ELG3336: Op Amp-based Active Filters

• **Advantages:**
  – Reduced size and weight, and therefore parasitics.
  – Increased reliability and improved performance.
  – Simpler design than for passive filters and can realize a wider range of functions as well as providing voltage gain.
  – In large quantities, the cost of an IC is less than its passive counterpart.

• **Disadvantages:**
  – Limited bandwidth of active devices limits the highest attainable pole frequency and therefore applications above 100 kHz (passive RLC filters can be used up to 500 MHz).
  – The achievable quality factor is also limited.
  – Require power supplies (unlike passive filters).
  – Increased sensitivity to variations in circuit parameters caused by environmental changes compared to passive filters.

• For applications, particularly in voice and data communications, the economic and performance advantages of active RC filters far outweigh their disadvantages.
A low-pass filter with a dc gain of $-R_f/R_i$. 

\[
H(f) = \frac{V_o}{V_i} = -\frac{Z_f}{Z_i}
\]

\[
\frac{1}{Z_f} = \frac{1}{R_f} + \frac{1}{1 + j2\pi f R_f C_f}
\]

\[
Z_f = \frac{R_f}{1 + j2\pi f R_f C_f}
\]

\[
H(f) = -\frac{Z_f}{Z_i} = -\left(\frac{R_f}{R_i}\right)\frac{1}{1 + j2\pi f R_f C_f}
\]

\[
= -\left(\frac{R_f}{R_i}\right)\left[\frac{1}{1 + j(f/f_B)}\right]
\]

\[
f_B = \frac{1}{2\pi R_f C_f}
\]
\[ v_o(t) = -\frac{1}{RC} \int_0^t v_{in}(t) \, dt \]
A high-pass filter with a high frequency gain of $-R_f/R_i$.
Higher Order Filters

\[
H(f) = H_1(f)H_2(f)\cdots H_n(f) \\
= (-1)^n \left( \frac{R_f}{R_i} \right)^n \left[ \frac{1}{1 + j(f / f_B)} \right]^n
\]
Single-Pole Active Filter Designs

High Pass

\[ \frac{v_{out}}{v_{in}} = \frac{1}{1/sCR + 1} = \frac{1}{1 + sRC} \]

\[ = \frac{sRC}{RC(s + 1/RC)} = \frac{s}{(s + 1/RC)} \]

Low Pass

\[ \frac{v_{out}}{v_{in}} = \frac{1/RC}{s + 1/RC} \]
Two-Pole (Sallen-Key) Filters

Low Pass Filter

High Pass Filter
Three-Pole Low-Pass Filter
Two-Stage Band-Pass Filter

Stage 1
Two-pole low-pass

Stage 2
Two-pole high-pass

$BW = f_2 - f_1$

$Q = f_0 / BW$
Multiple-Feedback Band-Pass Filter
Transfer function $H(j\omega)$

\[ H(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} \]

\[ H = \text{Re}(H) + j\text{Im}(H) \]

\[ |H| = \sqrt{\text{Re}(H)^2 + \text{Im}(H)^2} \]

\[ \angle H = \tan^{-1}\left(\frac{\text{Im}(H)}{\text{Re}(H)}\right) \quad \text{Re}(H) > 0 \]

\[ \angle H = 180^\circ + \tan^{-1}\left(\frac{\text{Im}(H)}{\text{Re}(H)}\right) \quad \text{Re}(H) < 0 \]
Corner Frequency

• The significance of the break frequency is that it represents the frequency where

\[ A_v(f) = 0.707 \angle -45^\circ \]

• This is where the output of the transfer function has an amplitude 3-\(dB\) below the input amplitude, and the output phase is shifted by \(-45^\circ\) relative to the input.

• Therefore, \(f_c\) is also known as the 3-\(dB\) frequency or the corner frequency.
Bode plots use a logarithmic scale for frequency, where a *decade* is defined as a range of frequencies where the highest and lowest frequencies differ by a factor of 10.

20 \log_{10} |A| = \text{Gain in dB}

\[ f_c = \text{Cut-off Frequency} \]

\[ \text{Gain at 3dB point (at } f_c \text{)} = \frac{|A|}{\sqrt{2}} \]

- Ripple in pass band causes non-linearity
- Possible to design with no ripple
- Ripple in stop band is less important
- Fall off dB / Decade (Gain in dB / Decade of f)
- Stop band attenuates (say -40dB)
Real Filters

• Butterworth Filters
  – Flat Pass-band.
  – $20n$ dB per decade roll-off.

• Chebyshev Filters
  – Pass-band ripple.
  – Sharper cut-off than Butterworth.

• Elliptic Filters
  – Pass-band and stop-band ripple.
  – Even sharper cut-off.

• Bessel Filters
  – Linear phase response – i.e. no signal distortion in pass-band.
Butterworth Filters

The magnitude response of a Butterworth filter.

Magnitude response for Butterworth filters of various order with $\varepsilon = 1$. Note that as the order increases, the response approaches the ideal brickwall type transmission.
Sketches of the transmission characteristics of a representative even- and odd-order Chebyshev filters.
# First-Order Filter Functions

| Filter Type and $T(s)$ | $s$-Plane Singularities | Bode Plot for $|T|$ | Passive Realization | Op Amp-RC Realization |
|------------------------|-------------------------|---------------------|---------------------|----------------------|
| (a) Low-Pass (LP) | ![s-plane singularity](image) | ![Bode plot](image) | $CR = \frac{1}{\omega_0}$ | ![Op Amp-RC circuit](image) |
| $T(s) = \frac{a_0}{s + \omega_0}$ | $j\omega$ | $20 \log \frac{a_0}{\omega_0}$ | dc gain = 1 | dc gain = $-\frac{R_2}{R_1}$ |
| (b) High-Pass (HP) | ![s-plane singularity](image) | ![Bode plot](image) | $CR = \frac{1}{\omega_0}$ | ![Op Amp-RC circuit](image) |
| $T(s) = \frac{a_1 s}{s + \omega_0}$ | $j\omega$ | $20 \log |a_1|$ | High-frequency gain = 1 | High-frequency gain = $-\frac{R_2}{R_1}$ |
| (c) General | ![s-plane singularity](image) | ![Bode plot](image) | $\frac{(C_1 + C_2) (R_1//R_2)}{\omega_0}$ | ![Op Amp-RC circuit](image) |
| $T(s) = \frac{a_1 s + a_0}{s + \omega_0}$ | $j\omega$ | $20 \log \frac{a_0}{\omega_0}$ | $C_1 R_1 = \frac{a_0}{a_1}$ | $C_1 R_1 = \frac{a_1}{a_0}$ |
| | | | dc gain = $\frac{R_2}{R_1 + R_2}$ | dc gain = $-\frac{R_2}{R_1}$ |
| | | | HF gain = $\frac{C_1}{C_1 + C_2}$ | HF gain = $-\frac{C_1}{C_2}$ |
# First-Order Filter Functions

| $T(s)$ | Singularities | $|T|$ and $\phi$ | Passive Realization | Op Amp-RC Realization |
|--------|---------------|-----------------|---------------------|----------------------|
| $T(s) = -a_1 \frac{s - \omega_0}{s + \omega_0}$ | ![Singularity Diagram] | ![Magnitude and Phase Diagram] | ![Passive Realization Diagram] | ![Op Amp-RC Realization Diagram] |
| $a_1 > 0$ | ![Singularities Diagram] | ![Magnitude Graph] 20 log $|a_1|$ | ![Passive Circuit] $CR = 1/\omega_0$ Flat gain ($a_1$) = 0.5 | ![Op Amp-RC Circuit] $CR = 1/\omega_0$ Flat gain ($a_1$) = 1 |
## Second-Order Filter Functions

| Filter Type and $T(s)$ | $s$-Plane Singularities | $|T| Functional Form |
|------------------------|-------------------------|---------------------|
| (a) Low-Pass (LP)      | $\frac{\omega_0}{Q}$   | $\frac{|a_0|}{\omega_0^2 \sqrt{1 - \frac{1}{4Q^2}}}$ |
| $T(s) = \frac{a_0}{s^2 + s \frac{\omega_0}{Q} + \omega_0^2}$ | $\omega_0$ at $\infty$ | $\frac{|a_0|}{\omega_0^2}$ |
| dc gain = $\frac{a_0}{\omega_0}$ | $\frac{\omega_0}{2Q}$ | $\omega_{max} = \frac{\omega_0}{\sqrt{1 - \frac{1}{2Q^2}}}$ |

| (b) High-Pass (HP)     | $\frac{\omega_0}{Q}$   | $\frac{|a_2|}{\sqrt{1 - \frac{1}{4Q^2}}}$ |
| $T(s) = \frac{a_2 s^2}{s^2 + s \frac{\omega_0}{Q} + \omega_0^2}$ | $\omega_0$ at $\infty$ | $\omega_{max} = \frac{\omega_0}{\sqrt{1 - \frac{1}{2Q^2}}}$ |
| High-frequency gain = $\frac{\omega_0}{2Q}$ | $\frac{\omega_0}{2Q}$ | |

| (c) Bandpass (BP)      | $\frac{\omega_0}{Q}$   | $\frac{(a_1 Q)}{\omega_0}$ |
| $T(s) = \frac{a_1 s}{s^2 + s \frac{\omega_0}{Q} + \omega_0^2}$ | $\omega_0$ at $\infty$ | $\omega_{max} = \frac{\omega_0}{\sqrt{1 + \frac{1}{4Q^2} + \frac{\omega_0^2}{2Q}}}$ |
| Center-frequency gain = $\frac{a_1 Q}{\omega_0}$ | $\frac{\omega_0}{2Q}$ | $\omega_1 \omega_2 = \omega_0^2$ |

0.707 $T_{max}$ = $\frac{(a_1 Q)}{\sqrt{2 \omega_0}}$
# Second-Order Filter Functions

## (d) Notch

The transfer function for the notch filter is given by:

\[ T(s) = a_2 \frac{s^2 + \omega_n^2}{s^2 + s \frac{\omega_0}{Q} + \omega_0^2} \]

- **dc gain**: \( \frac{\omega_0}{2Q} \)
- **high-frequency gain**: \( a_2 \)

![Notch Filter Diagram](image)

## (e) Low-Pass Notch (LPN)

The transfer function for the low-pass notch filter is given by:

\[ T(s) = a_2 \frac{s^2 + \omega_n^2}{s^2 + s \frac{\omega_0}{Q} + \omega_0^2} \]

- **\( \omega_n \)** \( \geq \omega_0 \)
- **dc gain**: \( a_2 \frac{\omega_n^2}{\omega_0^2} \)
- **high-frequency gain**: \( a_2 \)

![Low-Pass Notch Filter Diagram](image)

## (f) High-Pass Notch (HPN)

The transfer function for the high-pass notch filter is given by:

\[ T(s) = a_2 \frac{s^2 + \omega_n^2}{s^2 + s \frac{\omega_0}{Q} + \omega_0^2} \]

- **\( \omega_n \)** \( \leq \omega_0 \)
- **dc gain**: \( a_2 \frac{\omega_n^2}{\omega_0^2} \)
- **high-frequency gain**: \( a_2 \)

![High-Pass Notch Filter Diagram](image)
Second-Order Filter Functions

(g) All-Pass (AP)

\[ T(s) = a_2 \frac{s^2 - \frac{\omega_0}{Q} s + \frac{\omega_0^2}{Q}}{s^2 + s \frac{\omega_0}{Q} + \frac{\omega_0^2}{Q}} \]

Flat gain = \( a_2 \)
Second-Order LCR Resonator

(a) General structure
(b) LP
(c) HP
(d) BP
(e) Notch at $\omega_0$
(f) General notch
(g) LPN ($\omega_n > \omega_0$
(h) LPN as $s \to \infty$
(i) HPN ($\omega_n < \omega_0$)
Second-Order Active Filter: Two-Integrator-Loop

Derivation of an alternative two-integrator-loop biquad in which all op amps are used in a single-ended fashion. The resulting circuit in (b) is known as the Tow-Thomas biquad.
Low-Pass Active Filter Design

Design a fourth-order low-pass Butterworth filter having a frequency cut-off of 100 Hz.

Choose \( C = 0.1 \mu F \)

\[
R = \frac{1}{2\pi C f_b} = \frac{1}{2\pi (0.1 \times 10^{-6}) (100 \text{Hz})} = 15.92 \text{k}\Omega
\]
Low-Pass Active Filter Design

\[ 20 \log \left| \frac{V_o}{V_{in}} \right| \text{ (dB)} \]

-80 dB/decade slope

\( f \) (Hz)
Low-Pass Active Filter Design

Normalized gain

Second stage

First stage

Overall

$f$ (Hz)
Infinite-Gain Multiple-Feedback (IGMF)  
Negative Feedback Active Filter

\[ v_i^+ = 0 \quad v_i^- = 0 \quad \Rightarrow \quad V_o = -\frac{Z_5}{Z_3} V_x \quad \Rightarrow \quad V_x = -\frac{Z_3}{Z_5} V_o \]  
(1)

By KCL at \( V_x \),

\[ \frac{V_i - V_x}{Z_1} = \frac{V_x}{Z_2} + \frac{V_x}{Z_3} + \frac{V_x - V_o}{Z_4} \]  
(2)

Substitute (1) into (2) gives

\[ \frac{V_i}{Z_1} + \frac{Z_3}{Z_1 Z_5} V_o = -\frac{Z_3}{Z_5 Z_2} V_o - \frac{V_o}{Z_5} - \frac{Z_3}{Z_4 Z_5} V_o - \frac{V_o}{Z_4} \]  
(3)
rearranging equation (3), it gives,

\[
H = \frac{V_o}{V_i} = -\frac{1}{Z_1Z_3} \left( \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \frac{1}{Z_4} \right) + \frac{1}{Z_3Z_4}
\]

Or in admittance form:

\[
H = \frac{V_o}{V_i} = -\frac{Y_1Y_3}{Y_5\left(Y_1 + Y_2 + Y_3 + Y_4\right) + Y_3Y_4}
\]

<table>
<thead>
<tr>
<th>Filter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP</td>
<td>( R_1 ) ( C_2 ) ( R_3 ) ( R_4 ) ( C_5 )</td>
</tr>
<tr>
<td>HP</td>
<td>( C_1 ) ( R_2 ) ( C_3 ) ( C_4 ) ( R_5 )</td>
</tr>
<tr>
<td>BP</td>
<td>( R_1 ) ( R_2 ) ( C_3 ) ( C_4 ) ( R_5 )</td>
</tr>
</tbody>
</table>
IGMF Band-Pass Filter

Band-pass: \[ H(s) = K \frac{s}{s^2 + as + b} \]

To obtain the band-pass response, we let

\[ Z_1 = R_1 \quad Z_2 = R_2 \quad Z_3 = \frac{1}{j\omega C_3} = \frac{1}{sC_3} \quad Z_4 = \frac{1}{j\omega C_4} = \frac{1}{sC_4} \quad Z_5 = R_5 \]

\[ H(s) = -\frac{sC_3}{R_1} \frac{1}{s^2 C_3 C_4 + s \left( \frac{C_3 + C_4}{R_5} \right) + \frac{1}{R_5} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)} \]

This filter prototype has a very low sensitivity to component tolerance when compared with other prototypes.
Simplified Design (IGMF Filter)

\[ H(s) = -\frac{sC}{R_1} \left( \frac{1}{R_1R_5} + s \frac{2C}{R_5} + s^2 C^2 \right) \]

Comparing with the band-pass response

\[ H(s) = K \frac{s}{s^2 + \frac{\omega_p}{Q_p} s + \omega_p^2} \]

It gives,

\[ \omega_p = \frac{1}{C \sqrt{R_1R_5}} \quad Q_p = \frac{1}{2} \sqrt{\frac{R_5}{R_1}} \quad H(\omega_p) = -2Q^2 \]
Example: IGMF Band Pass Filter

To design a band-pass filter with \( f_o = 512 \text{Hz} \) and \( Q = 10 \)

\[
\omega_p = \frac{1}{C \sqrt{R_1 R_5}} = 2\pi(512 \text{Hz})
\]

\[
C = 100 \text{nF} \quad \rightarrow \quad R_1 R_5 = 9,662,741 \Omega^2
\]

\[
Q_p = \frac{1}{\frac{\sqrt{R_5}}{2 \sqrt{R_1}}} = 10
\]

\[
\rightarrow R_1 = 155.4 \Omega \quad R_5 = 62,170 \Omega
\]

With similar analysis, we can choose the following values:

\[
C = 10 \text{nF} \quad R_1 = 1,554 \Omega \quad \text{and} \quad R_5 = 621,700 \Omega
\]
Butterworth Response (Maximally Flat)

\[ |H(j\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_o}\right)^{2n}}} \]

where \( n \) is the order

Normalize to \( \omega_o = 1 \text{rad/s} \)

\[ |\hat{H}(j\omega)| = \frac{1}{\sqrt{1 + \omega^{2n}}} \]

Butterworth polynomials:

\[ B_1(s) = s + 1 \]
\[ B_2(s) = s^2 + \sqrt{2}s + 1 \]
\[ B_3(s) = s^3 + 2s^2 + 2s + 1 \]
\[ = (s + 1)(s^2 + s + 1) \]
\[ B_4(s) = s^4 + 2.61s^3 + 3.41s^2 + 2.61s + 1 \]
\[ = (s^2 + 0.77s + 1)(s^2 + 1.85s + 1) \]
\[ B_5(s) = s^5 + 3.24s^4 + 5.24s^3 + 5.24s^2 + 3.24s + 1 \]
\[ = (s + 1)(s^2 + 0.62s + 1)(s^2 + 1.62s + 1) \]
Second order Butterworth Filter

\[ H(s) = \frac{K}{1 + sC_4(R_1 + R_2) + sR_1C_3(1 - K) + s^2R_1R_2C_3C_4} = \frac{K'}{s^2 + \frac{\omega_p}{Q_p}s + \omega_p^2} \]

\[ Q_p = \frac{1}{\sqrt{R_1C_4} + \sqrt{R_2C_4} + (1 - K)\sqrt{R_1C_3}} \]

Setting \( R_1 = R_2 \) and \( C_1 = C_2 \)

\[ Q_p = \frac{1}{\sqrt{1 + \sqrt{1 + (1 - K)\sqrt{1}}} = \frac{1}{2 + (1 - K)} = \frac{1}{3 - K} \]

Now \( K = 1 + \frac{R_B}{R_A} \)

\[ Q_p = \frac{1}{3 - K} = \frac{1}{3 - \left(1 + \frac{R_B}{R_A}\right)} = \frac{1}{2 - \frac{R_B}{R_A}} \]

For Butterworth response:

\[ Q_p = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad Q_p = \frac{1}{\sqrt{2}} = \frac{1}{2 - \frac{R_B}{R_A}} \]

We have \( 2 - \frac{R_B}{R_A} = \sqrt{2} = 1.414 \)

We define Damping Factor (DF) as:

\[ DF = \frac{1}{Q_p} = 2 - \frac{R_B}{R_A} = 1.414 \]