ELG3150: Chapter 3

Problem E3.19

A single-input, single output system has the matrix equations

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 10 & 0 \end{bmatrix} X$$

Determine the transfer function $G(s) = \dfrac{Y(s)}{U(s)}$.

Solution

Take Laplace transform of both sides, to get

$$s\,X(s) = AX(s) + BU(s)$$
$$Y(s) = C\,X(s)$$

where A , B and C are given as follows

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 10 & 0 \end{bmatrix}$$

then

$$(sI - A)\,X(s) = BU(s)$$
$$X(s) = (sI - A)^{-1} BU(s)$$
$$Y(s) = C(sI - A)^{-1} BU(s)$$

Hence

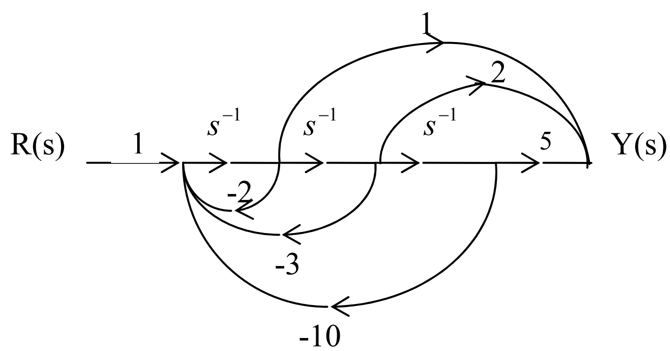
$$G(s) = C(sI - A)^{-1} B = \begin{bmatrix} 10 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 3 & s+4 \end{bmatrix}^{-1} . \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{10}{s^2 + 4s + 3}$$

Problem 3.4 (Solution)

$$T(s)=\frac{Y(s)}{R(s)}=\frac{5s^{-3}+2s^{-2}+s^{-1}}{10s^{-3}+3s^{-2}+2s^{-1}+1}$$

(a) Phase Variables

Note: In this set of exercises, we always take the output of integrators as state variables, with the convention that to start from the right hand side. So in the following figure, x_1 represents the output of the first integrator from right, and x_2 is the output of the middle one, and x_3 is the output of the left hand side integrator.



$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=x_3\\ \dot{x}_3=-10x_1-3x_2-2x_3+r(t)\\ y(t)=5x_1+2x_2+x_3\end{cases}$$

$$\dot{X}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -10 & -3 & -2\end{bmatrix}X+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}r(t)$$

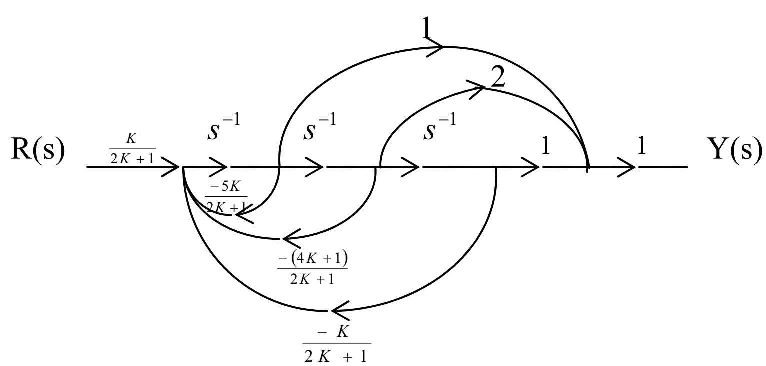
$$y(t)=\begin{bmatrix}5 & 2 & 1\end{bmatrix}X$$

Problem 3.7 (Solution)

(a)

$$\frac{Y(s)}{R(s)}=\frac{\frac{K}{s}G(s)}{1+\frac{K}{s}G(s)}=\frac{K\left(s^2+2s+1\right)}{(2K+1)s^3+5Ks^2+(4K+1)s+K}$$

$$\frac{Y(s)}{R(s)}=\frac{\frac{K}{2K+1}\left(s^{-3}+2s^{-2}+s^{-1}\right)}{\frac{K}{2K+1}s^{-3}+\frac{4K+1}{2K+1}s^{-2}+\frac{5K}{2K+1}s^{-1}+1}$$



$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=x_3\\ \dot{x}_3=\frac{-K}{2K+1}x_1-\frac{4K+1}{2K+1}x_2-\frac{5K}{2K+1}x_3+\frac{K}{2K+1}r\\ y(t)=\left(x_1+2x_2+x_3\right)\end{cases}$$

$$\dot{X}=\begin{bmatrix}0&1&0\\0&0&1\\ \frac{-K}{2K+1}&\frac{-(4K+1)}{2K+1}&\frac{-5K}{2K+1}\end{bmatrix}X+\begin{bmatrix}0\\0\\ \frac{K}{2K+1}\end{bmatrix}r(t)$$

$$y(t)=\begin{bmatrix}1&2&1\end{bmatrix}X$$

(b) Characteristic equation

$$\Delta(s) = (2K+1)s^3 + 5Ks^2 + (4K+1)s + K = 0 \qquad ; \quad K = 1$$

$$\Delta(s) = 3s^3 + 5s^2 + 5s + 1 = 0$$

$$\begin{cases} s_1 = -0.2551 \\ s_{2,3} = -0.7058 \pm 0.8991j \end{cases}$$

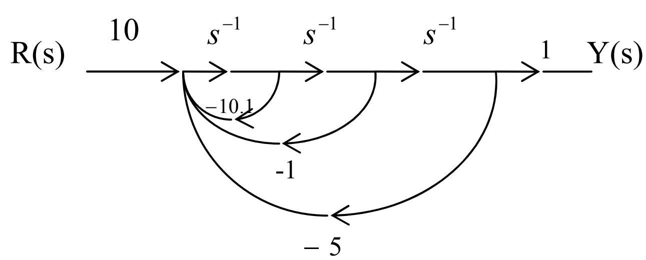
Problem 3.9 (Solution)

(a)

$$T(s) = \frac{10}{s^3 + 10.1s^2 + s + 5}$$

(b)

$$T(s) = \frac{10s^{-3}}{5s^{-3} + s^{-2} + 10.1s^{-1} + 1}$$



$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -5x_1 - x_2 - 10.1x_3 + 10r(t) \\ \omega(t) = x_1 \end{cases}$$

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -1 & -10.1 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} r(t)$$

$$\omega(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X$$

(c)

$$\Delta(s) = |sI - A| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 5 & 1 & s+10.1 \end{vmatrix} = s^3 + 10.1s^2 + s + 5 = 0$$

$$\begin{cases} s_1 = -10.05 \\ s_{2,3} = -0.025 \pm j0.7049 \end{cases}$$

Problem 3.18 (Solution)

$$\dot{X} = \begin{bmatrix} 1 & 1 & -1 \\ 4 & 3 & 0 \\ -2 & 1 & 10 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y(t) = \begin{bmatrix} 20 & 30 & 10 \end{bmatrix} X$$

$$\begin{cases} \dot{X} = AX + Bu \\ Y = CX \end{cases}$$

$$Y(s) = C(sI - A)^{-1} B.U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1} B = \begin{bmatrix} 20 & 30 & 10 \end{bmatrix} \begin{bmatrix} s-1 & -1 & +1 \\ -4 & s-3 & 0 \\ 2 & -1 & s-10 \end{bmatrix}^{-1} . \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$G(s) = \frac{10s^2 - 60s - 70}{s^3 - 14s^2 + 37s + 20}$$