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# ELG 3150

## Introduction to Control Systems

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# My agenda for this tutorial session

- I will introduce the Laplace Transforms as a useful tool for you to tackle control systems analysis.
- I will give examples on how to derive the transfer function of different systems represented by their mathematical models.
- I will talk about solving the system equation to obtain the system response and I will give examples regarding that.

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# Laplace Transforms

- We transform the system (model), which is identified by a differential equation (transfer function), from time domain to frequency domain.

$$\frac{d}{dt} = D = s \quad s = \sigma + j\omega$$

- We normally assume zero initial conditions at  $t=0$ . If any of the initial conditions are non-zero, then they must be added.

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- **How to implement time-to-frequency transformation**

$$f(s) = \int_0^{\infty} f(t) e^{-st} dt$$

where,

$f(t)$  = the function in terms of time  $t$

$f(s)$  = the function in terms of the Laplace  $s$

### Example

For  $f(t) = 5$ ,

$$f(s) = \int_0^{\infty} f(t) e^{-st} dt = \int_0^{\infty} 5e^{-st} dt = \left. \frac{5}{s} e^{-st} \right|_0^{\infty} = \left[ \frac{5}{s} e^{-s\infty} \right] - \left[ \frac{5e^{-s0}}{s} \right] = \frac{5}{s}$$

**However, in control systems analysis we would rather use tables to implement such transformation**

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- How can we carry out control system analysis using Laplace Transforms

1. We convert the system transfer function (differential equation) to the s-domain using Laplace Transform by replacing ' $d/dt$ ' or ' $D$ ' with ' $s$ '.
2. We convert the input function to the s-domain using the transform tables.
3. We combine algebraically the input and the transfer function to find out an output function.
4. We Use partial fractions to reduce the output function to simpler components.
5. We convert the output equation from the s-domain back to the time-domain to obtain the response using Inverse Laplace Transforms according to the tables.

# Laplace Transforms Properties

## L: Laplace Transform

TIME DOMAIN	FREQUENCY DOMAIN
$f(t)$	$f(s)$
$Kf(t)$	$KL[f(t)]$
$f_1(t) + f_2(t) - f_3(t) + \dots$	$f_1(s) + f_2(s) - f_3(s) + \dots$
$\frac{df(t)}{dt}$	$sL[f(t)] - f(0^-)$
$\frac{d^2f(t)}{dt^2}$	$s^2L[f(t)] - sf(0^-) - \frac{df(0^-)}{dt}$
$\frac{d^n f(t)}{dt^n}$	$s^n L[f(t)] - s^{n-1} f(0^-) - s^{n-2} \frac{df(0^-)}{dt} - \dots - \frac{d^{n-1} f(0^-)}{dt^{n-1}}$
$\int_0^t f(t) dt$	$\frac{L[f(t)]}{s}$
$f(t-a)u(t-a), a > 0$	$e^{-as}L[f(t)]$
$e^{-at}f(t)$	$f(s-a)$
$f(at), a > 0$	$\frac{1}{a}f\left(\frac{s}{a}\right)$
$tf(t)$	$\frac{-df(s)}{ds}$
$t^n f(t)$	$(-1)^n \frac{d^n f(s)}{ds^n}$
$\frac{f(t)}{t}$	$\int_s^\infty f(u) du$

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- Laplace Transforms make things easy, for example consider the following **convolution process:**

$$F(s) * G(s) = \int_{-\infty}^{+\infty} f(t - \tau)g(\tau)d\tau$$

which can be performed as simple multiplication in frequency domain using Laplace Transforms:

$$L[F(s) * G(s)] = F(s).G(s)$$

# Laplace Transforms Table

TIME DOMAIN		FREQUENCY DOMAIN
$\delta(t)$	unit impulse	1
A	step	$\frac{A}{s}$
t	ramp	$\frac{1}{s^2}$
$t^2$		$\frac{2}{s^3}$
$t^n, n > 0$		$\frac{n!}{s^{n+1}}$
$e^{-at}$	exponential decay	$\frac{1}{s+a}$
$\sin(\omega t)$		$\frac{\omega}{s^2 + \omega^2}$
$\cos(\omega t)$		$\frac{s}{s^2 + \omega^2}$
$te^{-at}$		$\frac{1}{(s+a)^2}$
$t^2 e^{-at}$		$\frac{2!}{(s+a)^3}$



TIME DOMAIN

FREQUENCY DOMAIN

$$e^{-at} \sin(\omega t)$$

$$\frac{\omega}{(s + a)^2 + \omega^2}$$

$$e^{-at} \cos(\omega t)$$

$$\frac{s + a}{(s + a)^2 + \omega^2}$$

$$e^{-at} \sin(\omega t)$$

$$\frac{\omega}{(s + a)^2 + \omega^2}$$

$$e^{-at} \left[ B \cos \omega t + \left( \frac{C - aB}{\omega} \right) \sin \omega t \right]$$

$$\frac{Bs + C}{(s + a)^2 + \omega^2}$$

$$2|A|e^{-\alpha t} \cos(\beta t + \theta)$$

$$\frac{A}{s + \alpha - \beta j} + \frac{A^{\text{complex conjugate}}}{s + \alpha + \beta j}$$

$$2t|A|e^{-\alpha t} \cos(\beta t + \theta)$$

$$\frac{A}{(s + \alpha - \beta j)^2} + \frac{A^{\text{complex conjugate}}}{(s + \alpha + \beta j)^2}$$

$$\frac{(c - a)e^{-at} - (c - b)e^{-bt}}{b - a}$$

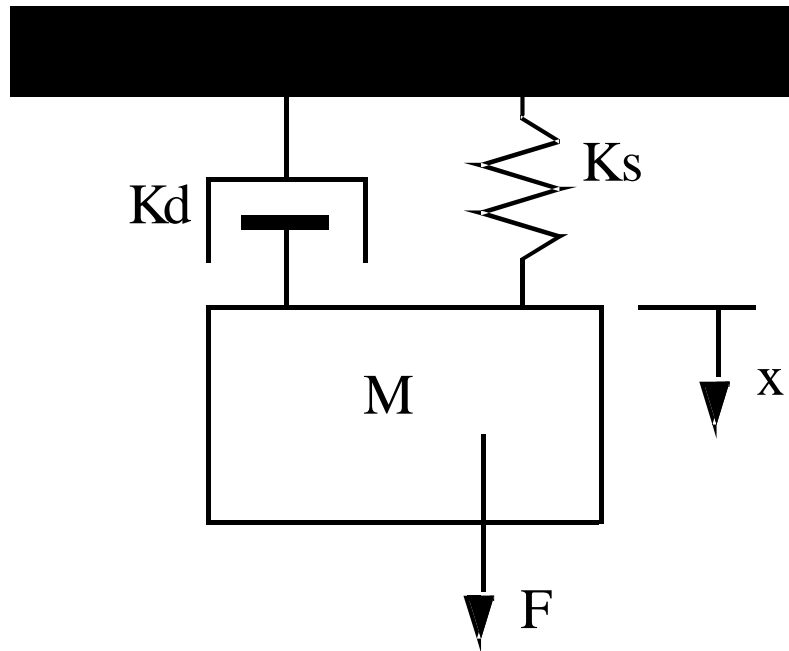
$$\frac{s + c}{(s + a)(s + b)}$$

$$\frac{e^{-at} - e^{-bt}}{b - a}$$

$$\frac{1}{(s + a)(s + b)}$$

## Example

For this simple mechanical system obtain the transfer function in s-domain (i.e. dynamical model)



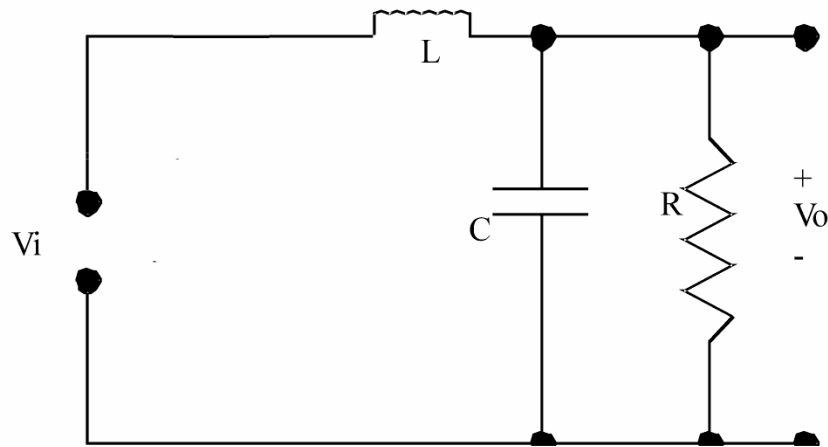
$$F = MD^2x + K_d Dx + K_s x$$

$$\frac{F(t)}{x(t)} = MD^2 + K_d D + K_s$$

$$L\left[\frac{F(t)}{x(t)}\right] = \frac{F(s)}{x(s)} = Ms^2 + K_d s + K_s$$

## Example

For this electrical circuit obtain the transfer function in s-domain



$$V_o = \frac{V_i \left( \frac{1}{DC + \frac{1}{R}} \right)}{DL + \left( \frac{1}{DC + \frac{1}{R}} \right)} = \frac{V_i \left( \frac{R}{1 + DCR} \right)}{DLR + \left( \frac{R}{1 + DCR} \right)} = V_i \left( \frac{R}{D^2 R^2 LC + DLR + R} \right)$$

$$\mathcal{L} \left[ \frac{V_o(t)}{V_i(t)} \right] = \frac{V_o(s)}{V_i(s)} = \left( \frac{R}{s^2 R^2 LC + sLR + R} \right)$$

Device	Time domain	s-domain	Impedance
Resistor	$V(t) = RI(t)$	$V(s) = RI(s)$	$Z = R$
Capacitor	$V(t) = \frac{1}{C} \int I(t) dt$	$V(s) = \left(\frac{1}{C}\right) \frac{I(s)}{s}$	$Z = \frac{1}{sC}$
Inductor	$V(t) = L \frac{d}{dt} I(t)$	$V(s) = LsI(s)$	$Z = Ls$

Impedances of electrical components

- Have more **examples** on how to obtain the transfer function in the s-domain for the given systems:

- Now back to our **simple mechanical system** to obtain its output response to a **step input** of magnitude 1000 N.

Given,

$$\frac{x(s)}{F(s)} = \frac{1}{Ms^2 + K_d s + K_s}$$

$$F(s) = \frac{A}{s}$$

Therefore,

$$x(s) = \left( \frac{x(s)}{F(s)} \right) F(s) = \left( \frac{1}{Ms^2 + K_d s + K_s} \right) \frac{A}{s}$$

Assume,

$$K_d = 3000 \frac{Ns}{m}$$

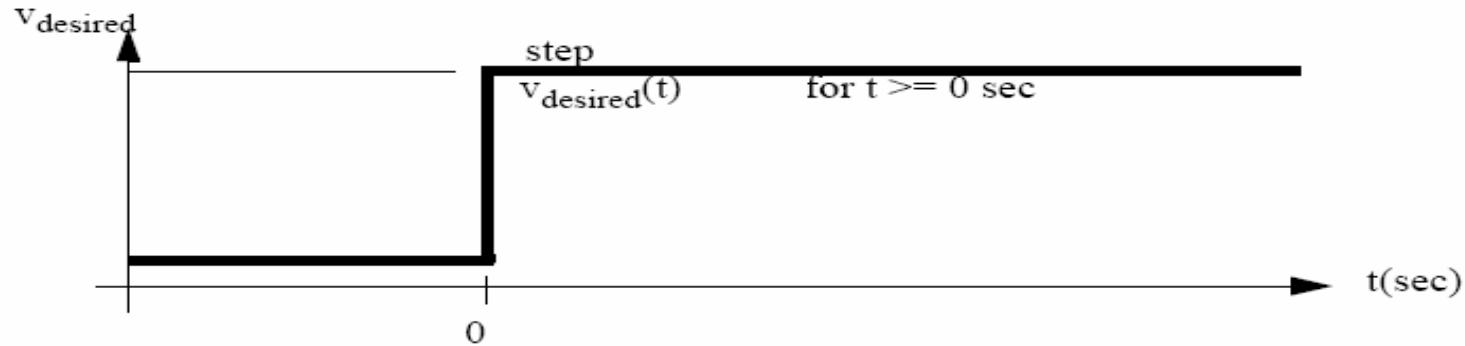
$$K_s = 2000 \frac{N}{m}$$




$$M = 1000 \text{kg}$$

$$A = 1000 \text{N}$$

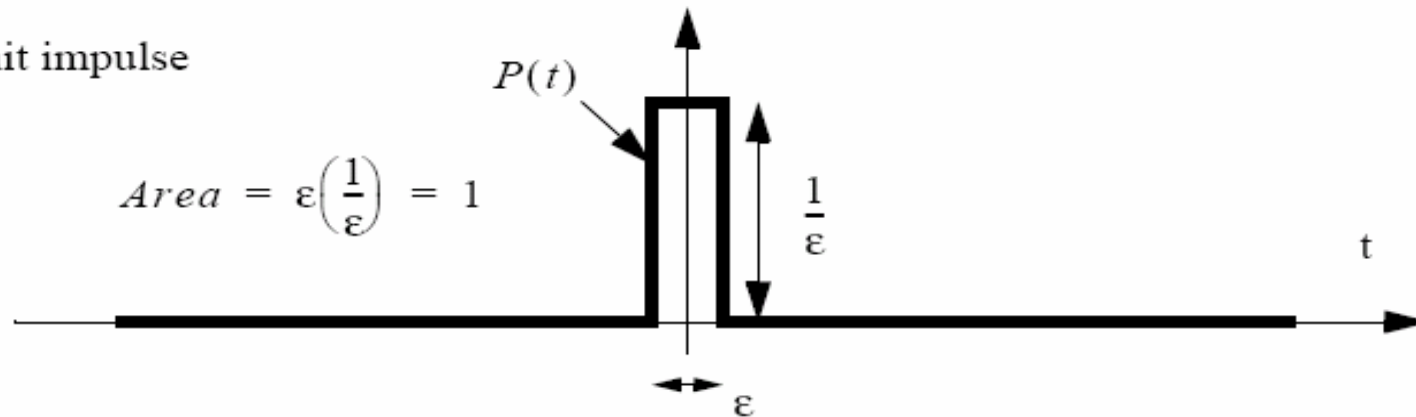
$$\therefore x(s) = \frac{1}{(s^2 + 3s + 2)s}$$

## ■ Types of inputs (driving force)



	Input type	Time function	Laplace function
	STEP	$f(t) = Au(t)$	$f(s) = \frac{A}{s}$
	RAMP	$f(t) = Atu(t)$	$f(s) = \frac{A}{s^2}$
	SINUSOID	$f(t) = A \sin(\omega t)u(t)$	$f(s) = \frac{A\omega}{s^2 + \omega^2}$

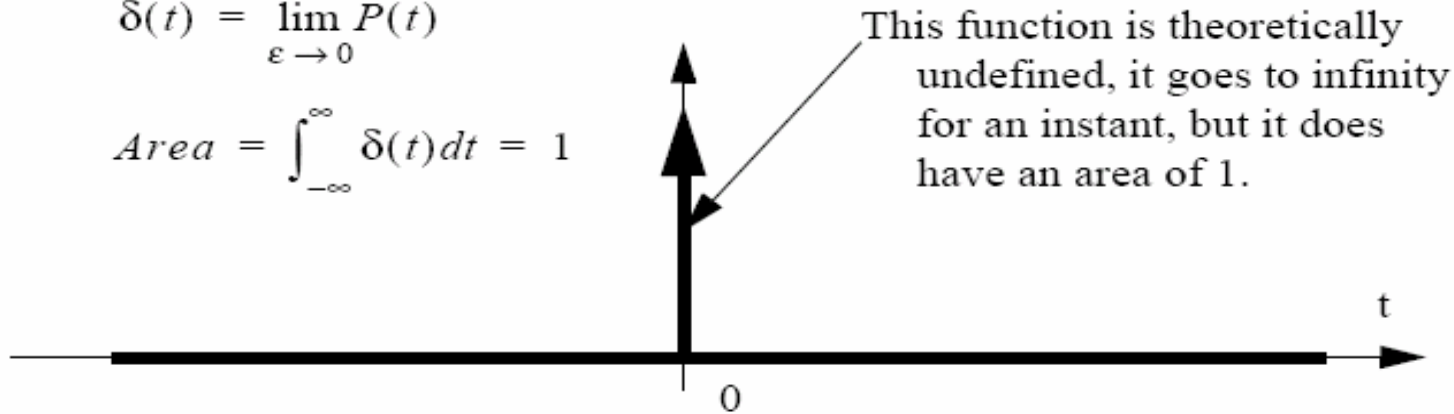
Unit impulse



Dirac delta function

$$\delta(t) = \lim_{\epsilon \rightarrow 0} P(t)$$

$$Area = \int_{-\infty}^{\infty} \delta(t) dt = 1$$



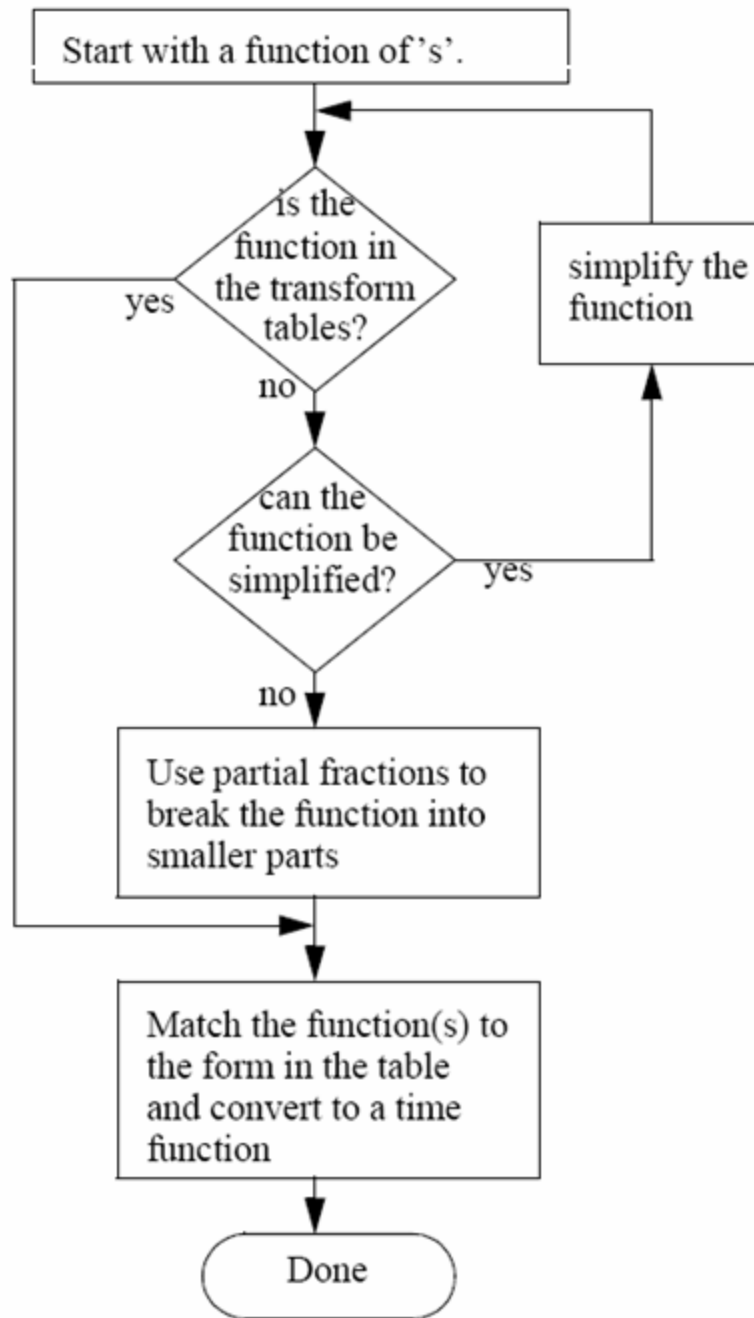
$$F(s) = L[\delta(t)] = 1$$

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# Solving for system response using Inverse Laplace Transforms

- Use the following flow chart to determine which way to go to do Inverse Laplace Transforms





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- Performing **partial fraction simplification**

$$x(s) = \frac{1}{(s^2 + 3s + 2)s} = \frac{1}{(s + 1)(s + 2)s} = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{s + 2}$$

$$A = \lim_{s \rightarrow 0} \left[ s \left( \frac{1}{(s + 1)(s + 2)s} \right) \right] = \frac{1}{2}$$

$$B = \lim_{s \rightarrow -1} \left[ (s + 1) \left( \frac{1}{(s + 1)(s + 2)s} \right) \right] = -1$$

$$C = \lim_{s \rightarrow -2} \left[ (s + 2) \left( \frac{1}{(s + 1)(s + 2)s} \right) \right] = \frac{1}{2}$$

$$x(s) = \frac{1}{(s^2 + 3s + 2)s} = \frac{0.5}{s} + \frac{-1}{s + 1} + \frac{0.5}{s + 2}$$

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- Now we proceed with the **Inverse Laplace Transforms** to obtain the system time response

$$x(t) = L^{-1}[x(s)] = L^{-1}\left[\frac{0.5}{s} + \frac{-1}{s+1} + \frac{0.5}{s+2}\right]$$

$$x(t) = L^{-1}\left[\frac{0.5}{s}\right] + L^{-1}\left[\frac{-1}{s+1}\right] + L^{-1}\left[\frac{0.5}{s+2}\right]$$

$$x(t) = [0.5] + [(-1)e^{-t}] + [(0.5)e^{-2t}]$$

$$x(t) = 0.5 - e^{-t} + 0.5e^{-2t}$$

- Try to think about the case where the driving force is an **impulse input**. So what will be the **impulse response** ?

$$F(s) = L[\delta(t)] = 1$$

$$x(s) = \frac{1}{(s^2 + 3s + 2)} = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

- What's about the **partial fractions simplification for the repeated roots.**

## Example

$$x(s) = \frac{1}{s^2(s+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}$$

$$C = \lim_{s \rightarrow -1} \left[ (s+1) \left( \frac{1}{s^2(s+1)} \right) \right] = 1$$

$$A = \lim_{s \rightarrow 0} \left[ s^2 \left( \frac{1}{s^2(s+1)} \right) \right] = \lim_{s \rightarrow 0} \left[ \frac{1}{s+1} \right] = 1$$

$$B = \lim_{s \rightarrow 0} \left[ \frac{d}{ds} \left[ s^2 \left( \frac{1}{s^2(s+1)} \right) \right] \right] = \lim_{s \rightarrow 0} \left[ \frac{d}{ds} \left( \frac{1}{s+1} \right) \right] = \lim_{s \rightarrow 0} [-(s+1)^{-2}] = -1$$

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Have another example

$$F(s) = \frac{5}{s^2(s+1)^3}$$

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

$$\begin{aligned}
\frac{5}{s^2(s+1)^3} &= \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)} \\
&= \frac{A(s+1)^3 + Bs(s+1)^3 + Cs^2 + Ds^2(s+1) + Es^2(s+1)^2}{s^2(s+1)^3} \\
&= \frac{s^4(B+E) + s^3(A+3B+D+2E) + s^2(3A+3B+C+D+E) + s(3A+B) + (A)}{s^2(s+1)^3}
\end{aligned}$$

$$\frac{5}{s^2(s+1)^3} = \frac{5}{s^2} + \frac{-15}{s} + \frac{5}{(s+1)^3} + \frac{10}{(s+1)^2} + \frac{15}{(s+1)}$$

$$\frac{5}{s^2(s+1)^3} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s+1)^3} + \frac{D}{(s+1)^2} + \frac{E}{(s+1)}$$

$$A = \lim_{s \rightarrow 0} \left[ \left( \frac{5}{s^2(s+1)^3} \right) s^2 \right] = \lim_{s \rightarrow 0} \left[ \frac{5}{(s+1)^3} \right] = 5$$

$$B = \lim_{s \rightarrow 0} \left[ \frac{d}{ds} \left( \frac{5}{s^2(s+1)^3} \right) s^2 \right] = \lim_{s \rightarrow 0} \left[ \frac{d}{ds} \left( \frac{5}{(s+1)^3} \right) \right] = \lim_{s \rightarrow 0} \left[ \frac{5(-3)}{(s+1)^4} \right] = -15$$

$$C = \lim_{s \rightarrow -1} \left[ \left( \frac{5}{s^2(s+1)^3} \right) (s+1)^3 \right] = \lim_{s \rightarrow -1} \left[ \frac{5}{s^2} \right] = 5$$

$$D = \lim_{s \rightarrow -1} \left[ \frac{1}{1!} \frac{d}{ds} \left( \frac{5}{s^2(s+1)^3} \right) (s+1)^3 \right] = \lim_{s \rightarrow -1} \left[ \frac{1}{1!} \frac{d}{ds} \frac{5}{s^2} \right] = \lim_{s \rightarrow -1} \left[ \frac{1}{1!} \frac{-2(5)}{s^3} \right] = 10$$

$$E = \lim_{s \rightarrow -1} \left[ \frac{1}{2!} \frac{d^2}{ds^2} \left( \frac{5}{s^2(s+1)^3} \right) (s+1)^3 \right] = \lim_{s \rightarrow -1} \left[ \frac{1}{2!} \frac{d^2}{ds^2} \frac{5}{s^2} \right] = \lim_{s \rightarrow -1} \left[ \frac{1}{2!} \frac{30}{s^4} \right] = 15$$

$$\frac{5}{s^2(s+1)^3} = \frac{5}{s^2} + \frac{-15}{s} + \frac{5}{(s+1)^3} + \frac{10}{(s+1)^2} + \frac{15}{(s+1)}$$



## Special case

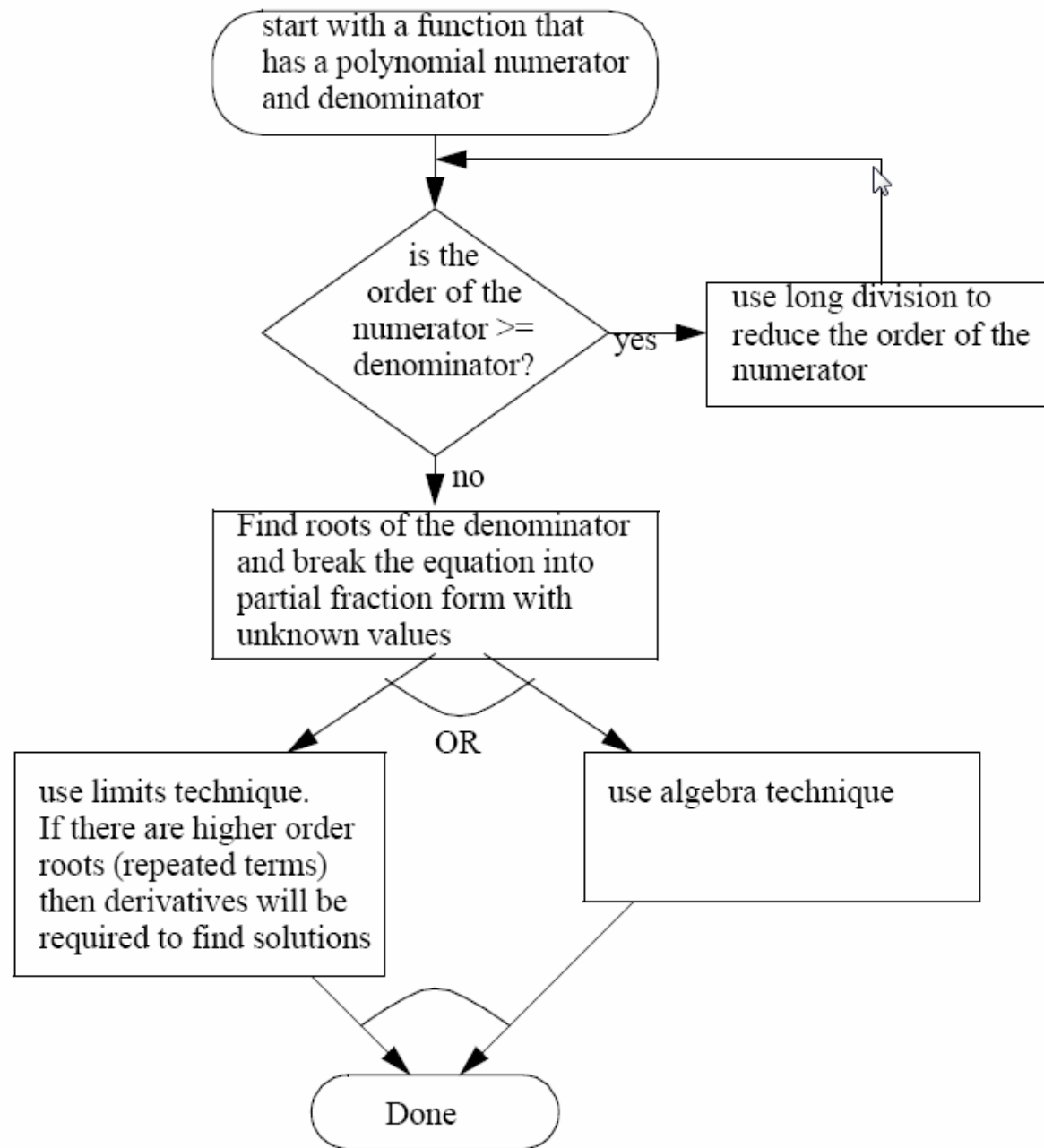
$$x(s) = \frac{5s^3 + 3s^2 + 8s + 6}{s^2 + 4}$$

This cannot be solved using partial fractions because the numerator is 3rd order and the denominator is only 2nd order. Therefore long division can be used to reduce the order of the equation.

$$\begin{array}{r} 5s + 3 \\ s^2 + 4 \overline{) 5s^3 + 3s^2 + 8s + 6} \\ \underline{5s^3 + 20s} \phantom{+ 6} \\ 3s^2 - 12s + 6 \\ \underline{3s^2 + 12} \\ -12s - 6 \end{array}$$

This can now be used to write a new function that has a reduced portion that can be solved with partial fractions.

$$x(s) = 5s + 3 + \frac{-12s - 6}{s^2 + 4} \quad \text{solve} \quad \frac{-12s - 6}{s^2 + 4} = \frac{A}{s + 2j} + \frac{B}{s - 2j}$$



## ■ Initial and Final Value Theorems

$$X(s) = \frac{1}{(s^2 + 3s + 2)s}$$

$$x(t \rightarrow \infty) = \lim_{s \rightarrow 0} [sX(s)] \quad \text{Final value theorem}$$

$$\therefore x(t \rightarrow \infty) = \lim_{s \rightarrow 0} \left[ \frac{1s}{(s^2 + 3s + 2)s} \right] = \lim_{s \rightarrow 0} \left[ \frac{1}{s^2 + 3s + 2} \right] = \frac{1}{(0)^2 + 3(0) + 2} = \frac{1}{2}$$

$$x(t \rightarrow 0) = \lim_{s \rightarrow \infty} [sX(s)] \quad \text{Initial value theorem}$$

$$\therefore x(t \rightarrow 0) = \lim_{s \rightarrow \infty} \left[ \frac{1(s)}{(s^2 + 3s + 2)s} \right] = \frac{1}{((\infty)^2 + 3(\infty) + 2)} = \frac{1}{\infty} = 0$$

Given the mass-spring-damper example

$$\therefore \frac{x(s)}{F(s)} = \frac{\frac{1}{M}}{s^2 + \frac{K_d}{M}s + \frac{K_s}{M}}$$

Component values are,

$$M = 1kg \quad K_s = 2\frac{N}{m} \quad K_d = 0.5\frac{Ns}{m}$$

The sinusoidal input is converted to the s-domain,

$$F(t) = 5 \cos(6t)N$$

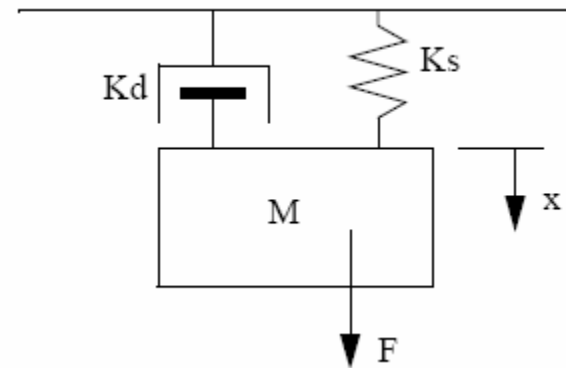
$$\therefore F(s) = \frac{5s}{s^2 + 6^2}$$

This can be combined with the transfer function to obtain the output function,

$$x(s) = F(s) \left( \frac{x(s)}{F(s)} \right) = \left( \frac{5s}{s^2 + 6^2} \right) \left( \frac{\frac{1}{1}}{s^2 + 0.5s + 2} \right)$$

$$\therefore x(s) = \frac{5s}{(s^2 + 36)(s^2 + 0.5s + 2)}$$

$$\therefore x(s) = \frac{A}{s + 6j} + \frac{B}{s - 6j} + \frac{C}{s - 0.5 + 1.39j} + \frac{D}{s - 0.5 - 1.39j}$$



$$A = \lim_{s \rightarrow -6j} \left[ \frac{(s+6j)(5s)}{(s-6j)(s^2+36)(s^2+0.5s+2)} \right] = \frac{-30j}{(-12j)(36-3j+2)}$$

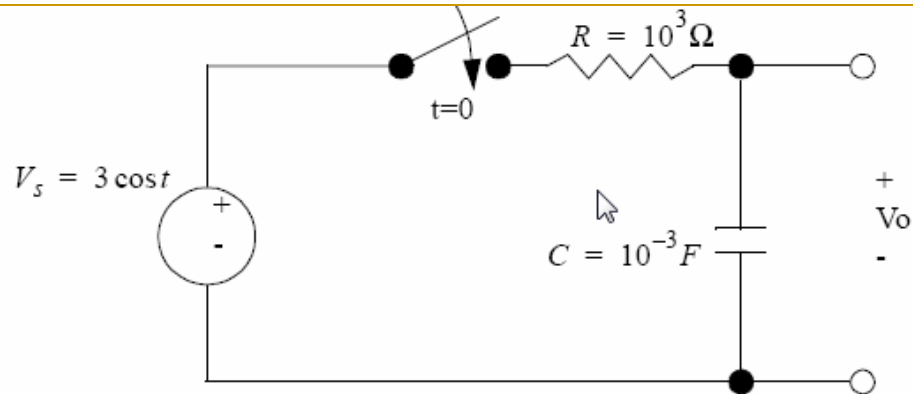
$$\therefore A = \frac{-30j}{-432j-36-24j} = \frac{30j}{36+456j} \times \frac{36-456j}{36-456j} = \frac{13680+1080j}{209,232} = 0.0654 + 0.00516j$$

--Continue on to find B, C, D same way

$$\therefore x(s) = \frac{0.0654 + 0.00516j}{s+6j} + \frac{0.0654 - 0.00516j}{s-6j} + \dots$$

Do inverse Laplace transform

$$x(t) = 2\sqrt{0.0654^2 + 0.00516^2} e^{-0t} \cos\left(0.00516t + \text{atan}\left(-\frac{0.00516}{0.0654}\right)\right) + \dots$$



As normal, relate the source voltage to the output voltage using component values in the s-domain.

$$V_o = V_s \left( \frac{Z_C}{Z_R + Z_C} \right) \quad V_s(s) = \frac{3s^2}{s^2 + 1} \quad Z_R = R \quad Z_C = \frac{1}{sC}$$

Next, equations are combined. The numerator of resulting output function must be reduced by long division.

$$V_o = \frac{3s^2}{s^2 + 1} \left( \frac{\frac{1}{sC}}{R + \frac{1}{sC}} \right) = \frac{3s^2}{(s^2 + 1)(1 + sRC)} = \frac{3s^2}{(s^2 + 1)(s10^3 10^{-3} + 1)}$$

The output function can be converted to a partial fraction form and the residues calculated.

$$V_o = \frac{3s^2}{(s^2 + 1)(s + 1)} = \frac{As + B}{s^2 + 1} + \frac{C}{s + 1} = \frac{As^2 + As + Bs + B + Cs^2 + C}{(s^2 + 1)(s + 1)}$$

$$V_o = \frac{3s^2}{(s^2 + 1)(s + 1)} = \frac{s^2(A + C) + s(A + B) + (B + C)}{(s^2 + 1)(s + 1)}$$

$$B + C = 0 \quad \therefore C = -B$$

$$A + B = 0 \quad \therefore A = -B$$

$$A + C = 3 \quad \therefore -B - B = 3 \quad \therefore B = -1.5 \quad \therefore A = 1.5 \quad \therefore C = 1.5$$

$$V_o = \frac{1.5s - 1.5}{s^2 + 1} + \frac{1.5}{s + 1}$$

The output function can be converted to a function of time using the transform tables, as shown below.

$$V_o(t) = L^{-1}[V_o(s)] = L^{-1}\left[\frac{1.5s - 1.5}{s^2 + 1} + \frac{1.5}{s + 1}\right] = L^{-1}\left[\frac{1.5s - 1.5}{s^2 + 1}\right] + L^{-1}\left[\frac{1.5}{s + 1}\right]$$

$$\therefore V_o(t) = 1.5L^{-1}\left[\frac{s}{s^2 + 1}\right] - 1.5L^{-1}\left[\frac{1}{s^2 + 1}\right] + 1.5e^{-t}$$

$$\therefore V_o(t) = 1.5 \cos t - 1.5 \sin t + 1.5e^{-t}$$

$$\therefore V_o(t) = \sqrt{1.5^2 + 1.5^2} \cos\left(t + \operatorname{atan}\left(\frac{-1.5}{1.5}\right)\right) + 1.5e^{-t}$$

$$\therefore V_o(t) = 2.121 \cos\left(t - \frac{\pi}{4}\right) + 1.5e^{-t}$$

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# Conclusion

- Laplace Transforms are simple, useful and powerful tools to solve for system response and hence analyse the control system
- **BUT we need to prove that !!!!**



# Solving the System Equation (Response) as differential equation

## Example

$$\dot{X} + 0.5X = 2\delta$$

The homogeneous solution  $\longrightarrow$  the deriving force (input) =0

$$\text{let } X_h = e^{At} \text{ so } \dot{X}_h = Ae^{At}$$

$$\dot{X} + 0.5X = 0$$

$$A + 0.5 = 0$$

$$X_h = Ce^{-0.5t}$$

The particular solution  $\longrightarrow$  time=0

$$\text{let } X_p = A \text{ so } \dot{X}_p = 0$$

$$\dot{X} + 0.5X = 2\delta(0)$$

$$0 + 0.5A = 2 \cdot 0$$

$$X_p = A = 0$$

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So equation solution (system response) = homogenous response + particular response

$$X = X_h + X_p$$
$$X = Ce^{-0.5t}$$

The initial condition caused by the impulse function

$$\left(\frac{1}{dt}\right)X_0 + 0.5(0) = 2\left(\frac{1}{dt}\right)$$

$$X_0 = 2$$

therefore

$$X(t) = Ce^{-0.5t}$$

$$X_0 = 2 = Ce^0 \longrightarrow C = 2$$

$$X(t) = 2e^{-0.5t}$$

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- While solving for system response using Laplace Transforms

$$(s + 0.5)X(s) = 2$$

$$X(s) = \frac{2}{s + 0.5}$$

$$X(t) = L^{-1}[X(s)] = 2e^{-0.5t}$$

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# Reference

- Electronic book

“Dynamic System Modeling and Control”  
by Hugh Jack at

[http://claymore.engineer.gvsu.edu/~jackh/books/model/pdf/model2\\_6.pdf](http://claymore.engineer.gvsu.edu/~jackh/books/model/pdf/model2_6.pdf)

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# Your Questions

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