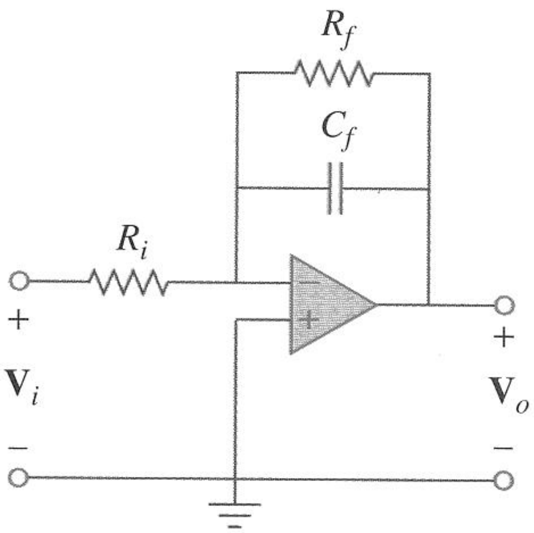
# ELG3336: Op Amp-based Active Filters

Filters that use active elements, e.g. amplifiers, power source.

- **Advantages:**
  - Reduced size and weight, and therefore parasitics.
  - Increased reliability and improved performance.
  - Simpler design than for passive filters and can realize a wider range of functions as well as providing voltage gain.
  - In large quantities, the cost of an IC is less than its passive counterpart.
- **Disadvantages:**
  - Limited bandwidth of active devices limits the highest attainable pole frequency and therefore applications above 100 kHz (passive RLC filters can be used up to 500 MHz).
  - The achievable quality factor is also limited.
  - Require power supplies (unlike passive filters).
  - Increased sensitivity to variations in circuit parameters caused by environmental changes compared to passive filters.
- For applications, particularly in voice and data communications, the economic and performance advantages of active RC filters far outweigh their disadvantages.

PF is a frequency when the filter starts to attenuate the output ripples, for example.

# First-Order Low-Pass Filter




$$H(f)=\frac{V_o}{V_i}=\boxed{-\frac{Z_f}{Z_i}}$$

$$\frac{1}{Z_f}=\frac{1}{R_f}+\frac{1}{\frac{1}{j2\pi fC_f}}=\frac{1}{R_f}+\frac{j2\pi fR_fC_f}{R_f}$$

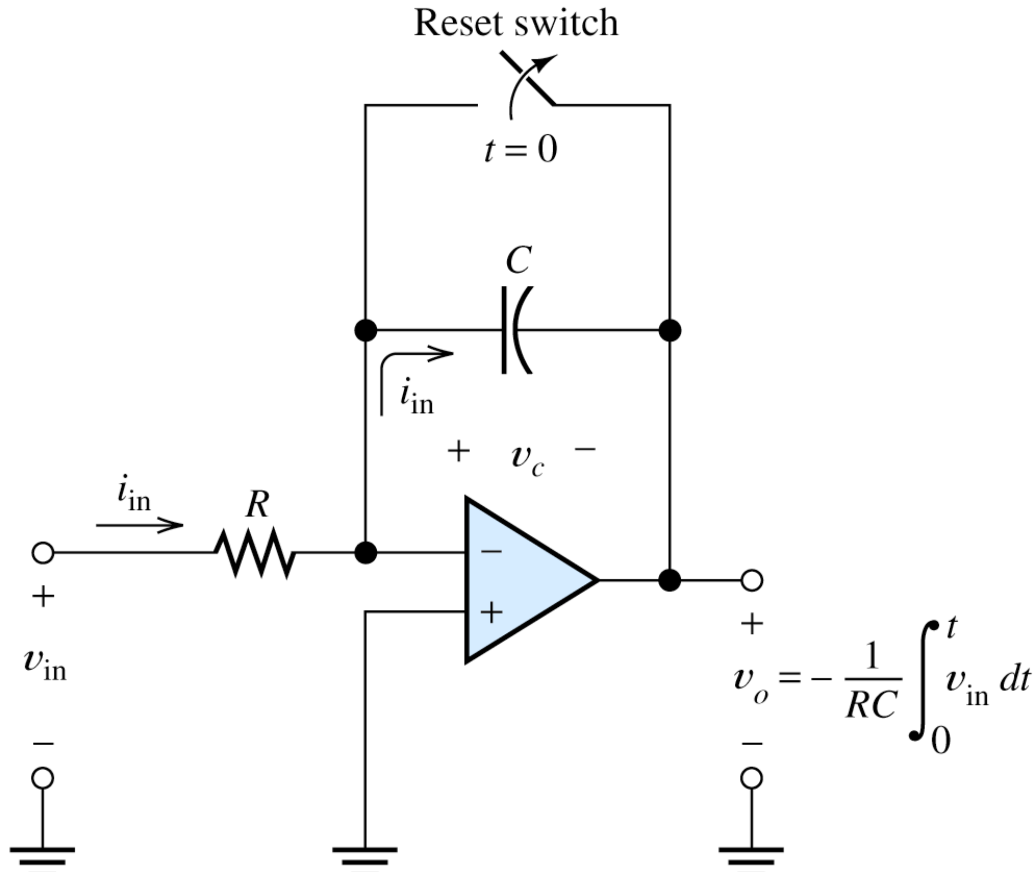
$$Z_f=\frac{R_f}{1+j2\pi fR_fC_f}$$

$$\boxed{H(f)}=-\frac{Z_f}{Z_i}=-\left(\frac{R_f}{R_i}\right)\frac{1}{1+j2\pi fR_fC_f}$$

$$\boxed{=-\left(\frac{R_f}{R_i}\right)\left[\frac{1}{1+j(f/f_B)}\right]}$$

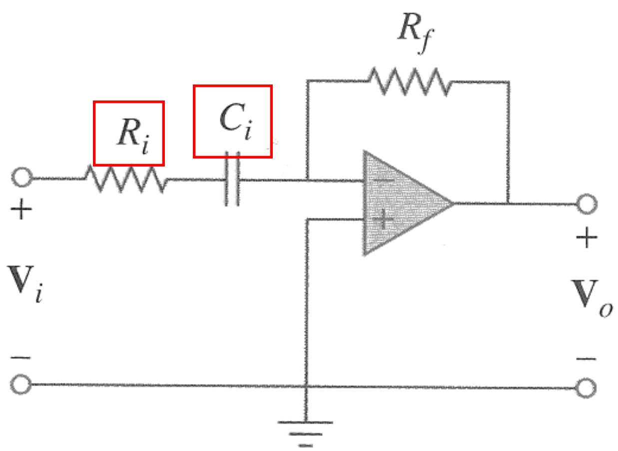
$$f_B=\frac{1}{2\pi R_fC_f}$$

A low-pass filter with a dc gain of $-R_f/R_i$




$$v_o(t) = -\frac{1}{RC}\int_0^t v_{in}(t)dt$$

# First-Order High-Pass Filter




$$H(f)=\frac{v_o}{v_i}=\boxed{-\frac{Z_f}{Z_i}}$$

$$Z_i=R_i+\frac{1}{j2\pi f C_i} \qquad Z_f=R_f$$

$$H(f)=-\frac{Z_f}{Z_i}=-\frac{R_f}{R_i+\frac{1}{j2\pi f C_i}}$$

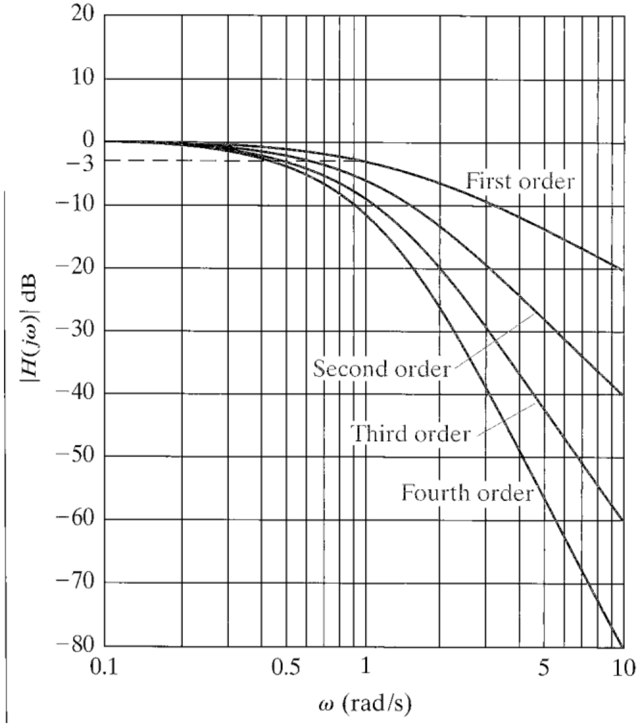
$$=-\frac{j2\pi f R_f C_i}{1+j2\pi f R_i C_i}=-\left(\frac{R_i}{R_i}\right)\frac{j2\pi f R_f C_i}{1+j2\pi f R_i C_i}$$

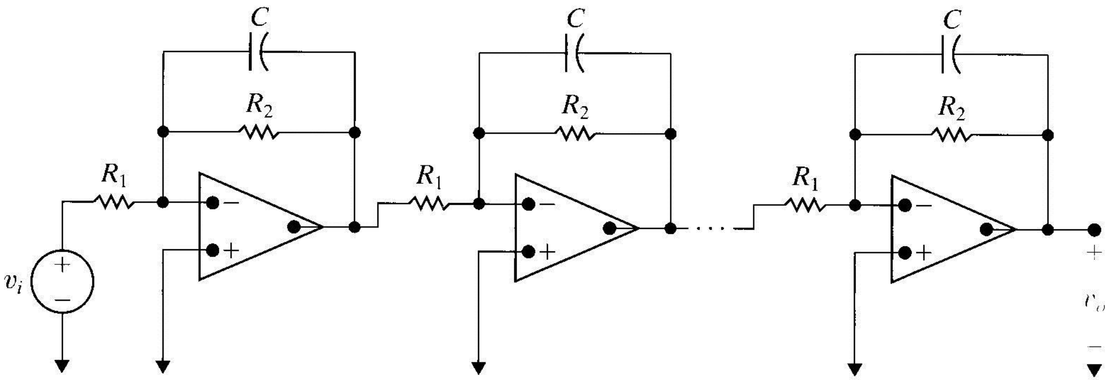
$$=-\left(\frac{R_f}{R_i}\right)\left[\frac{j(f/f_B)}{1+j(f/f_B)}\right]$$

$$\boxed{f_B=\frac{1}{2\pi R_i C_i}}$$

A high-pass filter with a high frequency gain of $-R_f/R_i$

# Higher Order Filters



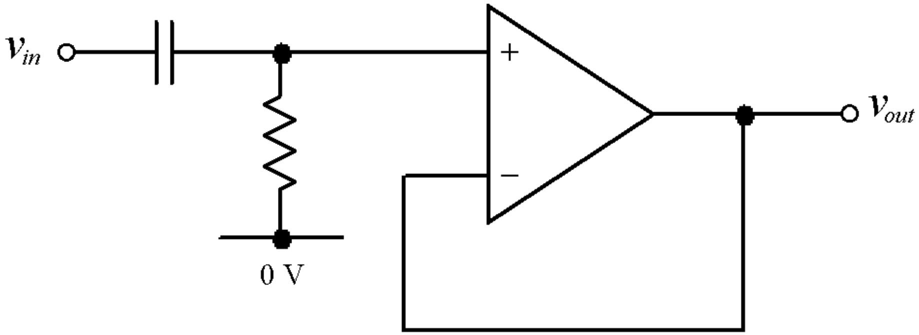




$$H(f) = H_1(f)H_2(f)***H_n(f)$$

$$= (-1)^n \left(\frac{R_f}{R_i}\right)^n \left[\frac{1}{1 + j(f / f_B)}\right]^n$$

# Single-Pole Active Filter Designs

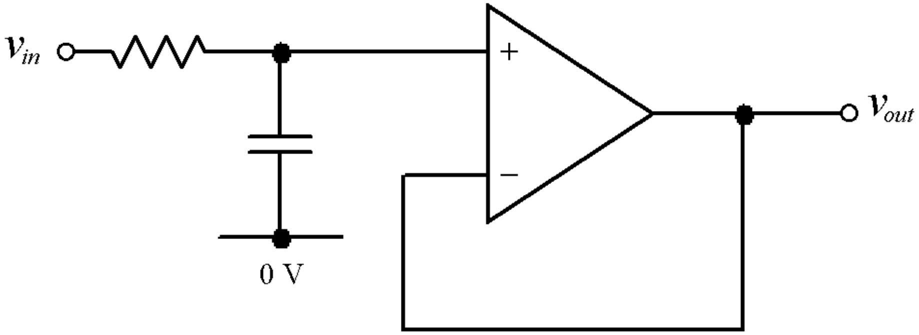
## High Pass




$$\frac{v_{out}}{v_{in}} = \frac{1}{\frac{1}{sCR}+1} = \frac{1}{\frac{1+sRC}{sCR}}$$

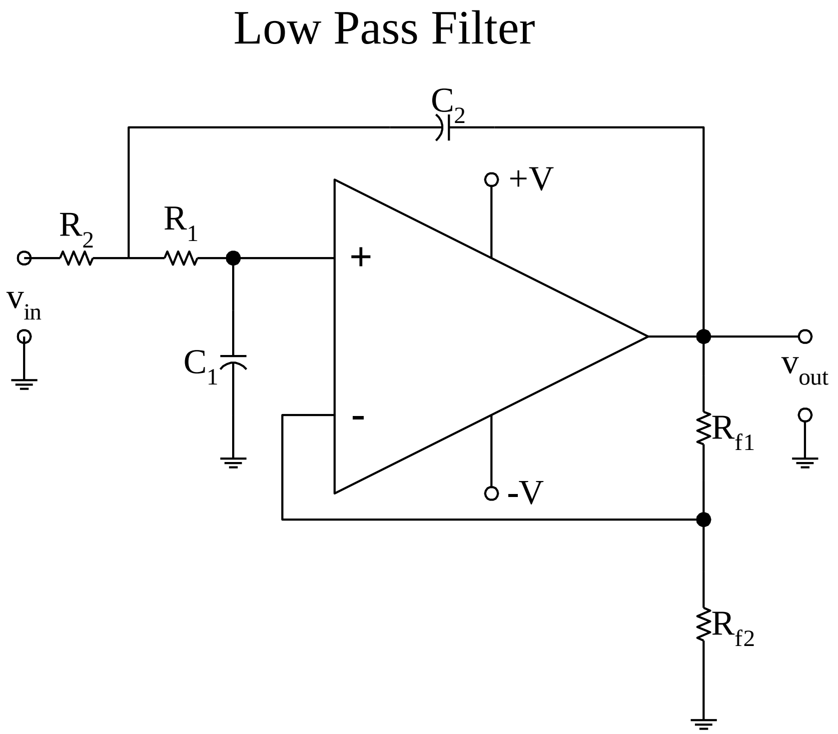
$$= \frac{sRC}{RC(s+1/RC)} = \frac{s}{(s+1/RC)}$$

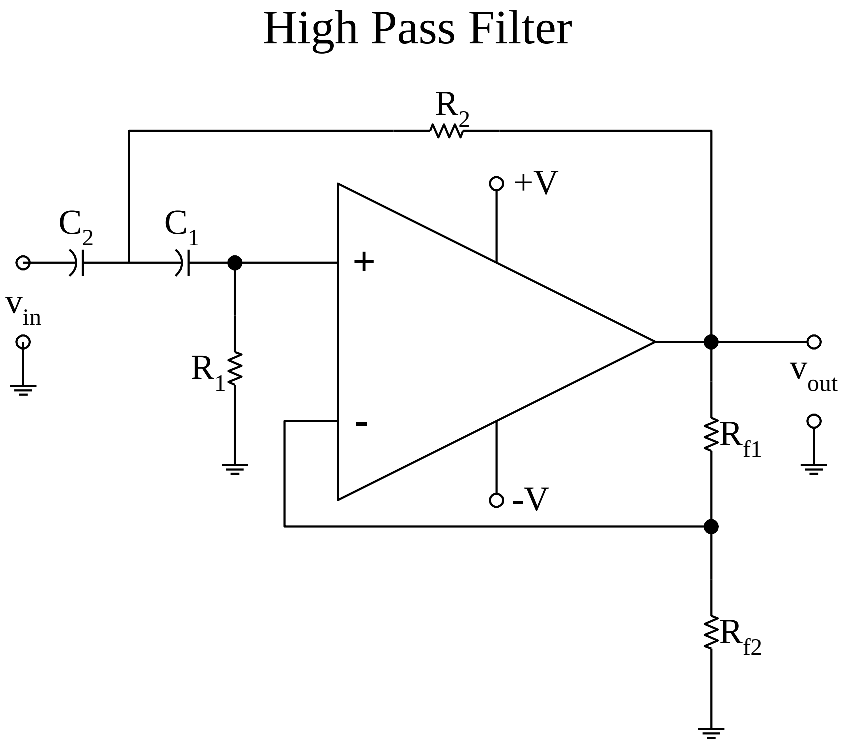
## Low Pass



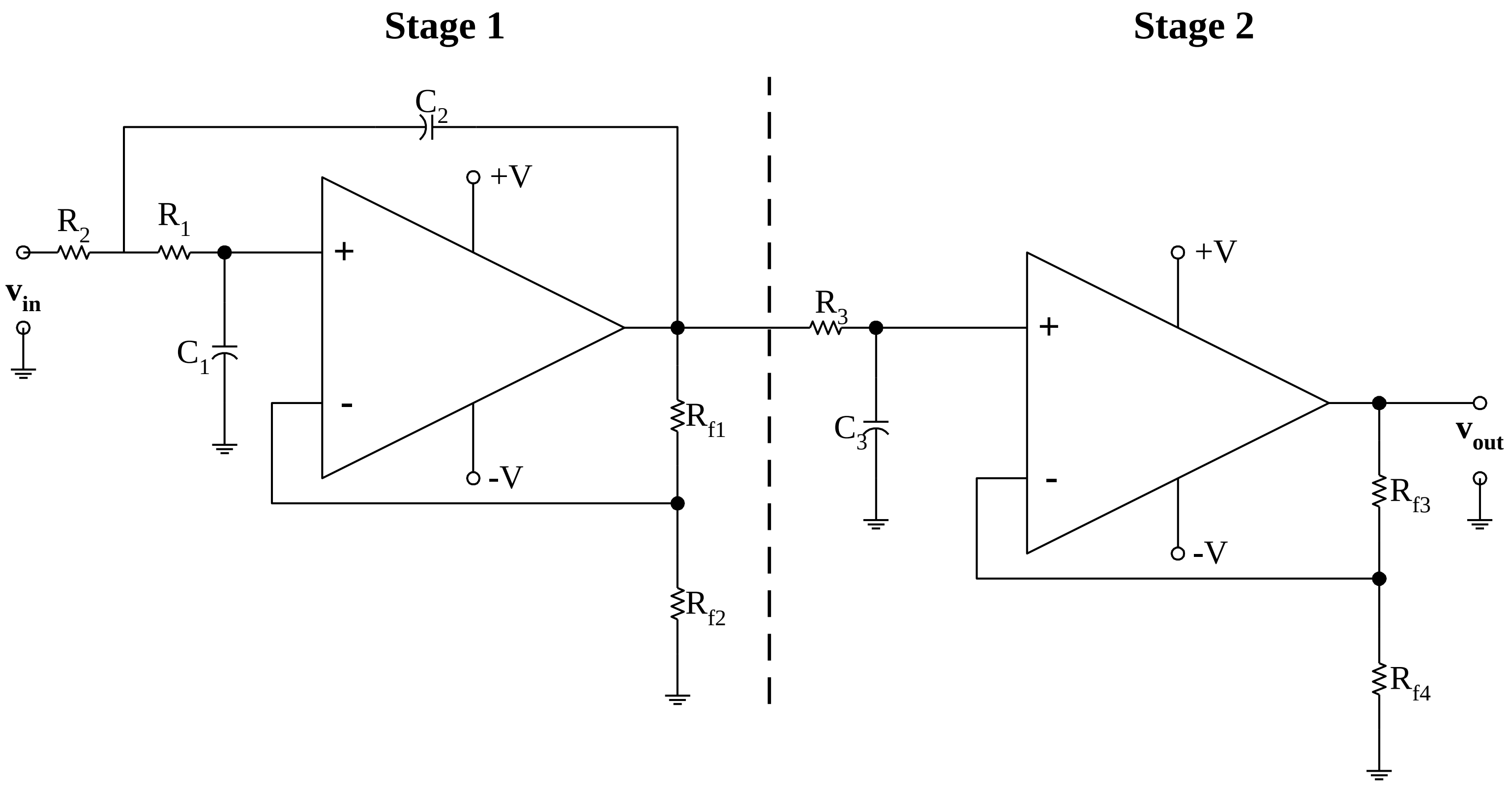

$$\boxed{\frac{v_{out}}{v_{in}} = \frac{1/RC}{s+1/RC}}$$

# Two-Pole (Sallen-Key) Filters

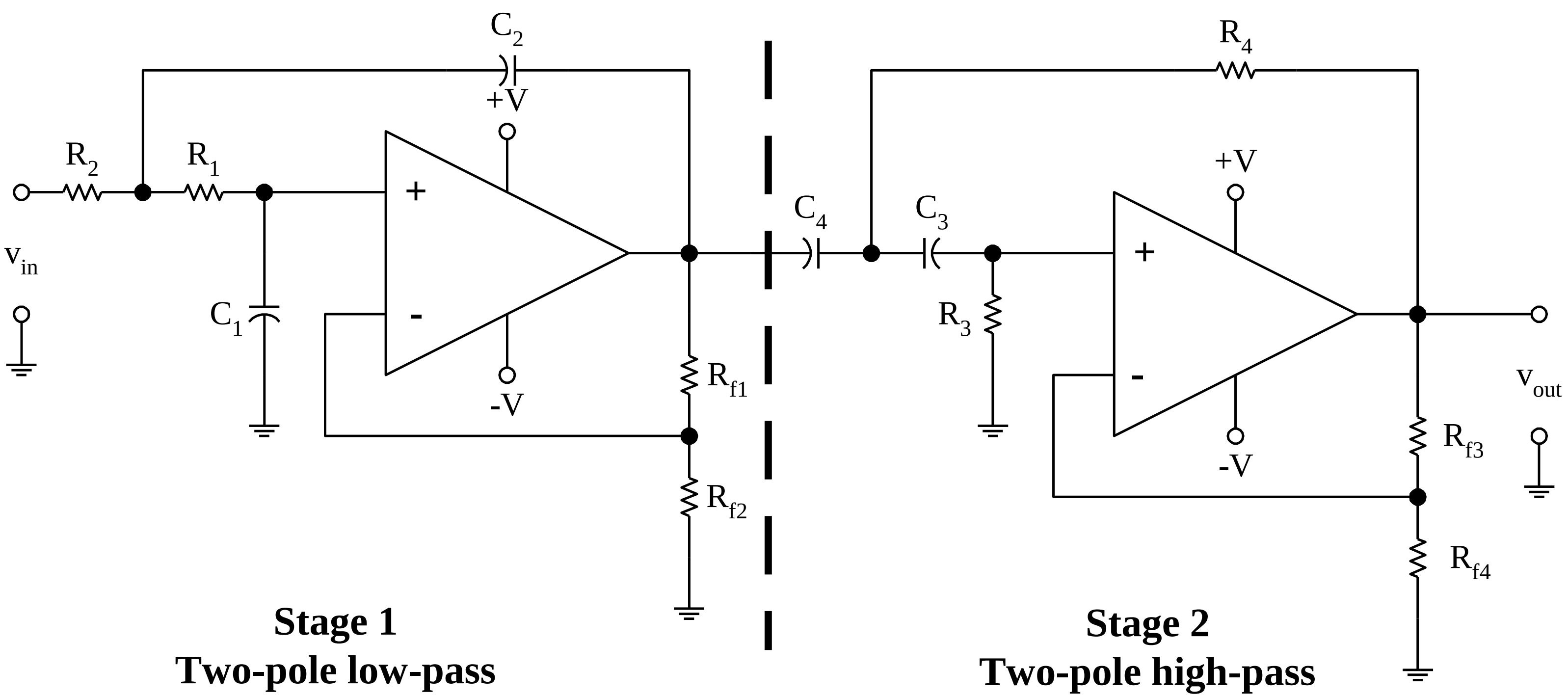


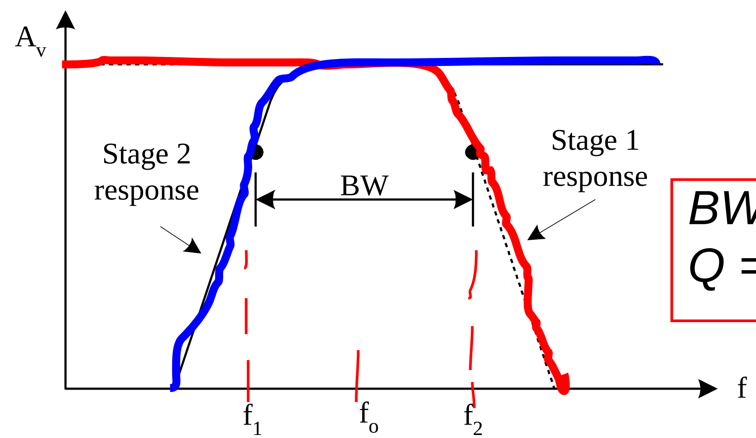



# Three-Pole Low-Pass Filter



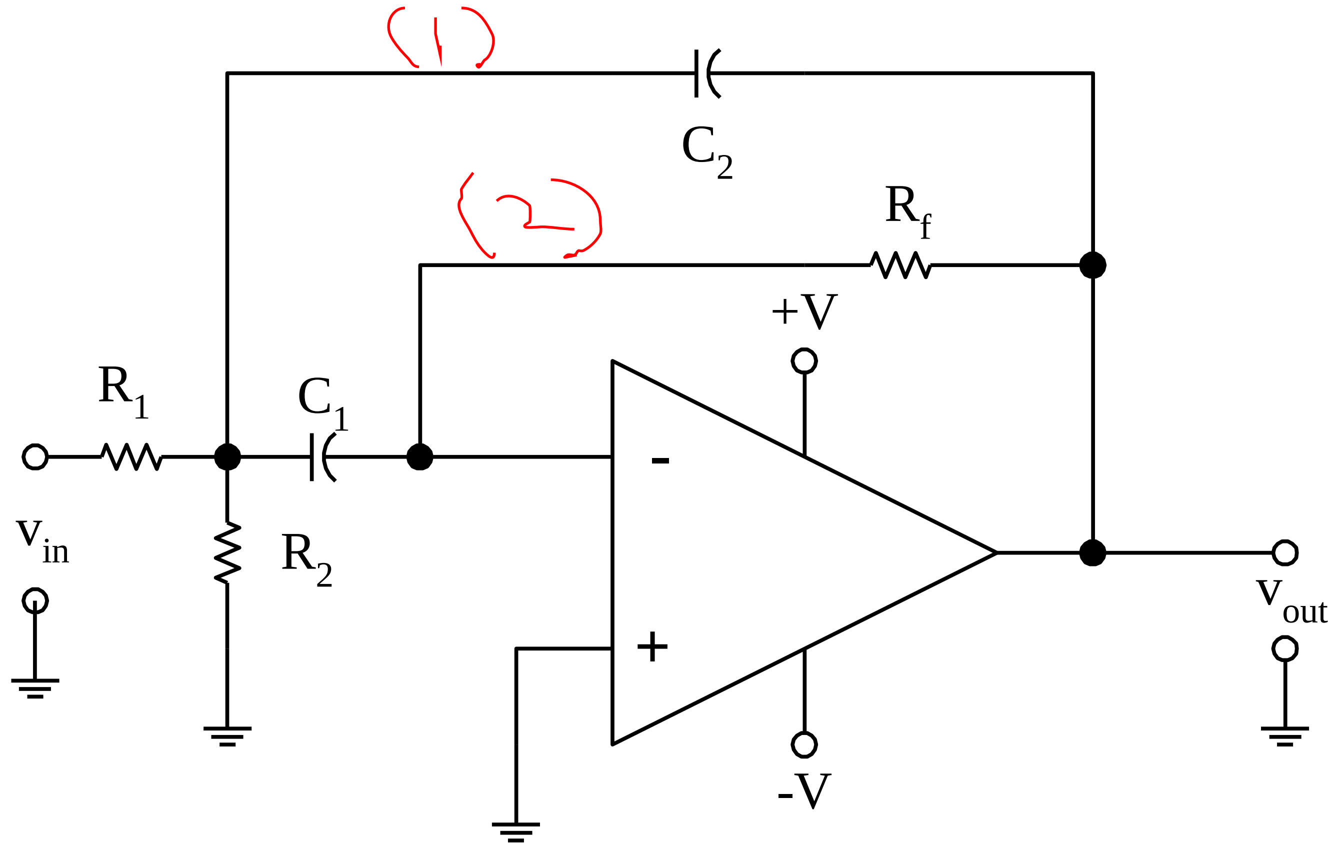
# Two-Stage Band-Pass Filter



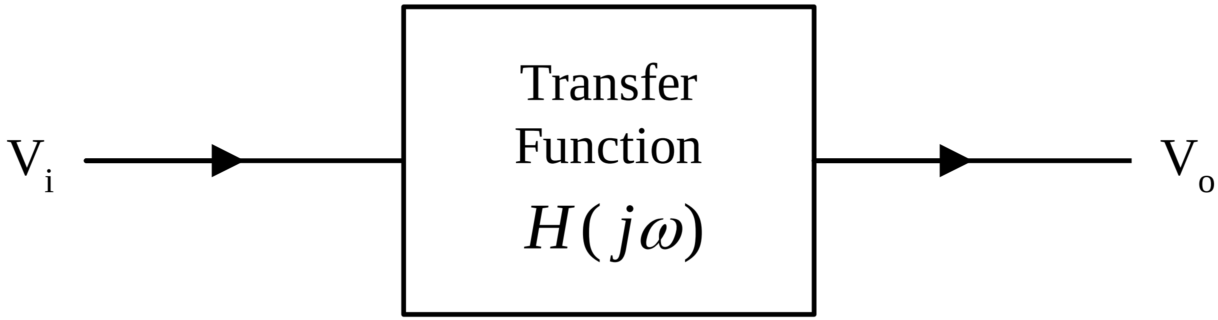




$$BW = f_2 - f_1$$
$$Q = f_0 / BW$$

# Multiple-Feedback Band-Pass Filter



# Transfer function $H(j\omega)$

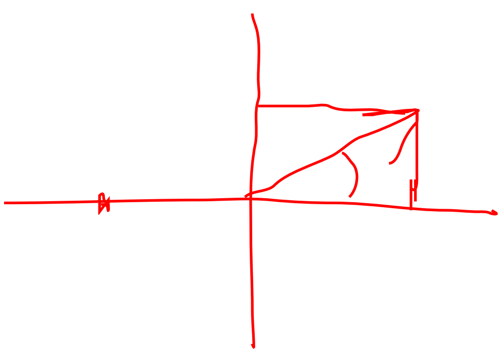



$$H(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)}$$

$$|H| = \sqrt{\mathrm{Re}(H)^2 + \mathrm{Im}(H)^2}$$

$$H = \mathrm{Re}(H) + j\,\mathrm{Im}(H)$$

$$\angle H = \tan^{-1}\left(\frac{\mathrm{Im}(H)}{\mathrm{Re}(H)}\right) \qquad \mathrm{Re}(H) \succ 0$$



$$\boxed{\angle H = 180^o + \tan^{-1}\left(\frac{\mathrm{Im}(H)}{\mathrm{Re}(H)}\right) \qquad \mathrm{Re}(H) \prec 0}$$

# Corner Frequency

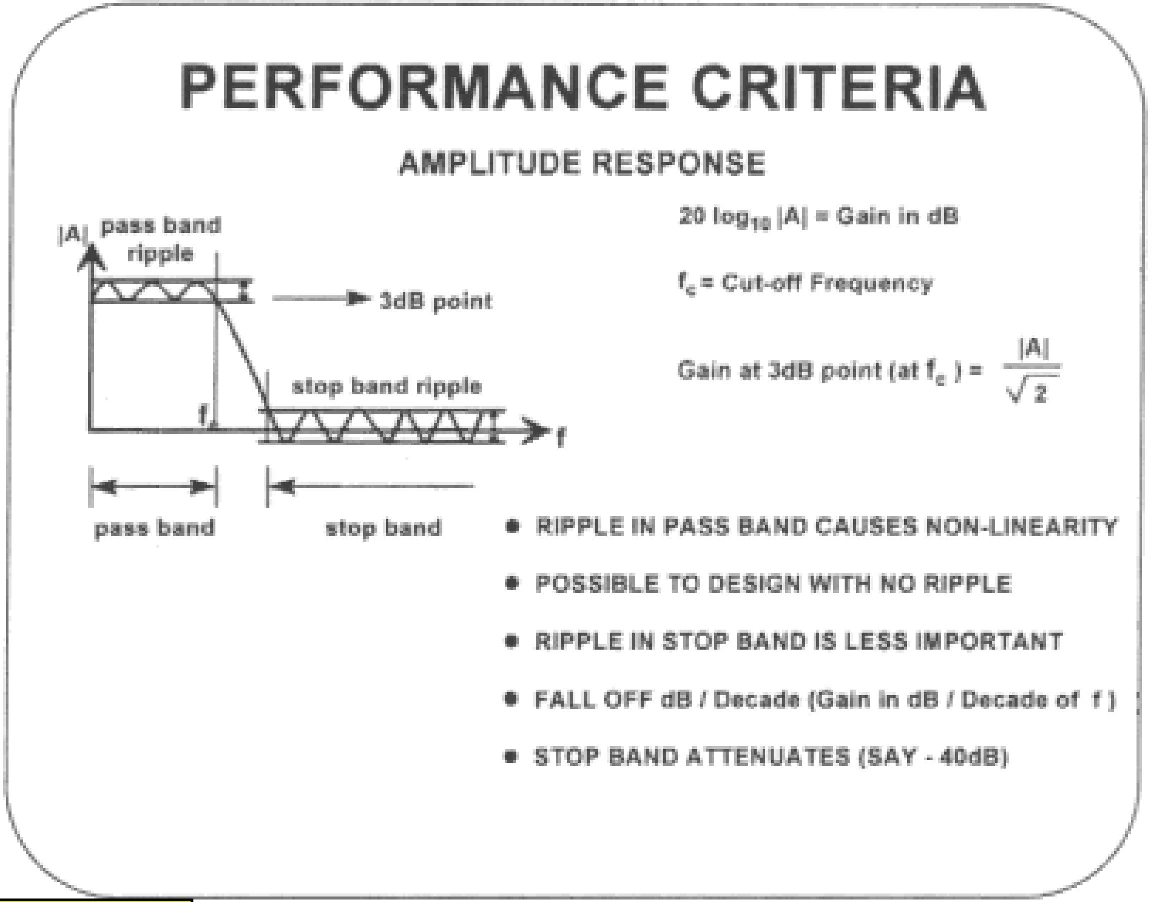
- The significance of the break frequency is that it represents the frequency where

$$\boxed{A_v(f) = 0 70.7\angle -45°}$$

Let us take the example of LPF.

- This is where the output of the transfer function has an amplitude *3-dB* below the input amplitude, and the output phase is shifted by -45° relative to the input.

- Therefore, $f_c$ is also known as the **3-dB frequency** or the **corner frequency**.



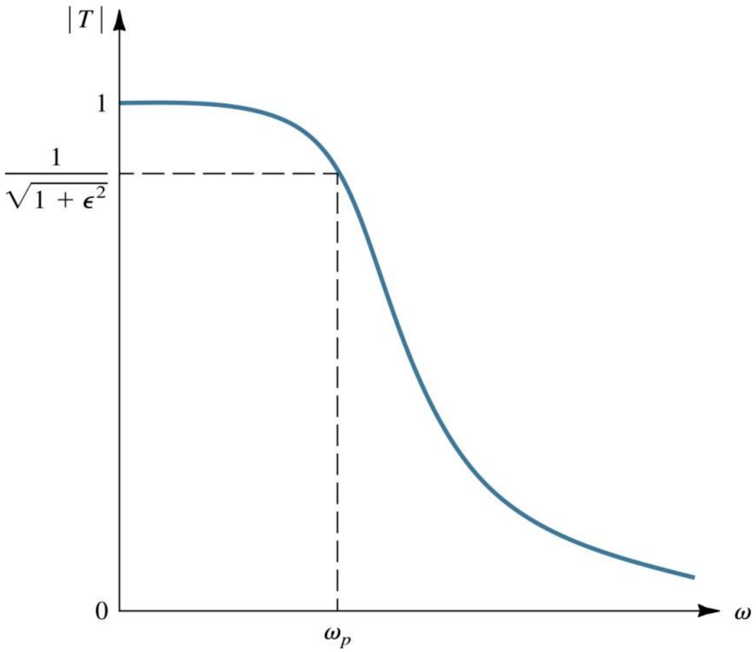

Freq. response mag. and phase versus frequency.

Bode plots use a logarithmic scale for frequency, where a *decade* is defined as a range of frequencies where the highest and lowest frequencies differ by a factor of 10.

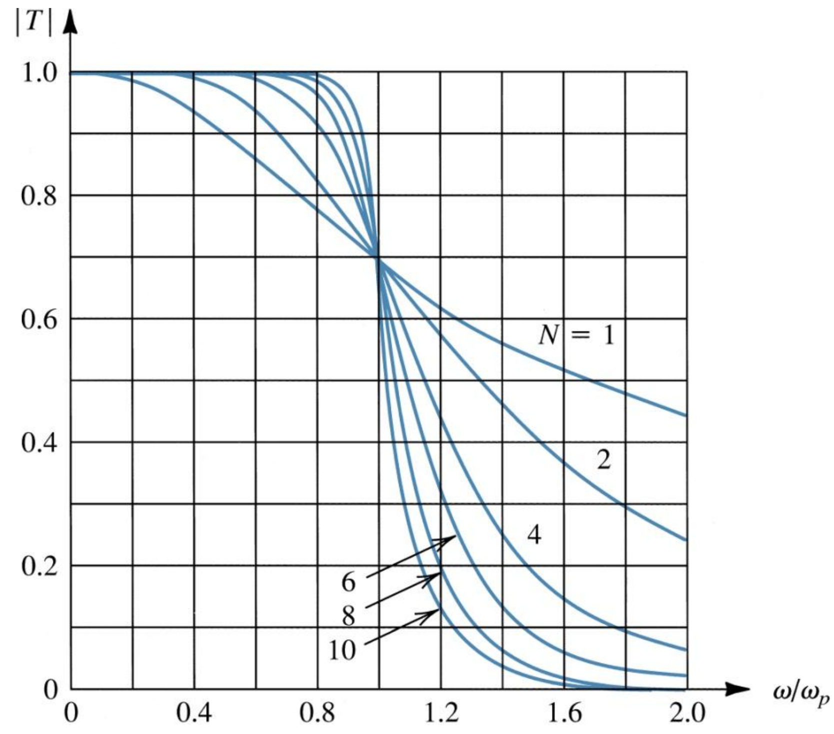
# Real Filters

- Butterworth Filters
  - Flat Pass-band.
  - 20*n* dB per decade roll-off.
- Chebyshev Filters
  - Pass-band ripple.
  - Sharper cut-off than Butterworth.
- Elliptic Filters
  - Pass-band and stop-band ripple.
  - Even sharper cut-off.
- Bessel Filters
  - Linear phase response – i.e. no signal distortion in pass-band.

# Butterworth Filters

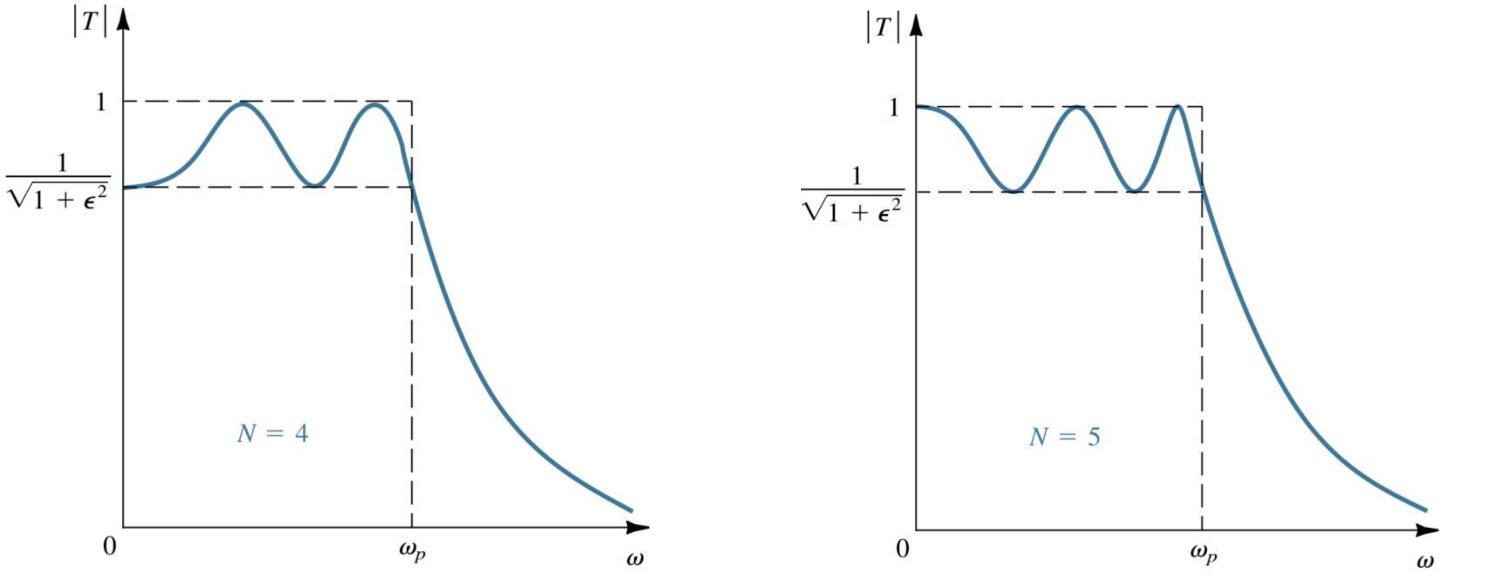



The magnitude response of a Butterworth filter.



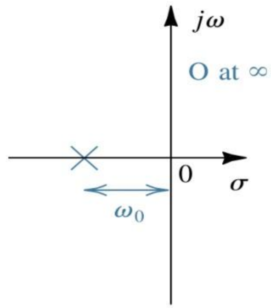
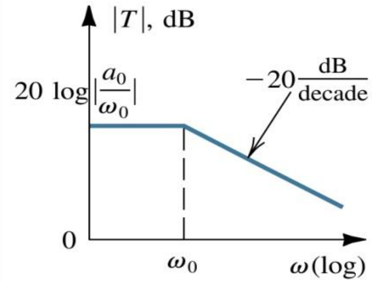
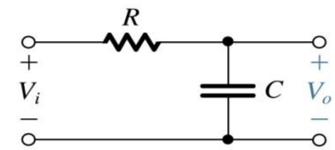
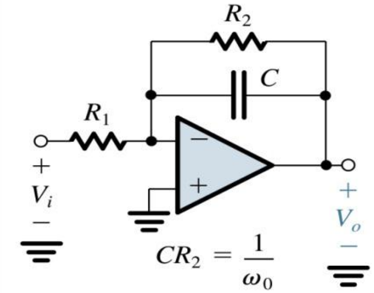
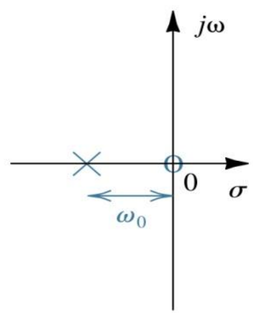
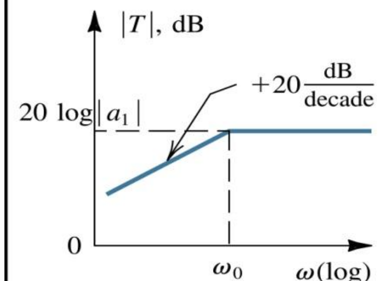
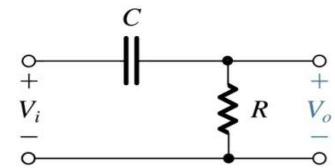
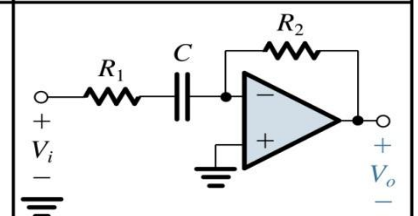
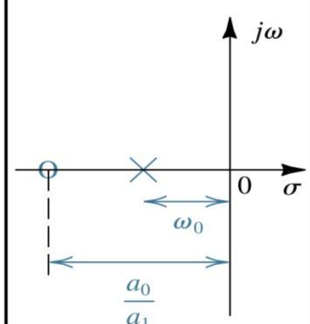
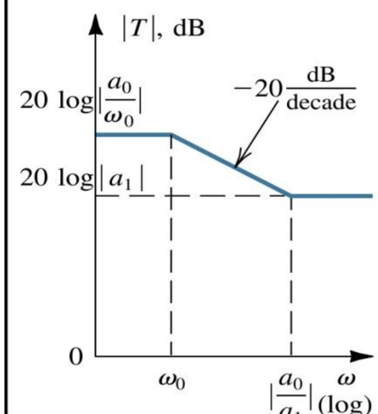
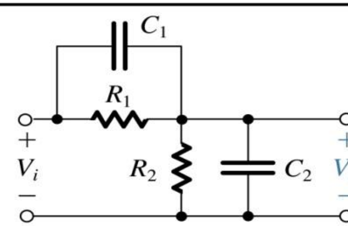
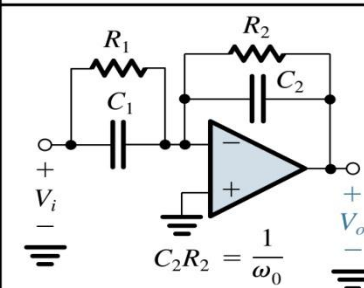

Magnitude response for Butterworth filters of various order with ε = 1. Note that as the order increases, the response approaches the ideal brickwall type transmission.

# Chebyshev Filters

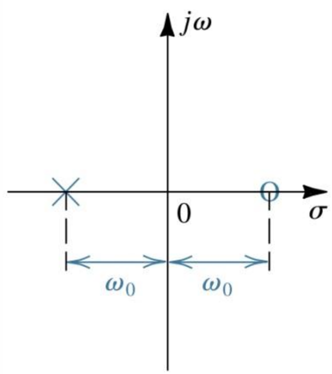
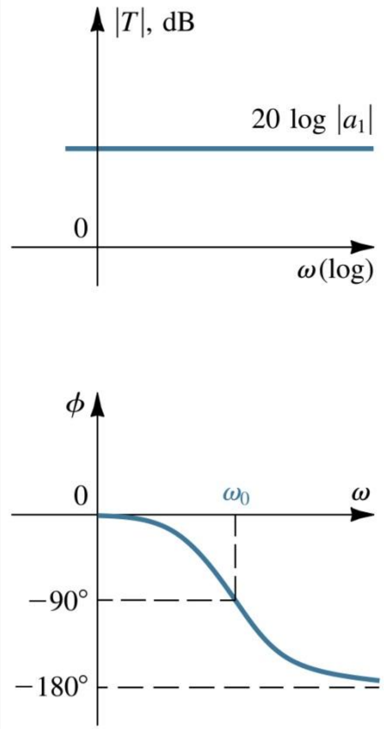
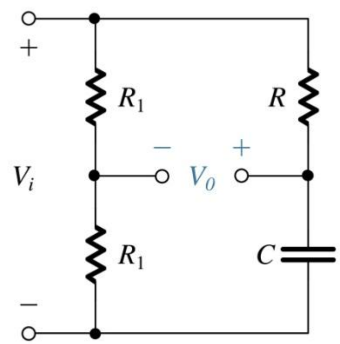
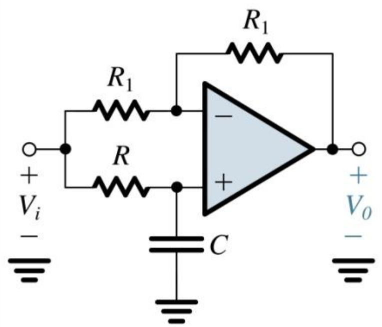



Sketches of the transmission characteristics of a representative even- and odd-order Chebyshev filters.

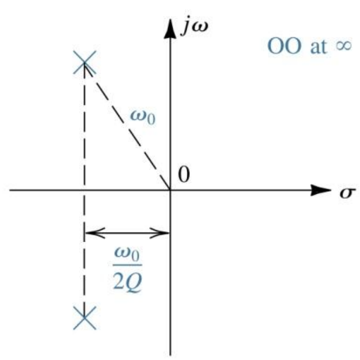
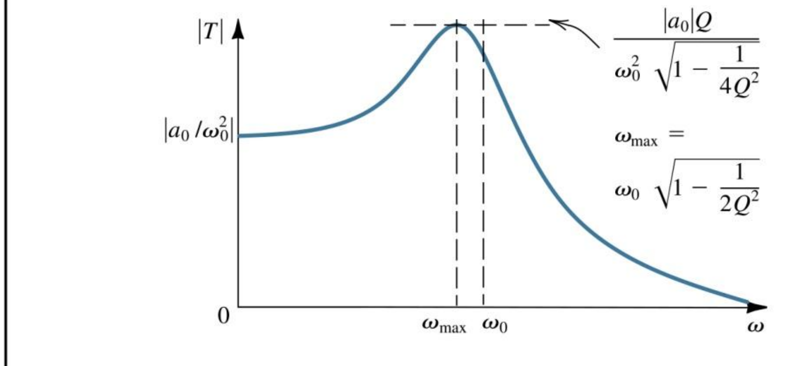
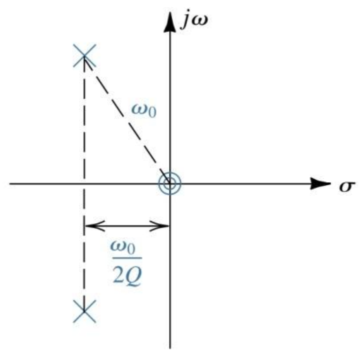
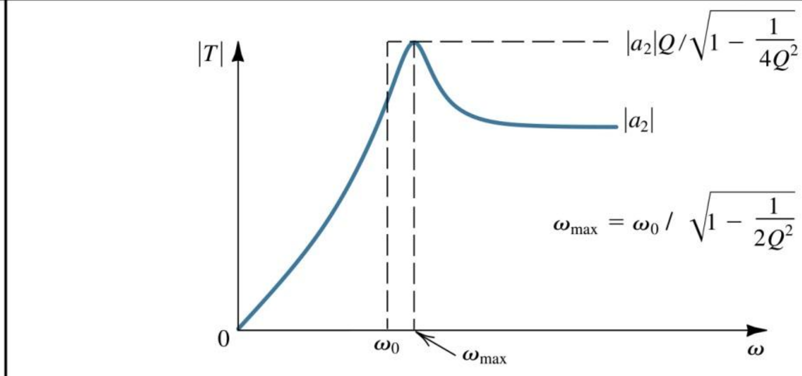
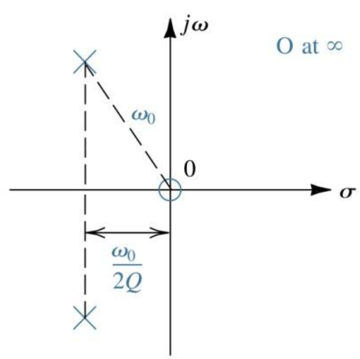
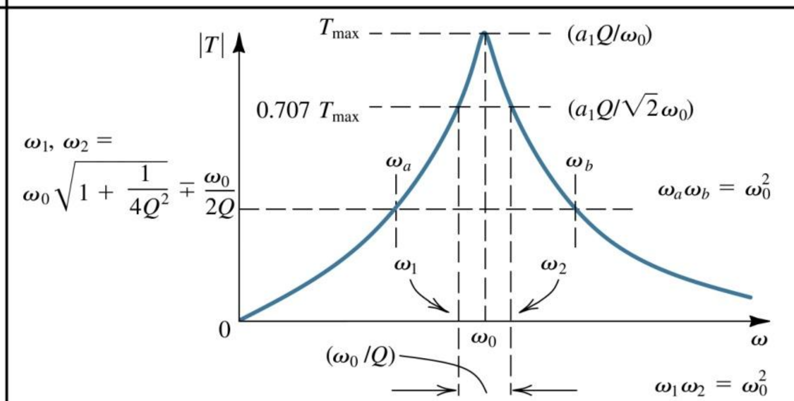
# First-Order Filter Functions

| Filter Type and $T(s)$ | $s$-Plane Singularities | Bode Plot for $\lvert T \rvert$ | Passive Realization | Op Amp-RC Realization |
|---|---|---|---|---|
| (a) Low-Pass (LP)<br>$T(s) = \frac{a_0}{s + \omega_0}$ |  |  | <br>$CR = \frac{1}{\omega_0}$<br>dc gain = 1 | <br>$CR_2 = \frac{1}{\omega_0}$<br>dc gain $= -\frac{R_2}{R_1}$ |
| (b) High-Pass (HP)<br>$T(s) = \frac{a_1 s}{s + \omega_0}$ |  |  | <br>$CR = \frac{1}{\omega_0}$<br>High-frequency gain = 1 | <br>$CR_1 = \frac{1}{\omega_0}$<br>High-frequency gain $= -\frac{R_2}{R_1}$ |
| (c) General<br>$T(s) = \frac{a_1 s + a_0}{s + \omega_0}$ |  |  | <br>$(C_1 + C_2)(R_1 // R_2) = \frac{1}{\omega_0}$<br>$C_1 R_1 = \frac{a_0}{a_1}$<br>dc gain $= \frac{R_2}{R_1 + R_2}$<br>HF gain $= \frac{C_1}{C_1 + C_2}$ | <br>$C_2 R_2 = \frac{1}{\omega_0}$<br>$C_1 R_1 = \frac{a_1}{a_0}$<br>dc gain $= -\frac{R_2}{R_1}$<br>HF gain $= -\frac{C_1}{C_2}$ |

# First-Order Filter Functions

| $T(s)$ | Singularities | $\lvert T \rvert$ and $\phi$ | Passive Realization | Op Amp-RC Realization |
|---|---|---|---|---|
| $T(s) = -a_1 \frac{s - \omega_0}{s + \omega_0}$ <br> $a_1 > 0$ |   |   |   $CR = 1/\omega_0$ <br> Flat gain $(a_1) = 0.5$ |   $CR = 1/\omega_0$ <br> Flat gain $(a_1) = 1$ |

# Second-Order Filter Functions

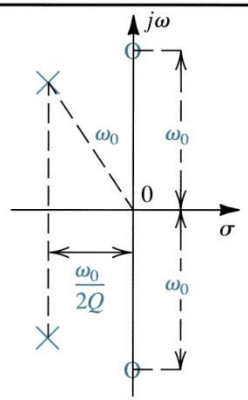
| Filter Type and $T(s)$ | $s$-Plane Singularities | $\lvert T\rvert$ |
|---|---|---|
| (a) Low-Pass (LP)<br>$T(s) = \dfrac{a_0}{s^2 + s\dfrac{\omega_0}{Q} + \omega_0^2}$<br>dc gain $= \dfrac{a_0}{\omega_0^2}$ |  |  |
| (b) High-Pass (HP)<br>$T(s) = \dfrac{a_2 s^2}{s^2 + s\dfrac{\omega_0}{Q} + \omega_0^2}$<br>High-frequency gain $= a_2$ |  |  |
| (c) Bandpass (BP)<br>$T(s) = \dfrac{a_1 s}{s^2 + s\dfrac{\omega_0}{Q} + \omega_0^2}$<br>Center-frequency gain $= \dfrac{a_1 Q}{\omega_0}$ |  |  |

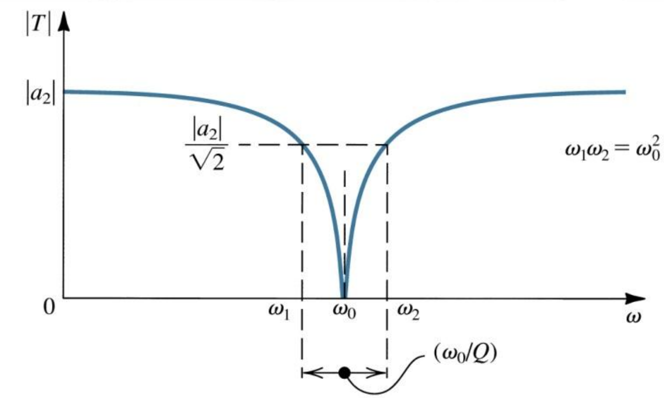
# Second-Order Filter Functions

**(d) Notch**

$$T(s) = a_2 \frac{s^2 + \omega_0^2}{s^2 + s\frac{\omega_0}{Q} + \omega_0^2}$$

dc gain = high-frequency gain = $a_2$



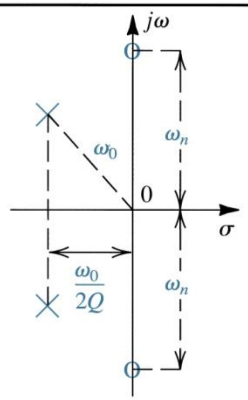

**(e) Low-Pass Notch (LPN)**

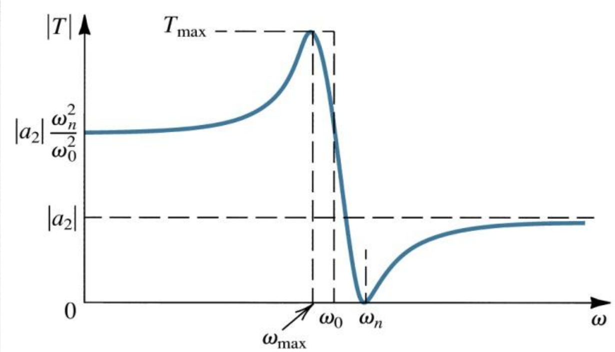
$$T(s) = a_2 \frac{s^2 + \omega_n^2}{s^2 + s\frac{\omega_0}{Q} + \omega_0^2}$$

$\omega_n \geqslant \omega_0$

dc gain $= a_2 \dfrac{\omega_n^2}{\omega_0^2}$

high-frequency gain = $a_2$



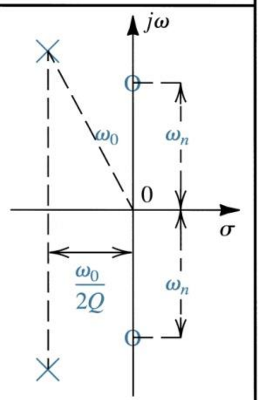

**(f) High-Pass Notch (HPN)**

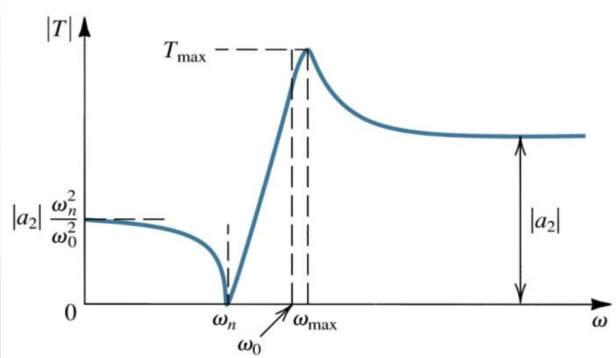
$$T(s) = a_2 \frac{s^2 + \omega_n^2}{s^2 + s\frac{\omega_0}{Q} + \omega_0^2}$$

$\omega_n \leqslant \omega_0$

dc gain $= a_2 \dfrac{\omega_n^2}{\omega_0^2}$

high-frequency gain = $a_2$




$$\omega_{\max} = \omega_0 \sqrt{\frac{\left(\frac{\omega_n^2}{\omega_0^2}\right)\left(1 - \frac{1}{2Q^2}\right) - 1}{\frac{\omega_n^2}{\omega_0^2} + \frac{1}{2Q^2} - 1}}$$

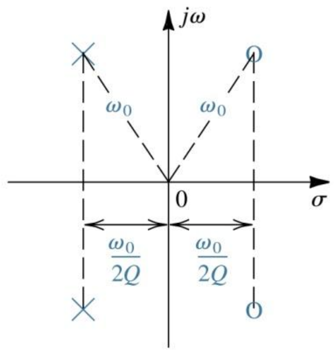
$$T_{\max} = \frac{|a_2| \, |\omega_n^2 - \omega_{\max}^2|}{\sqrt{(\omega_0^2 - \omega_{\max}^2)^2 + \left(\frac{\omega_0}{Q}\right)^2 \omega_{\max}^2}}$$

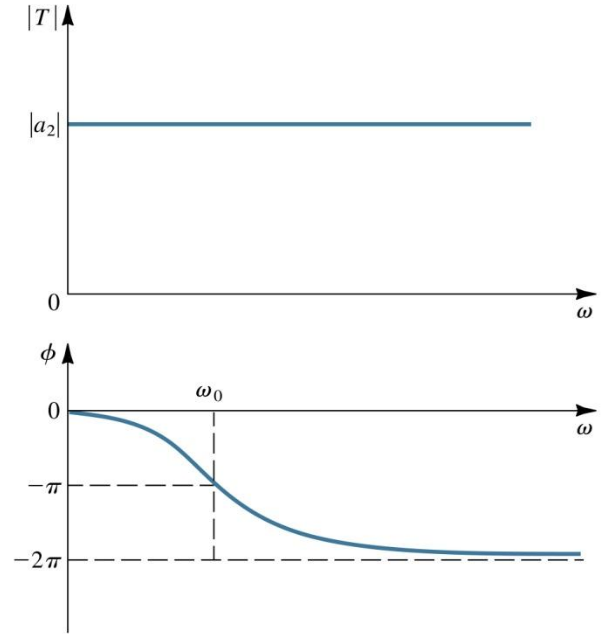
# Second-Order Filter Functions

(g) All-Pass (AP)

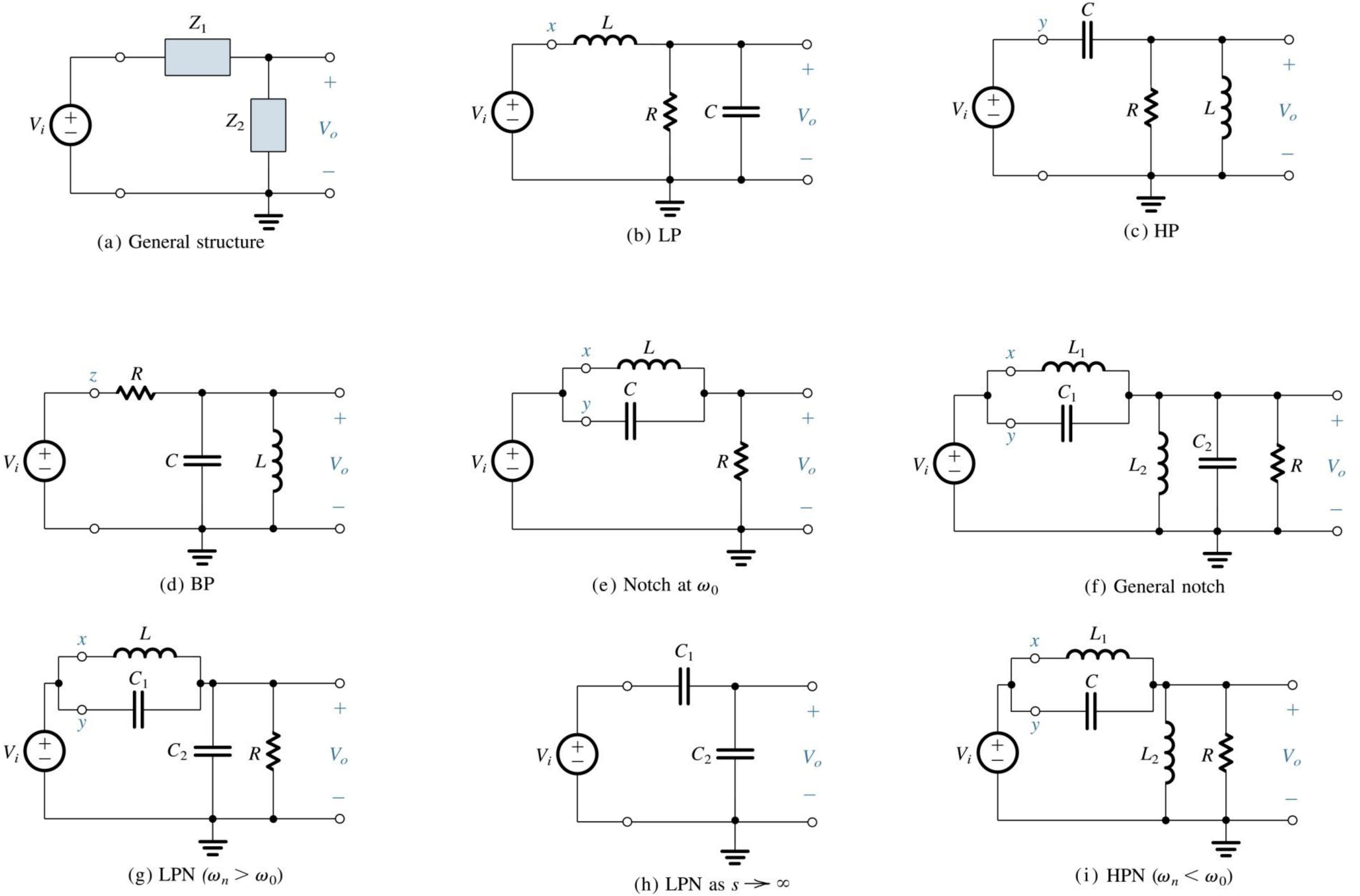
$$T(s) = a_2 \frac{s^2 - s\dfrac{\omega_0}{Q} + \omega_0^2}{s^2 + s\dfrac{\omega_0}{Q} + \omega_0^2}$$

Flat gain $= a_2$

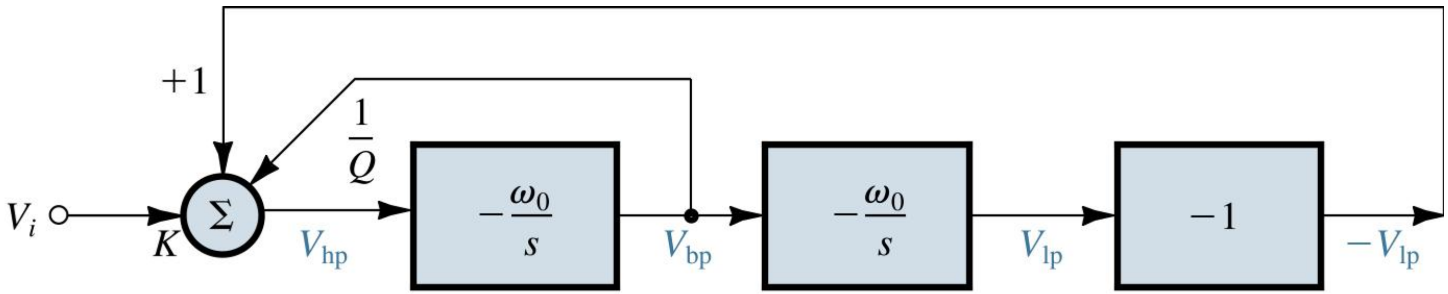


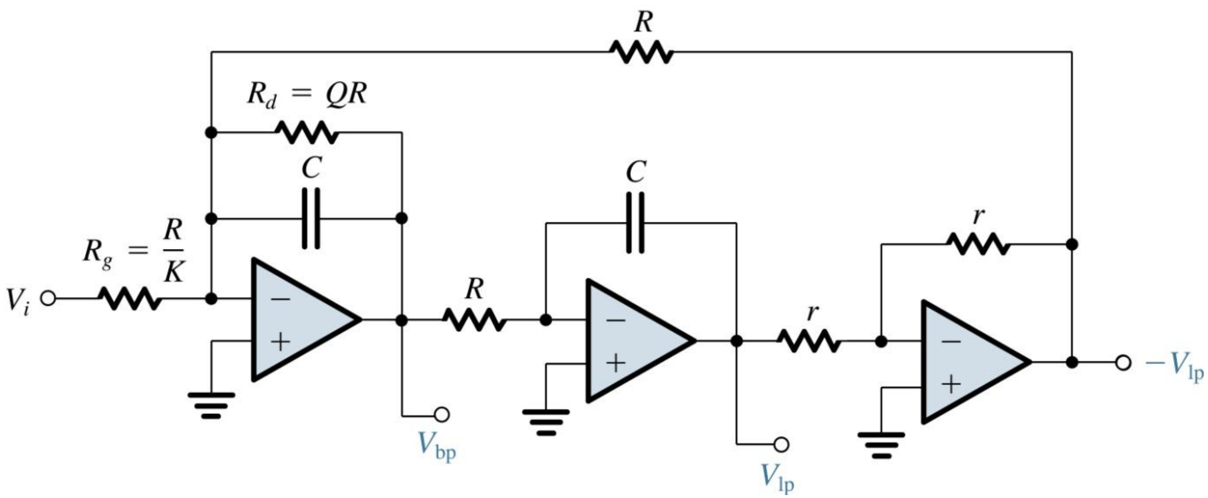



# Second-Order LCR Resonator



# Second-Order Active Filter: Two-Integrator-Loop

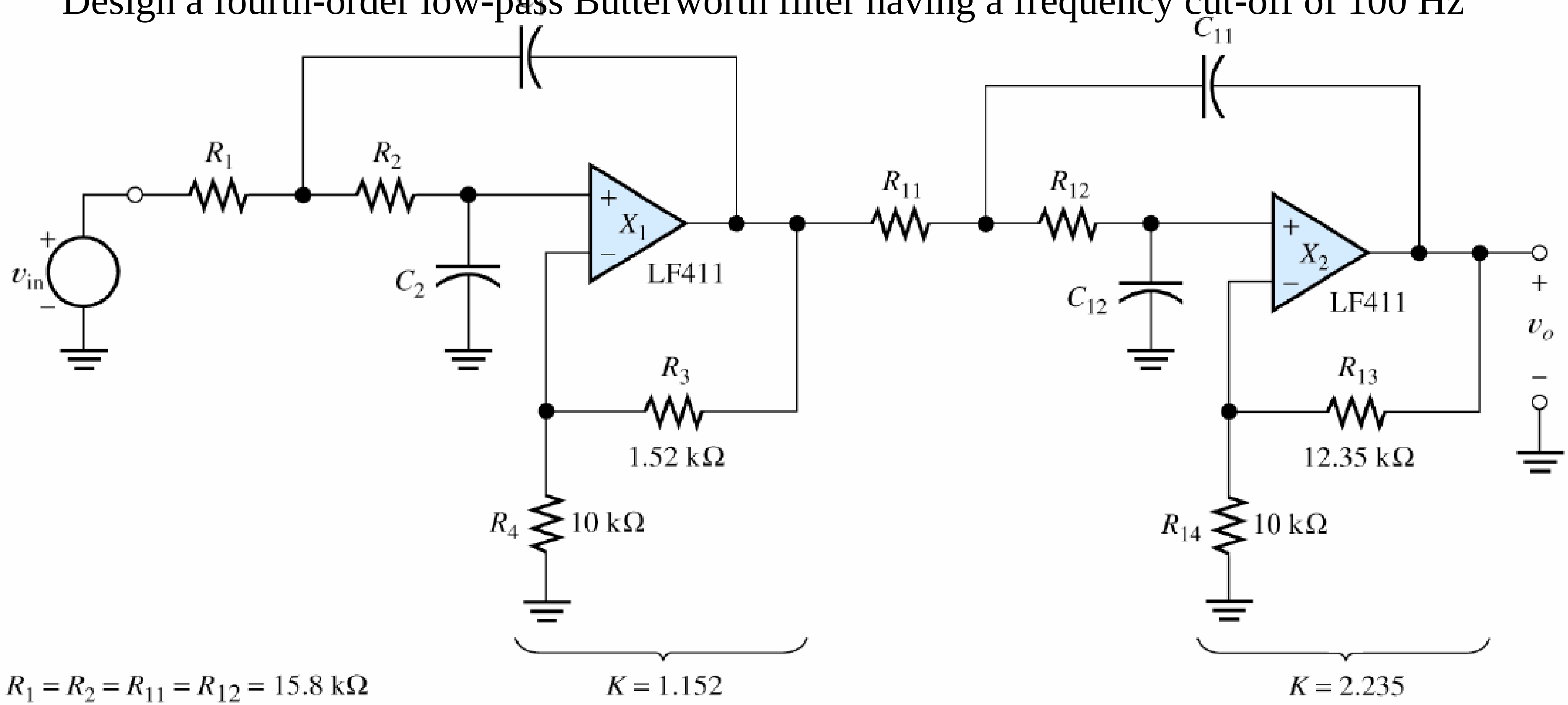



Derivation of an alternative two-integrator-loop biquad in which all op amps are used in a single-ended fashion. The resulting circuit in (b) is known as the Tow-Thomas biquad.

# Low-Pass Active Filter Design

Design a fourth-order low-pass Butterworth filter having a frequency cut-off of 100 Hz




$R_1 = R_2 = R_{11} = R_{12} = 15.8\ \text{k}\Omega$

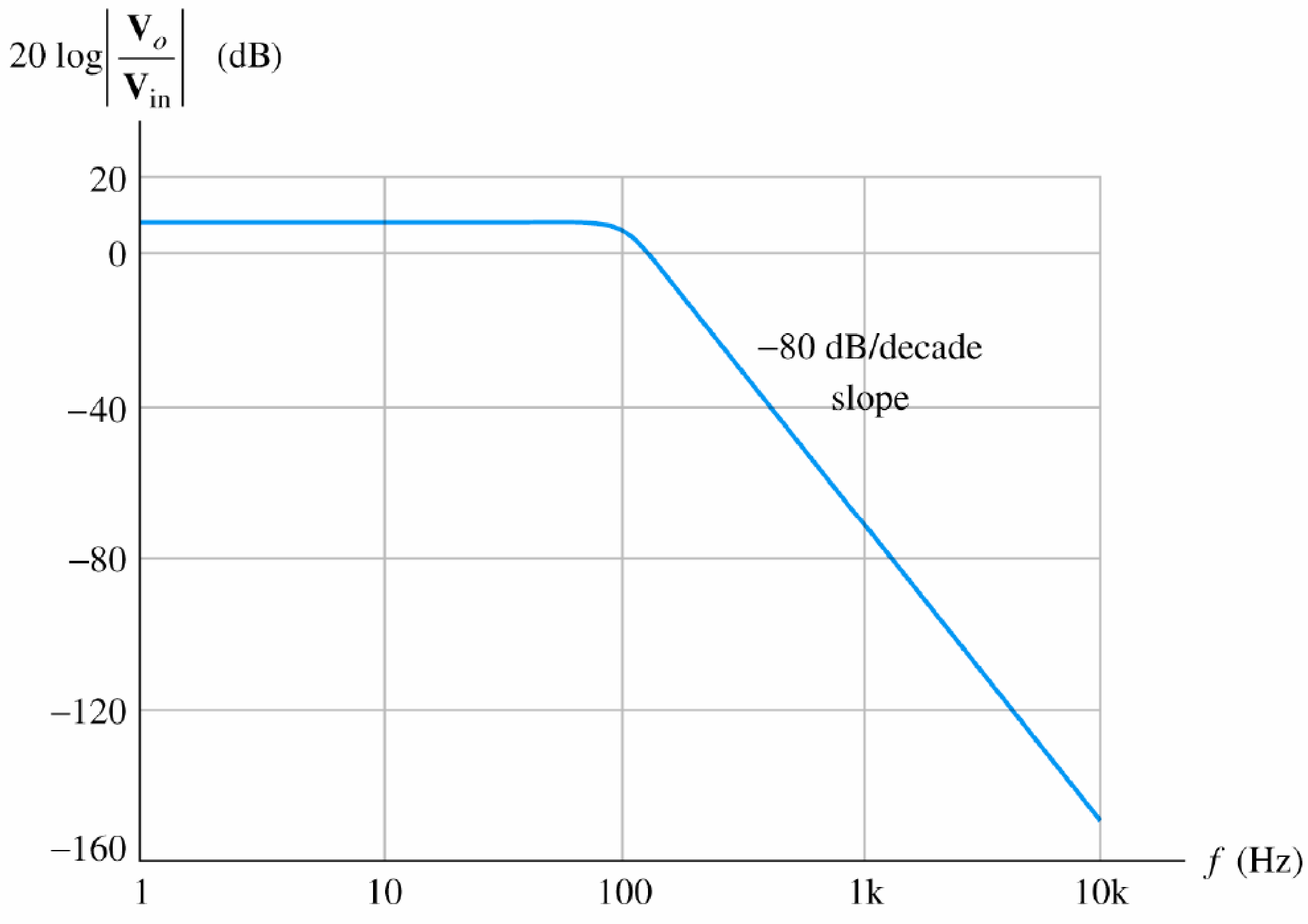
$C_1 = C_2 = C_{11} = C_{12} = 0.1\ \mu\text{F}$

$$\text{Choose C} = 0.1\mu\text{F}$$

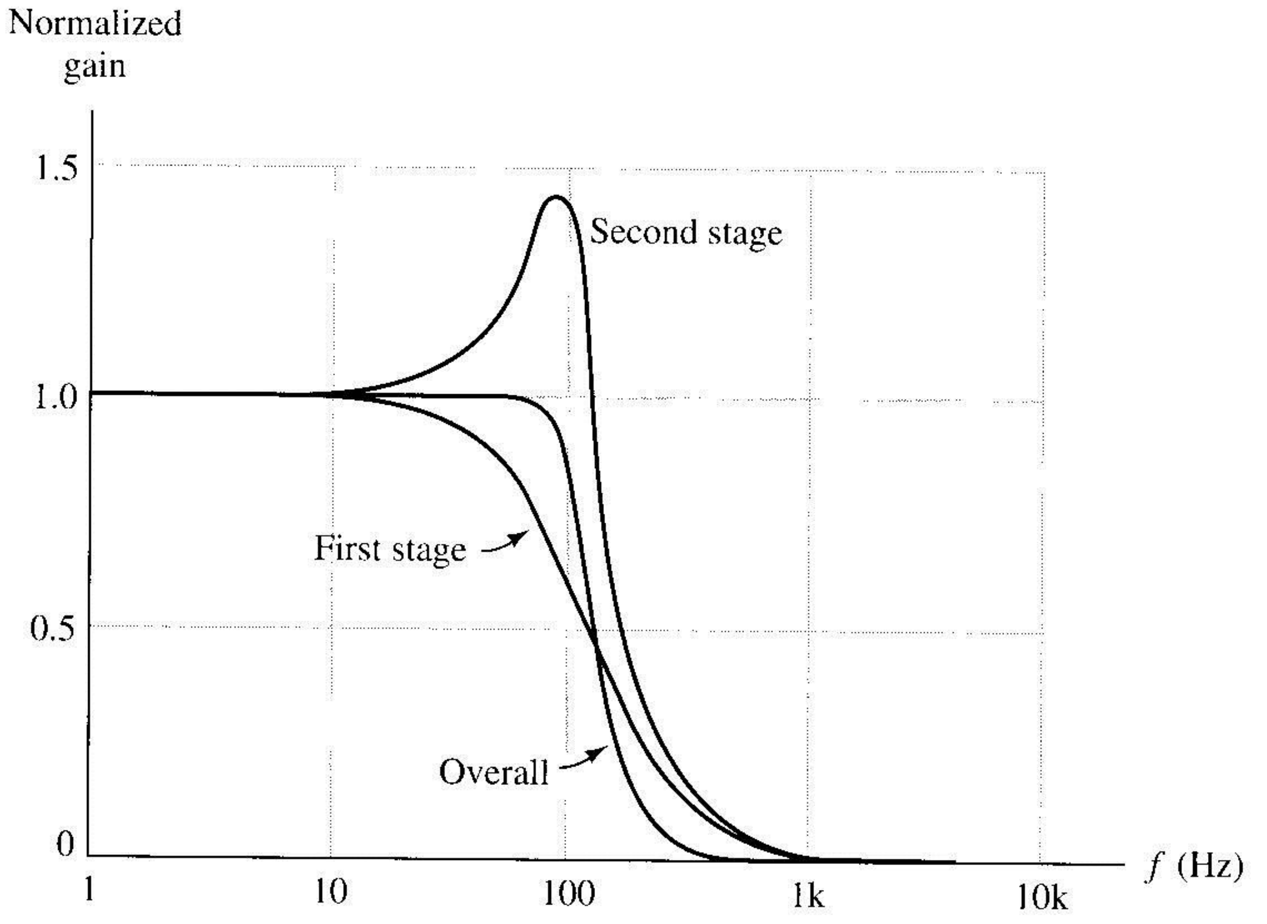
$$R = \frac{1}{2\pi C f_B} = \frac{1}{2\pi(0.1x10^{-6})(100Hz)}$$

$$= 15.92k\Omega$$

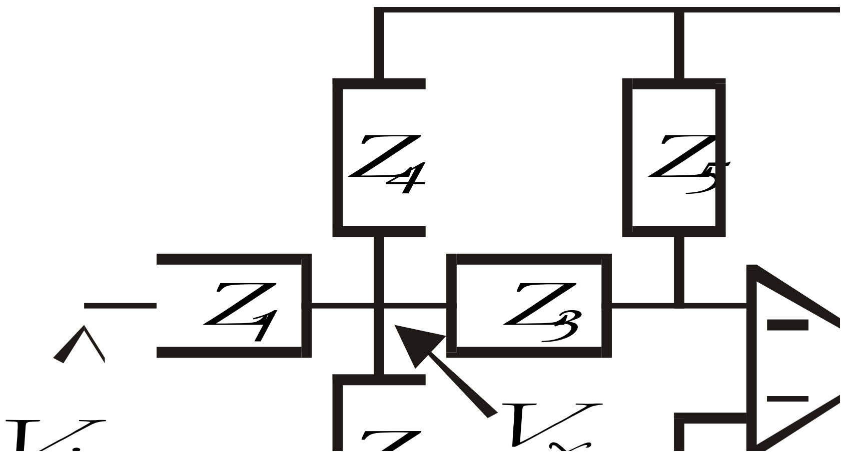
# Low-Pass Active Filter Design



# Low-Pass Active Filter Design



# Infinite-Gain Multiple-Feedback (IGMF) Negative Feedback Active Filter




$$v_i^+ = 0 \quad v_i^- = 0 \quad \Rightarrow \quad V_o = -\frac{Z_5}{Z_3} V_x \Rightarrow V_x = -\frac{Z_3}{Z_5} V_o \qquad (1)$$

By KCL at $V_x$,

$$\frac{V_i - V_x}{Z_1} = \frac{V_x}{Z_2} + \frac{V_x}{Z_3} + \frac{V_x - V_o}{Z_4} \qquad (2)$$

Substitute (1) into (2) gives

$$\frac{V_i}{Z_1} + \frac{Z_3}{Z_1 Z_5} V_o = -\frac{Z_3}{Z_5 Z_2} V_o - \frac{V_o}{Z_5} - \frac{Z_3}{Z_4 Z_5} V_o - \frac{V_o}{Z_4} \qquad (3)$$

rearranging equation (3), it gives,

$$H = \frac{V_o}{V_i} = -\frac{\dfrac{1}{Z_1 Z_3}}{\dfrac{1}{Z_5}\left(\dfrac{1}{Z_1} + \dfrac{1}{Z_2} + \dfrac{1}{Z_3} + \dfrac{1}{Z_4}\right) + \dfrac{1}{Z_3 Z_4}}$$

Or in admittance form:

$$H = \frac{V_o}{V_i} = -\frac{Y_1 Y_3}{Y_5\left(Y_1 + Y_2 + Y_3 + Y_4\right) + Y_3 Y_4}$$

| Filter \ Value | $Z_1$ | $Z_2$ | $Z_3$ | $Z_4$ | $Z_5$ |
|---|---|---|---|---|---|
| **LP** | $R_1$ | $C_2$ | $R_3$ | $R_4$ | $C_5$ |
| **HP** | $C_1$ | $R_2$ | $C_3$ | $C_4$ | $R_5$ |
| **BP** | $R_1$ | $R_2$ | $C_3$ | $C_4$ | $R_5$ |

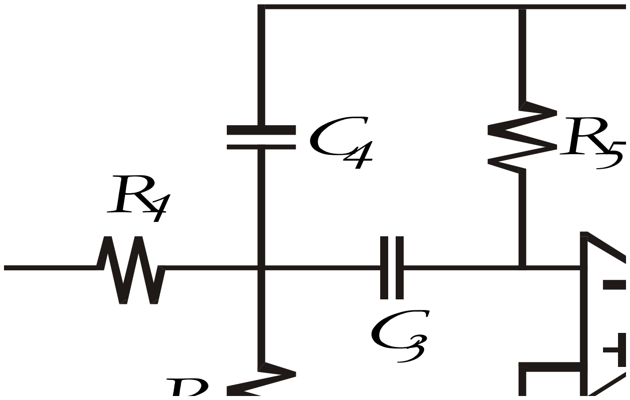
# IGMF Band-Pass Filter

Band-pass: $$H(s) = K\frac{s}{s^2 + as + b}$$

To obtain the band-pass response, we let

$$Z_1 = R_1 \quad Z_2 = R_2 \quad Z_3 = \frac{1}{j\omega C_3} = \frac{1}{sC_3} \quad Z_4 = \frac{1}{j\omega C_4} = \frac{1}{sC_4} \quad Z_5 = R_5$$

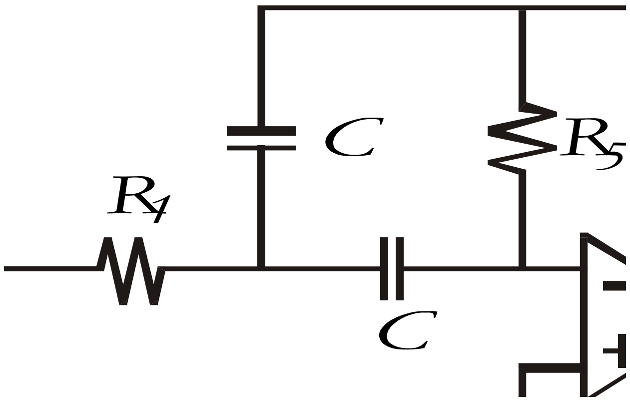
$$H(s) = -\frac{\dfrac{sC_3}{R_1}}{s^2 C_3 C_4 + s\dfrac{C_3 + C_4}{R_5} + \dfrac{1}{R_5}\left(\dfrac{1}{R_1} + \dfrac{1}{R_2}\right)}$$




This filter prototype has a very low sensitivity to component tolerance when compared with other prototypes.

# Simplified Design (IGMF Filter)

$$H(s) = -\frac{\frac{sC}{R_1}}{\frac{1}{R_1 R_5} + s\frac{2C}{R_5} + s^2 C^2}$$




Comparing with the band-pass response

$$H(s) = K\frac{s}{s^2 + \frac{\omega_P}{Q_P}s + \omega_P^2}$$

It gives,

$$\omega_p = \frac{1}{C\sqrt{R_1 R_5}} \qquad Q_p = \frac{1}{2}\sqrt{\frac{R_5}{R_1}} \qquad H(\omega_p) = -2Q^2$$

# Example: IGMF Band Pass Filter

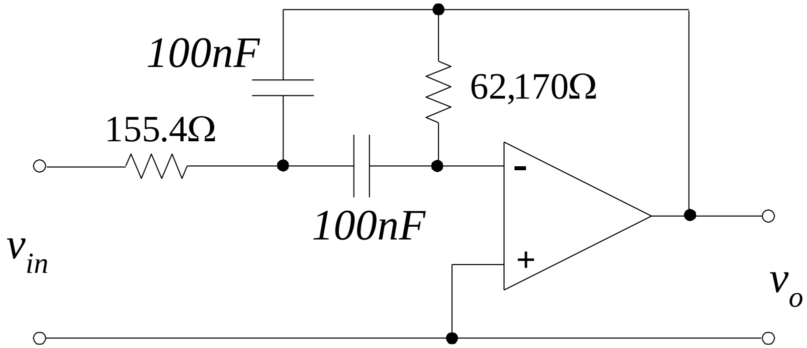
To design a band-pass filter with $f_O = 512Hz$ and $Q = 10$

$$\omega_P = \frac{1}{C\sqrt{R_1 R_5}} = 2\pi(512Hz)$$

$$C = 100\,nF \quad \rightarrow \quad R_1 R_5 = 9{,}662{,}741\Omega^2$$

$$Q_P = \frac{1}{2}\frac{\sqrt{R_5}}{\sqrt{R_1}} = 10$$

$$\rightarrow R_1 = 155.4\Omega \qquad R_5 = 62{,}170\Omega$$




With similar analysis, we can choose the following values:

$$C = 10\,nF \quad R_1 = 1{,}554\,\Omega \quad \text{and} \quad R_5 = 621{,}700\,\Omega$$

# Butterworth Response (Maximally Flat)

$$\left|H\left(j\omega\right)\right| = \frac{1}{\sqrt{1+\left(\dfrac{\omega}{\omega_o}\right)^{2n}}}$$

where n is the order

Normalize to $\omega_o$ = *1rad/s*

$$\left|\hat{H}(j\omega)\right| = \frac{1}{\sqrt{1+\omega^{2n}}}$$

Butterworth polynomials:

$$\left|\hat{H}(j\omega)\right| = \frac{1}{\left|B_n(j\omega)\right|}$$

Butterworth polynomials

$$B_1(s) = s+1$$

$$B_2(s) = s^2+\sqrt{2}s+1$$

$$B_3(s) = s^3+2s^2+2s+1$$
$$= (s+1)(s^2+s+1)$$

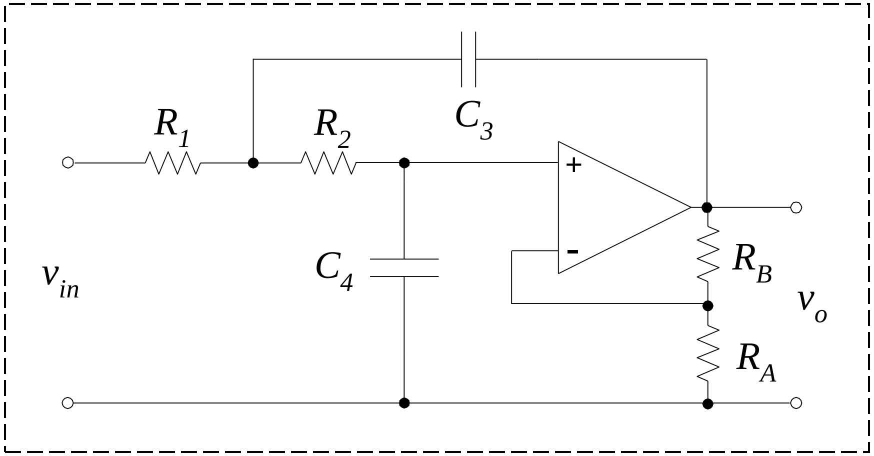
$$B_4(s) = s^4+2.61s^3+3.41s^2+2.61s+1$$
$$= (s^2+0.77s+1)(s^2+1.85s+1)$$

$$B_5(s) = s^5+3.24s^4+5.24s^3+5.24s^2+3.24s+1$$
$$= (s+1)(s^2+0.62s+1)(s^2+1.62s+1)$$

# Second order Butterworth Filter

$$H(s)=\frac{K}{1+sC_4\left(R_1+R_2\right)+sR_1C_3\left(1-K\right)+s^2R_1R_2C_3C_4}=\frac{K'}{s^2+\dfrac{\omega_P}{Q_P}s+\omega_P^2}$$

$$Q_P=\frac{1}{\sqrt{\dfrac{R_1C_4}{R_2C_3}}+\sqrt{\dfrac{R_2C_4}{R_1C_3}}+\left(1-K\right)\sqrt{\dfrac{R_1C_3}{R_2C_4}}}$$




Setting $R_1 = R_2$ and $C_1 = C_2$

$$Q_P=\frac{1}{\sqrt{1}+\sqrt{1}+\left(1-K\right)\sqrt{1}}=\frac{1}{2+\left(1-K\right)}=\frac{1}{3-K}$$

Now $K = 1 + R_B / R_A$

$$Q_P=\frac{1}{3-K}=\frac{1}{3-\left(1+\dfrac{R_B}{R_A}\right)}=\frac{1}{2-\dfrac{R_B}{R_A}}$$

For Butterworth response:

$$Q_P=\frac{1}{\sqrt{2}} \quad\Rightarrow\quad Q_P=\frac{1}{\sqrt{2}}=\frac{1}{2-\dfrac{R_B}{R_A}}$$

We have $$2-\frac{R_B}{R_A}=\sqrt{2}=1.414$$

We define Damping Factor (DF) as:

$$DF=\frac{1}{Q_P}=2-\frac{R_B}{R_A}=1.414$$