ELG3150 Tutorial for the Midterm

1. A system is described by the state variable equations

$$\dot{x} = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 5 & 0 \\ -6 & 1 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}$$

Determine *G(s) = Y(s)/U(s)*.

2. Determine the state variable matrix differential equation for the circuit shown in Fig. P2. The state variables are $x_1 = i, x_2 = v_1$, and $x_3 = v_2$. The input variable is *v(t)* and the output variable is $v_0(t)$.

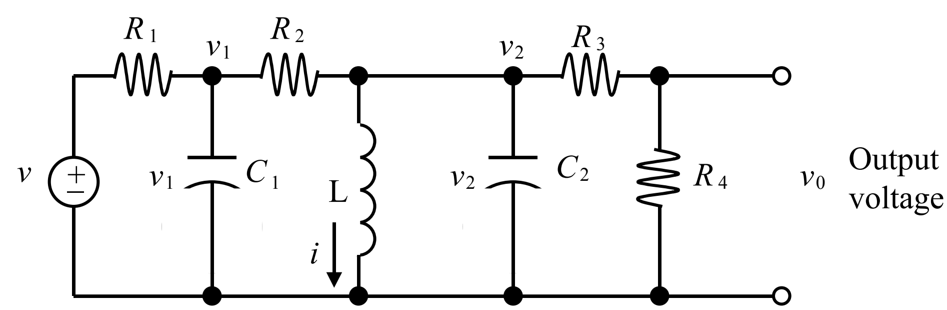


Figure P2. RLC circuit.

3. The speed control of a high-speed train is represented by the system shown in Fig. P3. Determine the equation for steady-state error for *K* for a unit step input *r(t)*. Consider the three values for *K* equal to 1, 10 and 100.

   (a) Determine the system transfer function in terms of *K*.
   (b) Determine and plot the response *y(t)* for (i) a unit step input *r(t)* and (ii) a unit step disturbance input *d(t)* for the three values of *K*.
   (c) Create a table showing overshoot, settling time (2% criterion), e_{ss} for *r(t)*, and $|y/d|_{\max}$ for the three values of *K*. Select the best compromise value.

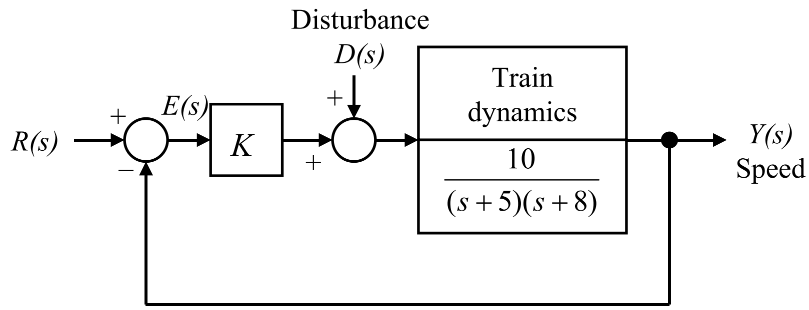


Figure P3. Speed Control.

4. The block diagram model of an armature-current-controlled dc motor is shown in Fig. 4.

   (a) Determine the steady-state error to a ramp input, $r(t) = t$, $t \geq 0$, in terms of K , K_b and K_m .
   (b) Let K_m =10 and K_b =0.054, and select K so that steady-state error is equal to 1.
   (c) Plot the responses to a unit step input and a unit ramp input for 20 seconds. Are the responses acceptable?

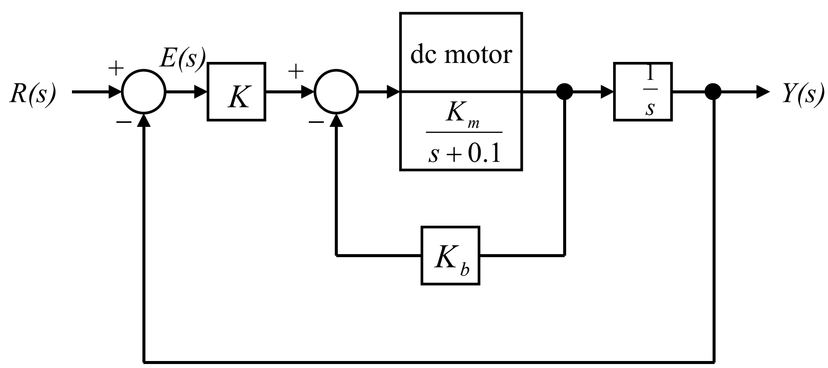


Figure P4. DC motor control.

Solutions

1. Solution

$$G(s) = C(sI - A)^{-1} B = \frac{s^2 - 9s + 4}{s^3 - 10s^2 + 21s + 16}$$

2. Solution

$$v_2 = L\frac{di}{dt}$$
$$0 = C_1\frac{dv_1}{dt} + \frac{v_1 - v}{R_1} + \frac{v_1 - v_2}{R_2}$$
$$0 = C_2\frac{dv_2}{dt} + \frac{v_2 - v_1}{R_2} + i + \frac{v_2}{R_3 + R_4}$$
$$v_0 = \frac{R_4}{R_3 + R_4}v_2$$

Let $x_1 = i, x_2 = v_1$, $x_3 = v_2$, $u = v$ and $y = v_0$. Then,

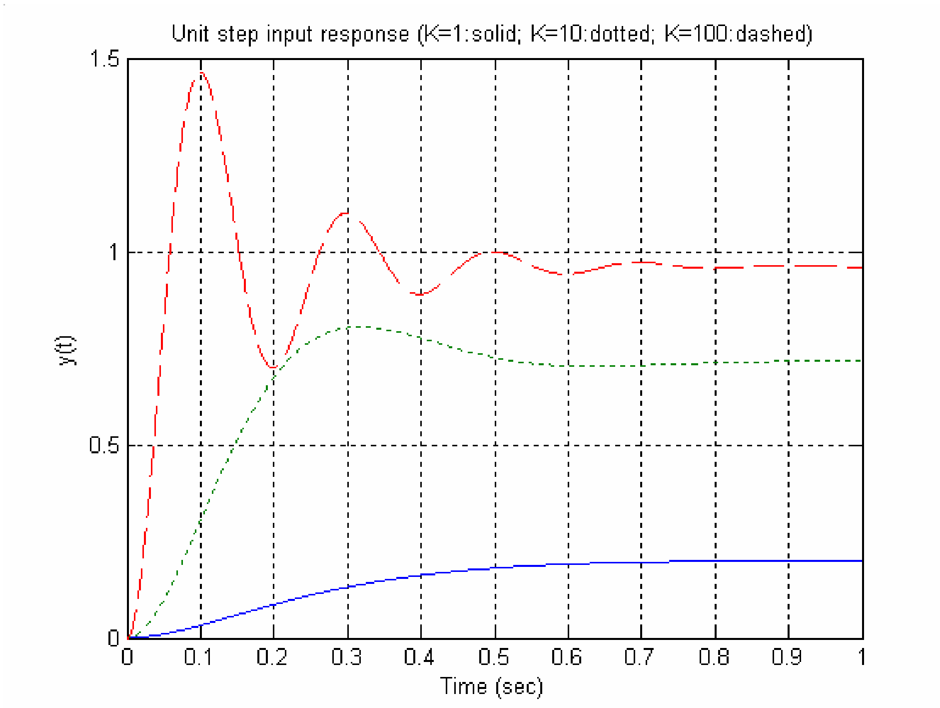
$$\dot{x} = \begin{bmatrix} 0 & 0 & \frac{1}{L} \\ 0 & -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) & \frac{1}{C_1 R_2} \\ -\frac{1}{C_2} & \frac{1}{C_2 R_2} & -\frac{1}{C_2}\left(\frac{1}{R_2} + \frac{1}{R_3 + R_4}\right) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{C_1 R_1} \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 0 & \frac{R_4}{R_3 + R_4} \end{bmatrix} x$$

3. Solution

The system transfer function is

$$Y(s) = \frac{10K}{(s+5)(s+8) + 10K}R(s) + \frac{10}{(s+5)(s+8) + 10K}D(s)$$

When considering the input response, we set *D(s)*=0, and similarly, when considering the disturbance response, we set *R(s)*=0. Te closed-loop step input and disturbance response for *K*=1, 10, 100 are shown in Figure P3. The performance results are summarized in Table P3. The best value of the gain is *K*=10, which is compromise between (i) percent overshoot, and (ii) disturbance rejection and tracking error.



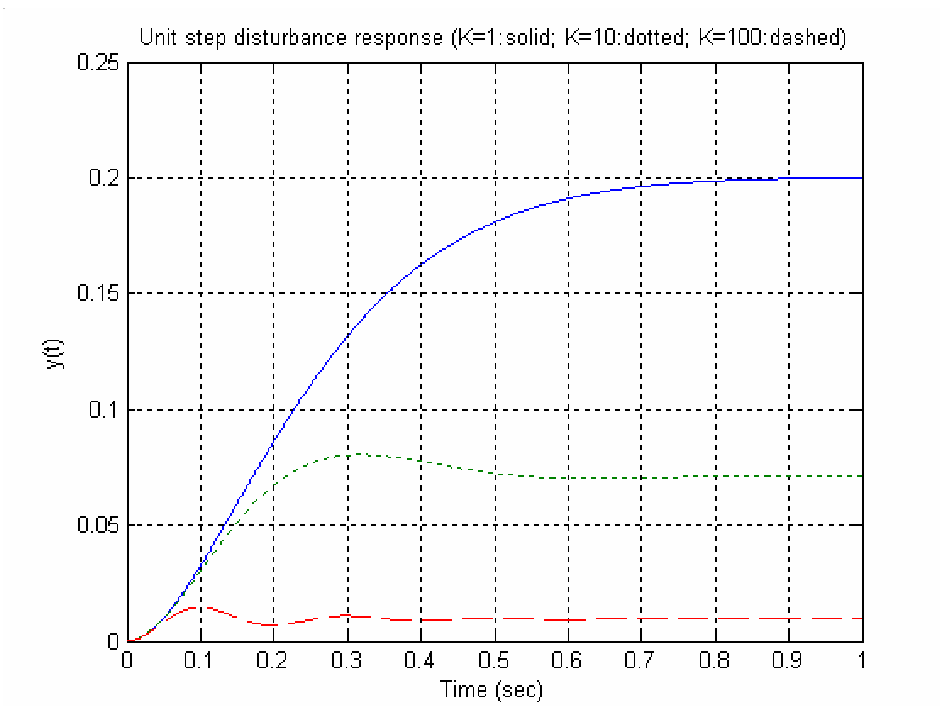


Figure P3. Closed-loop system input and disturbance responses (*K*=1: solid line, *K*=10: dotted line, and *K*=100: dashed line).

K	e_{ss}	T_s	$P.O.$	$\left\|y/d\right\|_{\max}$
1	0.8	0.615	0.0652	0.2
10	0.286	0.615	12.677	0.071
100	0.038	0.615	52.389	0.0096

Table P3. Performance summary

omega_n^2	omega_n	zeta	power	P.O.
50.0000	7.0711	0.9192	-7.3352	0.0652
140.0000	11.8322	0.5494	-2.0654	12.6767
1040.0000	32.2490	0.2016	-0.6465	52.3889

Table P3a. Spread sheet for $P.O.$ calculation

4. Solution

(a) The open-loop transfer function is

$$G(s) = \frac{KK_m}{s(s + 0.1 + K_b K_m)}$$

The closed-loop transfer function is

$$T(s) = \frac{KK_m}{s(s + 0.1 + K_b K_m) + KK_m}$$

The steady-state error tracking error for a ramp input $R(s) = 1/s^2$ is

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} \frac{KK_m s}{s(s + 0.1 + K_b K_m)} = \frac{KK_m}{0.1 + K_b K_m}$$

$$e_{ss} = \frac{1}{K_v} = \frac{0.1 + K_b K_m}{KK_m}$$

Another method to derive the steady-state error for a ramp input is

$$e_{ss} = \lim_{s\to 0} s[1 - T(s)]R(s)$$
$$= \lim_{s\to 0} \frac{s^2(s + 0.1 + K_b K_m)}{s(s + 0.1 + K_b K_m) + KK_m} \frac{1}{s^2}$$
$$= \frac{0.1 + K_b K_m}{KK_m}$$

(b) With K_m =10 and K_b =0.054, we have

$$\frac{0.1 + K_b K_m}{K K_m} = \frac{0.1 + 0.054(10)}{K(10)} = \frac{0.64}{10K} = 1$$

Solving for K yields

$$K = 0.064$$

Therefore, the closed-loop transfer function is

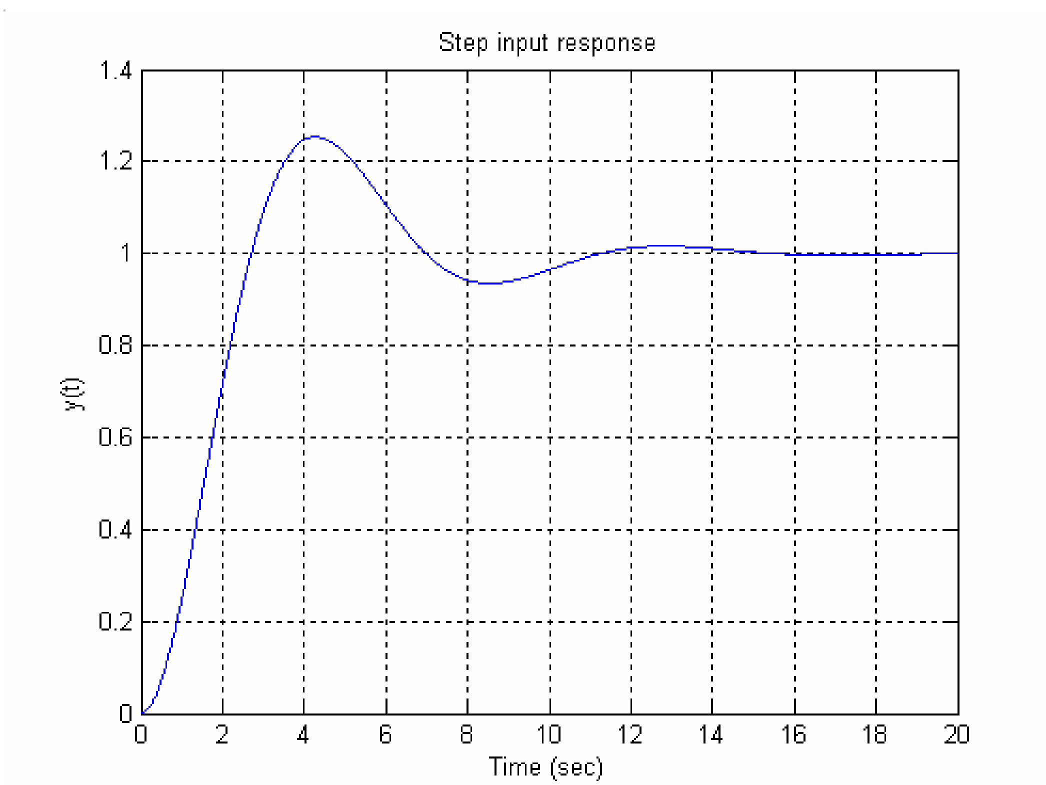
$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} = \frac{0.64}{s^2 + 0.64s + 0.64}$$

$\zeta = 0.4$ and $\omega_n = 0.8$

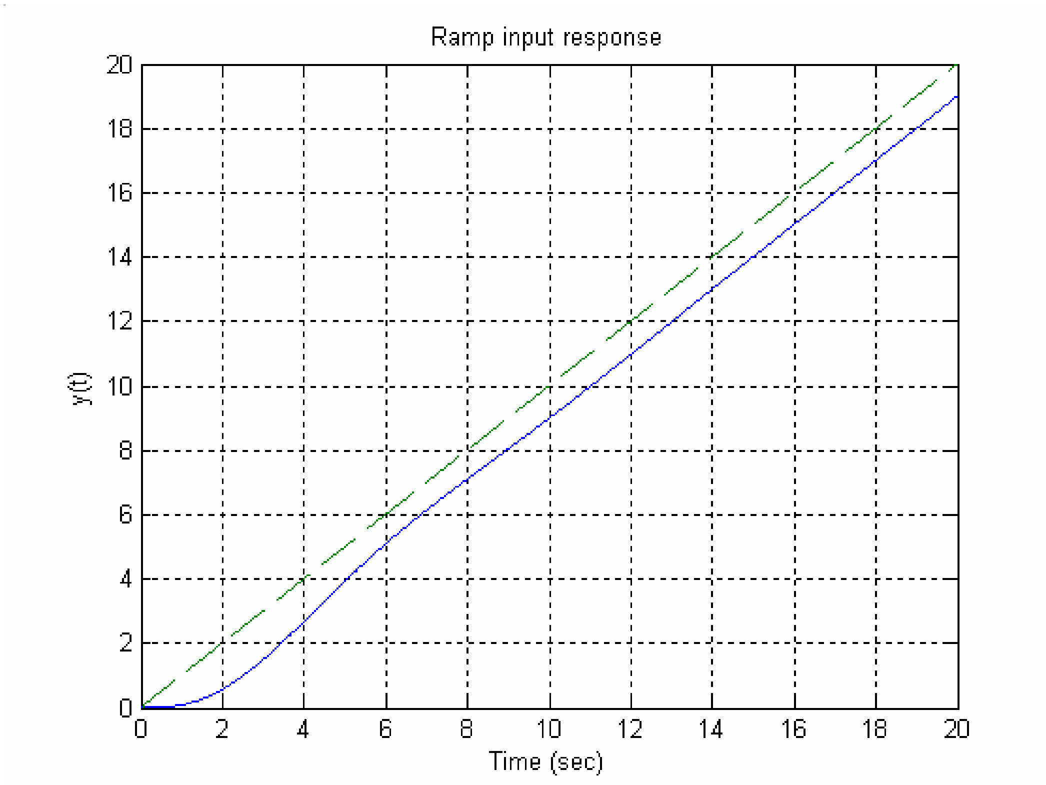
Thus, the overshoot and the settling time of the system are:

$P.O. = 25.4\%$ and $T_s = 4/0.32 = 12.5$ second

(c) The plot of a unit step input and unit ramp input are shown in Figure P4. The responses are acceptable.



(a)



(b)

Figure P4. Closed-loop system step and ramp responses.