></td>
<td>( R_1 )</td>
<td>( C_2 )</td>
<td>( R_3 )</td>
<td>( R_4 )</td>
<td>( C_5 )</td>
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<tr>
<td>HP</td>
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<td>( C_1 )</td>
<td>( R_2 )</td>
<td>( C_3 )</td>
<td>( C_4 )</td>
<td>( R_5 )</td>
</tr>
<tr>
<td>BP</td>
<td></td>
<td>( R_1 )</td>
<td>( R_2 )</td>
<td>( C_3 )</td>
<td>( C_4 )</td>
<td>( R_5 )</td>
</tr>
</tbody>
</table>
IGMF Band-Pass Filter

Band-pass: \[ H(s) = K \frac{s}{s^2 + as + b} \]

To obtain the band-pass response, we let

\[ Z_1 = R_1 \quad Z_2 = R_2 \quad Z_3 = \frac{1}{j\omega C_3} = \frac{1}{sC_3} \quad Z_4 = \frac{1}{j\omega C_4} = \frac{1}{sC_4} \quad Z_5 = R_5 \]

\[ H(s) = -\frac{sC_3}{R_1} \frac{sC_3C_4}{s^2C_3C_4 + \frac{C_3 + C_4}{R_5} + \frac{1}{R_5} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)} \]

This filter prototype has a very low sensitivity to component tolerance when compared with other prototypes.
Simplified Design (IGMF Filter)

\[
H(s) = -\frac{sC}{R_1} \left( \frac{1}{R_1R_5} + \frac{2C}{R_5} + s^2C^2 \right)
\]

Comparing with the band-pass response

\[
H(s) = K \frac{s}{s^2 + \frac{\omega_p}{Q_p}s + \omega_p^2}
\]

It gives,

\[
\omega_p = \frac{1}{C\sqrt{R_1R_5}} \quad Q_p = \frac{1}{2} \frac{\sqrt{R_5}}{R_1} \quad H(\omega_p) = -2Q^2
\]
Example: IGMF Band Pass Filter

To design a band-pass filter with \( f_o = 512 \text{Hz} \) and \( Q = 10 \)

\[
\omega_p = \frac{1}{C \sqrt{R_1 R_5}} = 2\pi(512\text{Hz})
\]

\[
C = 100 \text{nF} \quad \rightarrow \quad R_1 R_5 = 9,662,741\Omega^2
\]

\[
Q_p = \frac{1}{2} \frac{\sqrt{R_5}}{\sqrt{R_1}} = 10
\]

\[
\rightarrow R_1 = 155.4\Omega \quad R_5 = 62,170\Omega
\]

With similar analysis, we can choose the following values:

\[
C = 10 \text{nF} \quad R_1 = 1,554\Omega \quad \text{and} \quad R_5 = 621,700\Omega
\]
Butterworth Response (Maximally Flat)

\[
|H(j\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_o}\right)^{2n}}}
\]

where \( n \) is the order

Normalize to \( \omega_o = 1 \text{rad/s} \)

\[
|\hat{H}(j\omega)| = \frac{1}{\sqrt{1 + \omega^{2n}}}
\]

Butterworth polynomials:

\[
|\hat{H}(j\omega)| = \frac{1}{|B_n(j\omega)|}
\]

Butterworth polynomials:

\[
B_1(s) = s + 1
\]
\[
B_2(s) = s^2 + \sqrt{2}s + 1
\]
\[
B_3(s) = s^3 + 2s^2 + 2s + 1
\]
\[
= (s + 1)(s^2 + s + 1)
\]
\[
B_4(s) = s^4 + 2.61s^3 + 3.41s^2 + 2.61s + 1
\]
\[
= (s^2 + 0.77s + 1)(s^2 + 1.85s + 1)
\]
\[
B_5(s) = s^5 + 3.24s^4 + 5.24s^3 + 5.24s^2 + 3.24s + 1
\]
\[
= (s + 1)(s^2 + 0.62s + 1)(s^2 + 1.62s + 1)
\]
Second Order Butterworth Response

Started from the low-pass bi-quadratic function

\[ H(s) = K \frac{1}{s^2 + \frac{\omega_p}{Q_p} s + \omega_p^2} \]

For \( \omega_p = 1 \quad K = 1 \quad Q = \frac{1}{\sqrt{2}} \)

\[ H(s) = \frac{1}{s^2 + \sqrt{2} s + 1} \] (second order butterworth polynomial)

\[ H(j\omega) = \frac{1}{-\omega^2 + \sqrt{2} j\omega + 1} \]

\[ |H(j\omega)| = \frac{1}{\sqrt{(1 - \omega^2)^2 + (\sqrt{2}\omega)^2}} \]

\[ |H(j\omega)| = \frac{1}{\sqrt{1 - 2\omega^2 + \omega^4 + 2\omega^2}} \]

\[ |H(j\omega)| = \frac{1}{\sqrt{1 + \omega^4}} \]

\[ |H(j\omega)| = \frac{1}{\sqrt{1 + (\omega)^{2n}}} = \frac{1}{\sqrt{1 + (\omega)^{2n}}} \]
Second order Butterworth Filter

\[ H(s) = \frac{K}{1 + sC_4(R_1 + R_2) + sR_1C_3(1 - K) + s^2R_1R_2C_3C_4} = \frac{K'}{s^2 + \frac{\omega_p}{Q_p} s + \omega_p^2} \]

\[ Q_p = \frac{1}{\sqrt{R_1C_4} + \sqrt{R_2C_4} + (1 - K)\sqrt{R_1C_3}} \]

Setting \( R_1 = R_2 \) and \( C_1 = C_2 \)

\[ Q_p = \frac{1}{\sqrt{1 + \sqrt{1 + (1 - K)\sqrt{1}}} = \frac{1}{2 + (1 - K)} = \frac{1}{3 - K} \]

Now \( K = 1 + \frac{R_B}{R_A} \)

\[ Q_p = \frac{1}{3 - K} = \frac{1}{3 - \left(1 + \frac{R_B}{R_A}\right)} = \frac{1}{2 - \frac{R_B}{R_A}} \]

For Butterworth response:

\[ Q_p = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad Q_p = \frac{1}{\sqrt{2}} = \frac{1}{2 - \frac{R_B}{R_A}} \]

\[ \frac{2 - \frac{R_B}{R_A}}{Q_p} = \sqrt{2} = 1.414 \]

We define Damping Factor (DF) as:

\[ DF = \frac{1}{Q_p} = 2 - \frac{R_B}{R_A} = 1.414 \]